Heat Conduction in a Rod with a Non-uniform Thermal Diffusivity

J. L. Taylor[1]

Abstract

Let L be the length of a rod and $u(x,t)$ be its temperature for $(x,t) \in [0,L] \times [0,\infty)$. We assume that the initial and boundary temperatures of the rod are $f(x)$ and 0 respectively. This heat conduction problem is formulated as the following first initial-boundary value problem:

$$u_t = \frac{1}{\sigma} u_{xx} \text{ for } 0 < x < L,\ t > 0,$$

$$u(x,0) = f(x) \text{ for } 0 \leq x \leq L,\ u(0,t) = 0 = u(L,t) \text{ for } t > 0.$$

where σ is a positive function and f is a nonnegative function. In this paper, we study an approximated solution to the above problem.

1. Introduction

The heat equation can be used to describe the transfer of heat in a given material. The transfer of heat from one material to another creates problems in the machines' functions. This is the motivation to study heat conduction in materials.

Suppose that a rod is placed along the x-axis with $x = 0$ at the left end of the rod and $x = L$ at the right end. The heat conduction problem of the rod is described by (cf. Nagle et al. [3, p. 578]):

$$c\rho \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u(x,t)}{\partial x}\right) + Q(x,t)$$

where c, ρ, K_0, Q, and u are corresponding to specific heat, mass density, thermal conductivity, heat source, and temperature respectively. In the following discussion, we assume that there is no heat source, and K_0 is a positive constant. Then, the above expression becomes

$$u_t = \frac{1}{\sigma} u_{xx}$$

where $\sigma = c\rho / K_0$ and $1/\sigma$ is called the thermal diffusivity.

We assume that σ is a twice-differentiable positive function of x. Furthermore, the rod satisfies the homogeneous boundary condition. That is, the temperature of the rod at each end is 0. Also, the rod has an initial temperature

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Email Address: taylorseries1580@yahoo.com

$f(x)$ which is a differentiable function. Then, the following first initial-boundary value problem is formulated:

$$u_t = \frac{1}{\sigma} u_{xx} \text{ for } 0 < x < L,\ t > 0, \tag{1}$$

$$u(x,0) = f(x) \text{ for } 0 \leq x \leq L,\ u(0,t) = 0 = u(L,t) \text{ for } t > 0. \tag{2}$$

To solve the problem (1)-(2), the method of separation of variables is used. We assume that $u(x,t) = \phi(x)h(t)$. Substitute this expression into (1); it leads to the following two ordinary differential equations:

$$\frac{dh}{dt} + \lambda h = 0, \tag{3}$$

$$\frac{d^2\phi}{dx^2} + \lambda\sigma\phi = 0,\ \phi(0) = 0 = \phi(L), \tag{4}$$

where λ is a constant. Equation (4) is called Sturm-Liouville eigenvalue problem. Let λ_n and ϕ_n be the corresponding eigenvalues and eigenfunctions of (4) respectively. $\{\phi_n\}$ forms an orthogonal set with the weight function $\sigma(x)$. That is,

$$\int_0^L \sigma(x)\, \phi_n(x)\, \phi_m(x)\, dx = 0 \text{ for } n \neq m.$$

To each λ_n, the solution of (3) is $h_n(t) = e^{-\lambda_n t}$. From the result of Haberman [2, p. 142], the solution of the problem (1)-(2) is given by the following infinite series:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x)\, e^{-\lambda_n t} \tag{5}$$

where

$$a_n = \frac{\int_0^L \sigma(x)\, f(x)\, \phi_n(x)\, dx}{\int_0^L \sigma(x)\, \phi_n^2(x)\, dx}. \tag{6}$$

However, the analytic solution to $\phi_n(x)$ in (4) may not be obtained for any given $\sigma(x)$. In Section 2, we study the approximated solution to $\phi_n(x)$. In Section 3, we give an example to show the approximated solution of the problem (1)-(2) with given functions $\sigma(x)$ and $f(x)$.

2. Sturm-Liouville eigenvalue problem

The general form of Sturm-Liouville eigenvalue problem (cf. Gustafson [1, p.175]) is given by

$$\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi + \lambda r\phi = 0$$

where p, q, and r are functions of x, and λ is the eigenvalue. If we let $p = K_0$, $q = 0$, and $r = \sigma$, then it gives

$$\frac{d^2\phi}{dx^2} + \lambda\sigma\phi = 0. \tag{7}$$

The Liouville-Green approximation is used to determine the approximated value of $\phi(x)$ when λ is large. The following theorem is from Olver [4, pp. 190-191]. The book only provide the outline of the proof. We give the detail of the proof.

