

LINEAR N-GRAPHS

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ABSTRACT. We call a simple graph G a *linear N -graph* if its ordinary (vertex) chromatic number equals to the linear chromatic number of its neighborhood complex $\mathcal{N}(G)$. We prove that the linearity is preserved under taking joins and multiplication of vertices, and give a complete characterization of linear N -trees.

1. PRELIMINARIES

An (*abstract*) *simplicial complex* Δ on a finite set X is a family of subsets of X satisfying the following properties.

- (i) $\{x\} \in \Delta$ for all $x \in X$.
- (ii) If $F \in \Delta$ and $H \subset F$, then $H \in \Delta$.

The elements of Δ are called *faces* of it; the *dimension* of a face F is $\dim(F) := |F| - 1$, and the *dimension* of Δ is defined to be $\dim(\Delta) := \max\{\dim(F) : F \in \Delta\}$. The 0 and 1-dimensional faces of Δ are called *vertices* and *edges* while maximal faces are called *facets*. We denote by \mathcal{F}_Δ and $\mathcal{F}_\Delta(x)$ the set of facets of Δ and those containing a given vertex $x \in X$ respectively.

We next introduce some basics of graph theory, and we refer readers to [3] for more details.

By a simple graph G , we will mean an undirected graph without loops or multiple edges. If G is a graph, $V(G)$ and $E(G)$ (or simply V and E) denote its vertex and edge sets. An edge between u and v is denoted by $e = uv$. If $U \subset V$, the graph induced on U is written G_U , and $G - U$ denotes the graph induced on $V - U$. In particular, we abbreviate $G - \{v\}$ to $G - v$. We denote by $|G|$ and $\|G\|$ the order and size of G , while $d(v)$ denotes the degree of a given vertex $v \in V$. A subset $U \subseteq V$ is called an *independent set* if no two vertices in U forms an edge in G .

Throughout C_n and P_n denote the (chordless) cycle and path with n vertices, while K_n and K_{n_1, n_2, \dots, n_r} the complete graph and complete r -partite graph respectively. We recall that a graph G is said to be a *tree* if it is connected and contains no cycle.

There are various ways to associate a simplicial complex to a given simple graph $G = (V, E)$; however, we here only consider one of them, known as the neighborhood complex of G . The set $\mathcal{N}(u) := \{v \in V : uv \in E\}$ is called the (open) *neighborhood* of u in G , and the *neighborhood complex* $\mathcal{N}(G)$ of $G = (V, E)$ is defined to be the

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simplicial complex on V such that a subset $F \subseteq V$ is a face of $\mathcal{N}(G)$ if and only if

$$\bigcap_{u \in F} \mathcal{N}(u) \neq \emptyset.$$

We illustrate in Figure 1, a simple graph and its neighborhood complex.

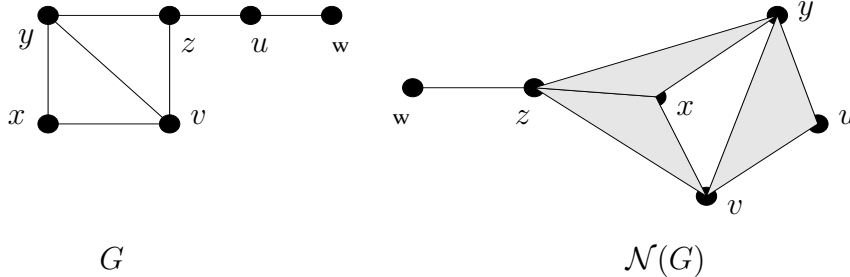


FIGURE 1. A simple graph and its neighborhood complex.

2. LINEAR COLORINGS OF SIMPLICIAL COMPLEXES AND LINEAR N-GRAPHS

The notion of linear colorings of simplicial complexes was introduced by Civan and Yalçın [1] as a combinatorial tool to study the topology of simplicial complexes. Since it is possible to associate simplicial complexes to simple graphs, the method of linear coloring seems to be useful for understanding the structures of graphs as well. We here explore this connection in terms of the neighborhood complexes of graphs. We refer readers to [1] for more details.

Definition 2.1. Let Δ be a simplicial complex with vertex set X . A mapping $\alpha: X \rightarrow [k]$ is called a *linear coloring* of Δ if for every $i \in [k]$, the set $\mathcal{F}_i = \{\mathcal{F}(v) : \alpha(v) = i\}$ is linearly ordered by inclusion, where $[k] = \{1, \dots, k\}$. In this case, the map α is said to be a k -linear coloring of Δ . The least integer k for which Δ admits a k -linear coloring is called the *linear chromatic number* of Δ and denoted by $\text{lchr}(\Delta)$.

An alternative way to decide whether a given coloring of a simplicial complex Δ is linear can be stated as follows. A coloring $\alpha: X \rightarrow [k]$ is a linear coloring of Δ if and only if whenever $\alpha(x) = \alpha(y)$ for some $x, y \in X$, then one of the inclusions $\mathcal{F}_\Delta(x) \subseteq \mathcal{F}_\Delta(y)$ or $\mathcal{F}_\Delta(y) \subseteq \mathcal{F}_\Delta(x)$ holds. In Figure 2, we present a linear coloring of a simplicial complex Δ and a coloring of it which is not linear.

Let $G = (V, E)$ be a simple graph. We recall that a (vertex) coloring of G is a surjective mapping $\nu: V \rightarrow [n]$ such that $\nu(x) \neq \nu(y)$ whenever $(x, y) \in E$, and the least integer n for which G admits a (vertex) coloring is called the *vertex chromatic number* and denoted by $\chi(G)$.

Civan and Yalçın [1] proved that any linear coloring of the neighborhood complex of a given graph G is in fact a vertex coloring of it.

Proposition 2.2. ([1]) *Let $G = (V, E)$ be a simple graph and let $\mathcal{N}(G)$ denote its neighborhood complex. If $\kappa: V \rightarrow [k]$ is a k -linear coloring of $\mathcal{N}(G)$, then κ is a coloring of the underlying graph G .*

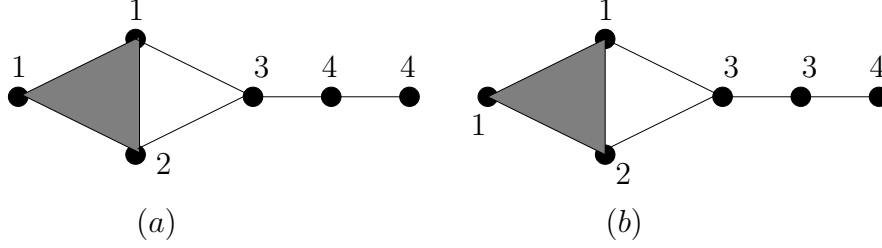


FIGURE 2. A linear coloring (a) of a simplicial complex Δ and a non-linear coloring (b).

It follows that for any graph G , we have $\text{lchr}(\mathcal{N}(G)) \geq \chi(G)$. Furthermore, it can be seen that not every vertex coloring of a graph G turns out to be a linear coloring of $\mathcal{N}(G)$; however, they have also provided a characterization when a vertex coloring of a simple graph is linear whose converse does not hold in general.

Proposition 2.3. ([1]) *A coloring $\nu: V \rightarrow [k]$ of $G = (V, E)$ is a k -linear coloring of $\mathcal{N}(G)$ if either $\mathcal{N}(v) \subseteq \mathcal{N}(u)$ or $\mathcal{N}(u) \subseteq \mathcal{N}(v)$ holds for every $x, y \in V$ with $\nu(x) = \nu(y)$.*

The central objects of our study are those simple graphs whose chromatic numbers equal to the linear chromatic numbers of their neighborhood complexes.