Theorem 1. *If $\lambda \gg 1$, the approximation for $\phi(x)$ is given by*

$$\phi(x) \approx \tilde{A}\sigma^{-\frac{1}{4}} e^{\int_0^x (-\lambda\sigma)^{1/2} ds} + \tilde{B}\sigma^{-\frac{1}{4}} e^{-\int_0^x (-\lambda\sigma)^{1/2} ds}$$

where $\tilde{A}$ and $\tilde{B}$ are arbitrary constants.

Proof. Let $g(x) = -\lambda\sigma(x)$, then (7) becomes

$$\frac{d^2\phi}{dx^2} = g(x)\phi. \tag{8}$$

Let $\xi(x) = \int_0^x g^{1/2}(s)\,ds$ and $\phi(x) = (\xi'(x))^{-1/2} W(\xi)$. Then,

$$\frac{d\xi}{dx} = g^{1/2}$$

and

$$\frac{d\phi}{dx} = \frac{dW}{dx}(\xi')^{\frac{-1}{2}} - \frac{W}{2}(\xi')^{\frac{-3}{2}}\xi''.$$

Also,

$$\begin{aligned}
\frac{d^2\phi}{dx^2} &= \frac{d^2W}{dx^2}(\xi')^{\frac{-1}{2}} - \frac{dW}{dx}(\xi')^{\frac{-3}{2}}\xi'' - \frac{W}{2}\frac{d}{dx}\left[(\xi')^{\frac{-3}{2}}\xi''\right] \\
&= \frac{d^2W}{dx^2}(\xi')^{\frac{-1}{2}} - \frac{dW}{dx}(\xi')^{\frac{-3}{2}}\xi'' - \frac{W}{2}\left[\frac{-3}{2}(\xi')^{\frac{-5}{2}}(\xi'')^2 + (\xi')^{\frac{-3}{2}}\xi'''\right] \\
&= \frac{d}{d\xi}\left(\frac{dW}{d\xi}\frac{d\xi}{dx}\right)\left(\frac{d\xi}{dx}\right)(\xi')^{\frac{-1}{2}} - \frac{dW}{d\xi}\frac{d\xi}{dx}\left[(\xi')^{\frac{-3}{2}}\xi''\right] \\
&+ \frac{3W}{4}(\xi')^{\frac{-5}{2}}(\xi'')^2 - \frac{W}{2}(\xi')^{\frac{-3}{2}}\xi'''.
\end{aligned}$$

By $\frac{d\xi}{dx} = g^{1/2}$, $\frac{d^2\xi}{dx^2} = \frac{1}{2}g^{-1/2}g'$, and $\frac{d^3\xi}{dx^3} = \frac{-1}{4}g^{-3/2}(g')^2 + \frac{1}{2}g^{-1/2}g''$, we have

$$\begin{aligned}
\frac{d^2\phi}{dx^2} &= \frac{d}{d\xi}\left(\frac{dW}{d\xi}g^{\frac{1}{2}}\right)g^{\frac{1}{4}} - \frac{dW}{d\xi}g^{-\frac{1}{4}}\left(\frac{1}{2}g^{\frac{-1}{2}}g'\right) + \frac{3W}{4}g^{\frac{-5}{4}}\left[\frac{1}{4}g^{-1}(g')^2\right] \\
&- \frac{W}{2}g^{\frac{-3}{4}}\left[\frac{-1}{4}g^{\frac{-3}{2}}(g')^2 + \frac{1}{2}g^{\frac{-1}{2}}g''\right] \\
&= \frac{d^2W}{d\xi^2}g^{\frac{3}{4}} + \frac{dW}{d\xi}g^{-\frac{1}{4}}\frac{1}{2}\frac{dg}{d\xi} - \frac{1}{2}g^{\frac{-3}{4}}\frac{dg}{dx}\frac{dW}{d\xi} \\
&- \frac{W}{2}\left[\frac{-3}{8}g^{\frac{-9}{4}}(g')^2 - \frac{1}{4}g^{\frac{-9}{4}}(g')^2 + \frac{1}{2}g^{\frac{-5}{4}}g''\right] \\
&= \frac{d^2W}{d\xi^2}g^{\frac{3}{4}} + \frac{1}{2}g^{-\frac{1}{4}}\frac{dW}{d\xi}\frac{dg}{d\xi} - \frac{1}{2}g^{\frac{-3}{4}}\frac{dg}{d\xi}\frac{dW}{d\xi}\frac{d\xi}{dx} \\
&- \frac{W}{2}\left[\frac{-5}{8}g^{\frac{-9}{4}}(g')^2 + \frac{1}{2}g^{\frac{-5}{4}}g''\right] \\
&= \frac{d^2W}{d\xi^2}g^{\frac{3}{4}} - \frac{W}{2}\left[\frac{-5}{8}g^{\frac{-9}{4}}(g')^2 + \frac{1}{2}g^{\frac{-5}{4}}g''\right].
\end{aligned}$$