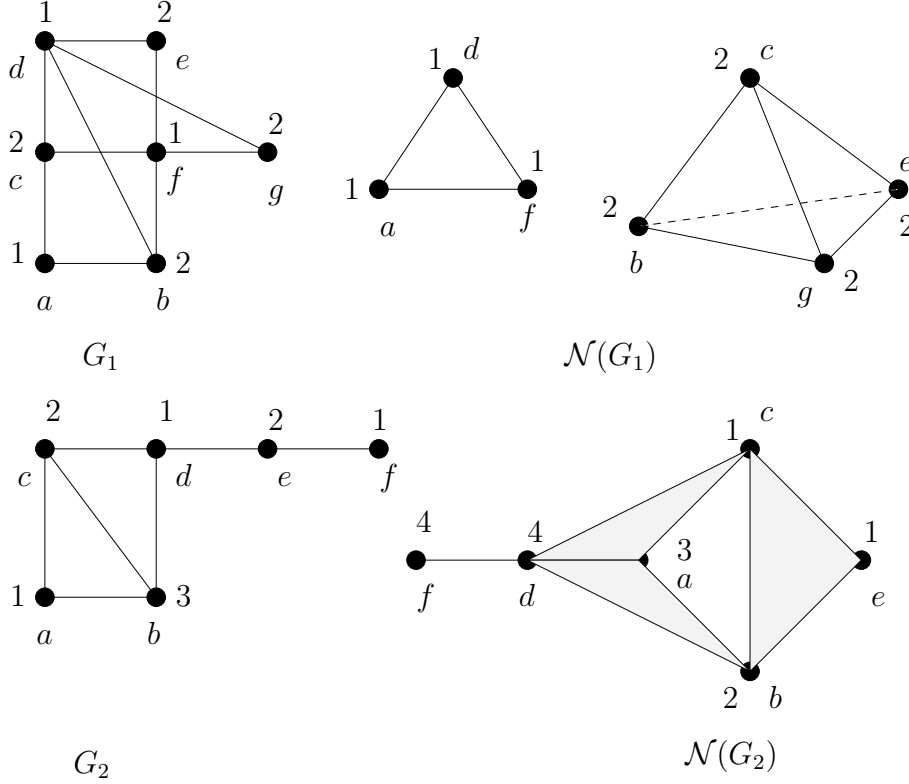
Definition 2.4. A simple graph G is called a *linear N -graph* if $\chi(G) = \text{lchr}(\mathcal{N}(G))$.

Example 2.5. The complete graphs and complete r -partite graphs are clearly examples of linear N -graphs. The graphs depicted in Figure 3, G_1 is a linear N -graph with $\chi(G_1) = \text{lchr}(\mathcal{N}(G_1)) = 2$, while G_2 is not a linear N -graph, since $\chi(G_2) = 3$ and $\text{lchr}(G_2) = 4$.

We next show that the linearity is preserved under some operations on graphs. We first recall that for any two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their join $G_1 * G_2$ is obtained from the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ by joining all the vertices of G_1 to all the vertices of G_2 . For example, $K_2 * K_3 = K_5$.

Proposition 2.6. *If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are linear N -graphs, so is their join $G_1 * G_2$.*

Proof. We let $V_1 = \{v_1, \dots, v_n\}$ and $V_2 = \{u_1, \dots, u_m\}$. Since G_1 and G_2 are both linear N -graphs, we have the equalities $\chi(G_1) = \text{lchr}(\mathcal{N}(G_1)) =: k_1$ and $\chi(G_2) = \text{lchr}(\mathcal{N}(G_2)) =: k_2$. On the other hand, it is well known that $\chi(G_1 * G_2) = \chi(G_1) + \chi(G_2)$; hence, we need only to verify that $\text{lchr}(\mathcal{N}(G_1 * G_2)) = k_1 + k_2$. Since the linear chromatic number is an upper bound for the vertex chromatic number, it is enough to show that the number of $k_1 + k_2$ colors is sufficient to color $\mathcal{N}(G_1 * G_2)$ linearly. We note that any facet of $\mathcal{N}(G_1 * G_2)$ is of the form $F \cup V_i$ for some $F \in \mathcal{F}_{\mathcal{N}(G_j)}$, where $1 \leq i, j \leq 2$ and $i \neq j$. Therefore, any two vertices of $\mathcal{N}(G_i)$ having the same color can be linearly colored by the same color in $\mathcal{N}(G_1 * G_2)$, which proves that $\text{lchr}(\mathcal{N}(G_1 * G_2)) \leq k_1 + k_2$. \square

FIGURE 3. A linear N -graph G_1 and a non-linear N -graph G_2 .

Let $G = (V, E)$ be a simple graph with $|V| = n$, and let \mathbb{N} denote the set of positive integers. For a given vector $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$, we define $G \circ \mathbf{m}$ to be the graph constructed by substituting for each $v_i \in V$ an independent set of m_i vertices $v_i^1, v_i^2, \dots, v_i^{m_i}$ and joining v_i^s with v_j^t if and only if v_i and v_j are adjacent in G . It is more customary to allow the vectors \mathbf{m} to have zero coordinates to construct induced subgraphs of a given graph G by multiplication of vertices from G . However, we here only consider the restricted case ([2]).

Proposition 2.7. *If $G = (V, E)$ is a linear N -graph, so is $G \circ \mathbf{m}$ for all $\mathbf{m} \in \mathbb{N}^n$.*

Proof. We note that $\chi(G) = \chi(G \circ \mathbf{m})$ for any $\mathbf{m} \in \mathbb{N}^n$, where $|G| = n$. Therefore, it is enough to show that if $\chi(G) = k$, then the number of k colors is sufficient to color $\mathcal{N}(G \circ \mathbf{m})$ linearly. Let κ be a k -linear coloring of $\mathcal{N}(G)$ and let $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ be given. We write $V = \{x_1, \dots, x_n\}$. Now, for any two $1 \leq i, j \leq m_t$, we have $\mathcal{N}(x_i^t) = \mathcal{N}(x_j^t)$ in $G \circ \mathbf{m}$. Thus, the mapping $\bar{\kappa}: V(G \circ \mathbf{m}) \rightarrow [k]$ defined by $\bar{\kappa}(x_i^t) := \kappa(x_i)$ is linear. This completes the proof. \square

Theorem 2.8. *A tree T is a linear N -graph if and only if $|\{v \in V(T) : d(v) > 1\}| \leq 2$.*

Proof. Since there is nothing to prove if $|E| = 0$ or 1 , we suppose that $|E| > 1$. Assume first that $T = (V, E)$ is a linear N -graph so that $\chi(T) = \text{lchr}(\mathcal{N}(T)) = 2$, and let $V = V_1 \cup V_2$ be the bipartition of vertices of T with respect to a 2-linear coloring

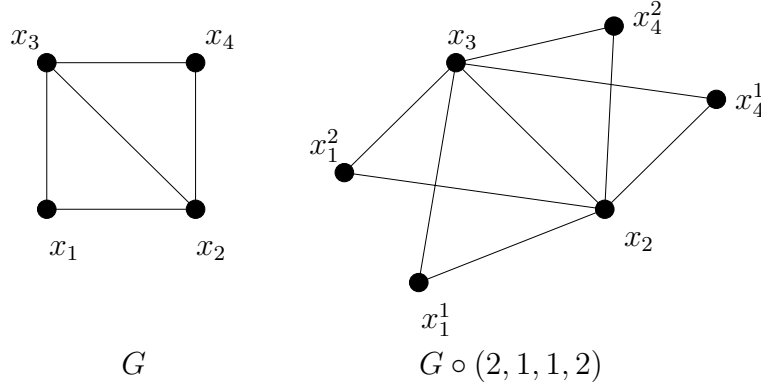


FIGURE 4. A multiplication of vertices of a graph.

$\kappa: V \rightarrow [2]$, where we write $V_1 = \{v_1^1, \dots, v_k^1\}$ and $V_2 = \{v_1^2, \dots, v_l^2\}$ such that $\kappa(v_j^i) = i$ for any $i = 1, 2$ and $j \in [k]$ or $j \in [l]$.

By the linearity of κ , we may assume without loss of generality that the inclusions

$$\begin{aligned} \mathcal{F}_{\mathcal{N}(T)}(v_1^1) &\subseteq \mathcal{F}_{\mathcal{N}(T)}(v_2^1) \subseteq \dots \subseteq \mathcal{F}_{\mathcal{N}(T)}(v_k^1), \\ \mathcal{F}_{\mathcal{N}(T)}(v_1^2) &\subseteq \mathcal{F}_{\mathcal{N}(T)}(v_2^2) \subseteq \dots \subseteq \mathcal{F}_{\mathcal{N}(T)}(v_l^2) \end{aligned}$$

hold. It follows that if F_1 and F_2 are facets of $\mathcal{N}(T)$ containing v_1^1 and v_1^2 respectively, then we must have $V_1 \subseteq F_1$ and $V_2 \subseteq F_2$ by the above inclusions. We note that $F_1 \cap F_2 = \emptyset$ for any such two facets, since κ is also a vertex coloring of T . We further claim that any of these facets is the neighborhood of only one vertex of T . Otherwise, assume that $F_1 = \mathcal{N}(v_r^2)$ and $F_1 = \mathcal{N}(v_s^2)$ for some $r, s \in [l]$ with $r \neq s$. Then, for example, the set $\{v_1^1, v_r^2, v_k^1, v_s^2, v_1^1\}$ forms a cycle in T which is impossible. Therefore, the sets $F_1 = V_1$ and $F_2 = V_2$ are the only facets of $\mathcal{N}(T)$, i.e., $\mathcal{F}(\mathcal{N}(T)) = \{V_1, V_2\}$ which forces $|\{v \in V(T) : d(v) > 1\}| \leq 2$.