Then (8) becomes,

$$\frac{d^2W}{d\xi^2}g^{\frac{3}{4}} - \frac{W}{2}\left[\frac{-5}{8}g^{\frac{-9}{4}}(g')^2 + \frac{1}{2}g^{\frac{-5}{4}}g''\right] = g^{\frac{3}{4}}W.$$

It is equivalent to

$$\frac{d^2W}{d\xi^2} = W\left[\frac{-5}{16}g^{-3}(g')^2 + \frac{1}{4}g^{-2}g'' + 1\right]. \tag{9}$$

Let $\varphi = -g^{-3/4}\frac{d^2}{dx^2}g^{-1/4}$. By $g = -\lambda\sigma$, we have

$$\begin{aligned}
\varphi &= \frac{-5}{16}g^{-3}(g')^2 + \frac{1}{4}g^{-2}g'' \\
&= \frac{5(\sigma')^2}{16\lambda\sigma^3} - \frac{\sigma''}{4\lambda\sigma^2}.
\end{aligned}$$

Therefore, (9) becomes

$$\frac{d^2W}{d\xi^2} = (1+\varphi)W. \tag{10}$$

Now, if $\lambda \gg 1$, then φ is negligible. The approximated equation of (10) is

$$\frac{d^2W}{d\xi^2} \approx W.$$

Therefore, the approximated solution to W is

$$W \approx Ae^{\xi} + Be^{-\xi}$$

where A and B are arbitrary constants. By $\xi(x) = \int_0^x g^{1/2}(s)\,ds$ and $\phi(x) = (\xi'(x))^{-1/2} W$, we obtain

$$\phi(x) \approx A g^{-\frac{1}{4}} e^{\int_0^x g^{1/2} ds} + B g^{-\frac{1}{4}} e^{-\int_0^x g^{1/2} ds}.$$

By $g = -\lambda\sigma$, we get

$$\phi(x) \approx \tilde{A}\sigma^{-\frac{1}{4}} e^{\int_0^x (-\lambda\sigma)^{1/2} ds} + \tilde{B}\sigma^{-\frac{1}{4}} e^{-\int_0^x (-\lambda\sigma)^{1/2} ds},$$

where $\tilde{A} = A(-\lambda)^{-1/4}$ and $\tilde{B} = B(-\lambda)^{-1/4}$. The proof is completed. □

Adjustment can be made to the Liouville-Green approximation. This adjustment is from Haberman [2, p. 184], and it deals with the envelope function, namely $\tilde{A}\sigma^{-1/4}$ or $\tilde{B}\sigma^{-1/4}$. The idea is to set the envelope function as an infinite series.

Theorem 2. *If $\lambda \gg 1$, there exists an infinite series*

$$E(x) = \sum_{n=0}^{\infty} \lambda^{-\frac{n}{2}} E_n(x) \tag{11}$$

where E_n satisfies

$$E_n(x) = \frac{i}{2}\sigma^{-\frac{1}{4}} \int \sigma^{-\frac{1}{4}} E''_{n-1}(x)\,dx$$

with $E_0 = \sigma^{-1/4}$ such that $E(x)e^{\pm i\lambda^{1/2}\int_0^x \sigma^{1/2} ds}$ is an approximation solution to (7).

Proof. Let $\omega = i\lambda^{1/2}\int_0^x \sigma^{1/2}(s)ds$, then

$$\frac{d\omega}{dx} = i\lambda^{\frac{1}{2}}\sigma^{\frac{1}{2}}, \tag{12}$$

and

$$\frac{d^2\omega}{dx^2} = i\lambda^{\frac{1}{2}}\frac{\sigma'}{2\sigma^{\frac{1}{2}}}. \tag{13}$$

If $\phi(x) \approx E(x)e^{\omega}$, then

$$\frac{d\phi}{dx} \approx E'e^{\omega} + Ee^{\omega}\frac{d\omega}{dx},$$

and

$$\frac{d^2\phi}{dx^2} \approx E''e^{\omega} + 2E'e^{\omega}\frac{d\omega}{dx} + Ee^{\omega}\left(\frac{d\omega}{dx}\right)^2 + Ee^{\omega}\frac{d^2\omega}{dx^2}.$$