For the converse, suppose that $|\{v \in V(T) : d(v) > 1\}| \leq 2$. We examine each possibility separately. If $|\{v \in V(T) : d(v) > 1\}| = 0$, then we must have either $|E| = 0$ or 1 so that the equality $\chi(T) = \text{lchr}(\mathcal{N}(T))$ is clear. Let $|\{v \in V(T) : d(v) > 1\}| = 1$, and let $v \in V$ be the vertex with $d(v) > 1$. It follows that $\mathcal{N}(v) = V \setminus \{v\}$, since T is connected. Thus, $\mathcal{F}(\mathcal{N}(T)) = \{\{v\}, V \setminus \{v\}\}$ from which we deduce that $\chi(T) = \text{lchr}(\mathcal{N}(T)) = 2$. Finally, let $|\{v \in V(T) : d(v) > 1\}| = 2$, and let $u, v \in V$ be the vertices satisfying $d(u) > 1$ and $d(v) > 1$. We note in this case that $uv \in E$ and $\mathcal{F}(\mathcal{N}(T)) = \{\mathcal{N}(u), \mathcal{N}(v)\}$; hence, $\chi(T) = \text{lchr}(\mathcal{N}(T)) = 2$. \square

Example 2.9. In Figure 5, we present linear and non-linear N -trees with their neighborhood complexes.

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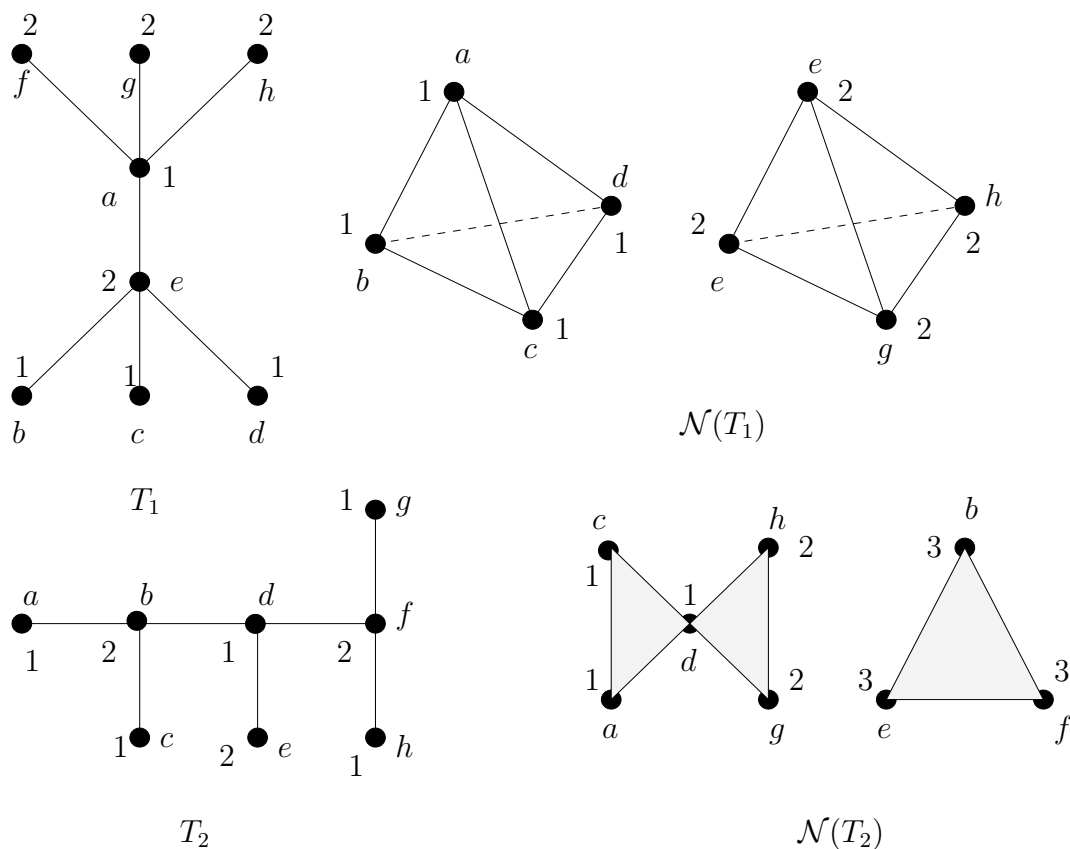


FIGURE 5. A linear N -tree T_1 and a non-linear N -tree T_2 .

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