Thus, (7) is approximated by

$$E''e^{\omega} + 2E'e^{\omega}\frac{d\omega}{dx} + E\left[e^{\omega}\left(\frac{d\omega}{dx}\right)^2 + e^{\omega}\frac{d^2\omega}{dx^2}\right] + \lambda\sigma E e^{\omega} \approx 0.$$

By (12), we have

$$E''e^{\omega} + 2i\lambda^{\frac{1}{2}}\sigma^{\frac{1}{2}}E'e^{\omega} + Ee^{\omega}\left(-\lambda\sigma + \frac{d^2\omega}{dx^2} + \lambda\sigma\right) \approx 0.$$

By (13), the above expression becomes

$$E'' + 2i\lambda^{\frac{1}{2}}\sigma^{\frac{1}{2}}E' + \frac{i\lambda^{\frac{1}{2}}\sigma'}{2\sigma^{\frac{1}{2}}}E \approx 0.$$

Now, use $E(x) = \sum\limits_{n=0}^{\infty} \lambda^{-\frac{n}{2}}E_n(x)$, we get

$$\left.\begin{array}{l} (E_0'' + \lambda^{-\frac{1}{2}}E_1'' + \lambda^{-1}E_2'' + ... + \lambda^{-\frac{n-1}{2}}E_{n-1}'' + \lambda^{-\frac{n}{2}}E_n'' + ...) \\ +2i\lambda^{\frac{1}{2}}\sigma^{\frac{1}{2}}(E_0' + \lambda^{-\frac{1}{2}}E_1' + \lambda^{-1}E_2' + ... + \lambda^{-\frac{n-1}{2}}E_{n-1}' + \lambda^{-\frac{n}{2}}E_n' + ...) \\ +i\lambda^{\frac{1}{2}}\frac{\sigma'}{2\sigma^{\frac{1}{2}}}(E_0 + \lambda^{-\frac{1}{2}}E_1 + \lambda^{-1}E_2 + ... + \lambda^{-\frac{n-1}{2}}E_{n-1} + \lambda^{-\frac{n}{2}}E_n + ...) \\ \approx 0. \end{array}\right\} \quad (14)$$

Again, using the fact that $\lambda \gg 1$, we can say that higher powers of λ will give the most important terms in our approximation. Equate the coefficient of $\lambda^{1/2}$ to 0, we have

$$2\sigma^{\frac{1}{2}}E_0' + \frac{\sigma'}{2\sigma^{\frac{1}{2}}}E_0 = 0.$$

Integrate the above expression with respect to x, then

$$E_0 = \sigma^{-1/4}.$$

In (14), equate the coefficient of λ^0 to 0, it gives

$$E_0'' + 2i\sigma^{\frac{1}{2}}E_1' + \frac{i\sigma'}{2\sigma^{\frac{1}{2}}}E_1 = 0.$$

Similarly, equate the coefficient of $\lambda^{-\frac{n-1}{2}}$ to 0, we obtain

$$E_{n-1}'' + 2i\sigma^{\frac{1}{2}}E_n' + \frac{i\sigma'}{2\sigma^{\frac{1}{2}}}E_n = 0.$$

The above expression is equivalent to

$$\begin{aligned} 2i\sigma^{\frac{1}{2}}E_n' + i\frac{\sigma'}{2\sigma^{\frac{1}{2}}}E_n &= -E_{n-1}'' \\ E_n' + \frac{\sigma'}{4\sigma}E_n &= \frac{iE_{n-1}''}{2\sigma^{\frac{1}{2}}}. \end{aligned}$$

Multiplying both side by the integrating factor $\sigma^{\frac{1}{4}}$, we have

$$\frac{d}{dx}\left(E_n\sigma^{1/4}\right) = \frac{i\sigma^{1/4}}{2\sigma^{1/2}}E_{n-1}''.$$

Integrating both side with respect to x,

$$E_n = \frac{i}{2}\sigma^{-\frac{1}{4}} \int \sigma^{-\frac{1}{4}} E''_{n-1} dx.$$

Similarly, the above proof is true for $\omega = -i\lambda^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}}(t)dt$. Thus, the proof is completed. □

Note that $E(x)$ is precisely the Liouville-Green approximation if the first term in the infinite series in (11) is used. We remark that a better approximation of $\phi(x)$ can be obtained if more terms in the infinite series of (11) is used.

From the result of Theorem 1,

$$\phi(x) \approx \tilde{A}\sigma^{-1/4} e^{i\lambda^{1/2}\int_0^x \sigma^{1/2} ds} + \tilde{B}\sigma^{-1/4} e^{-i\lambda^{1/2}\int_0^x \sigma^{1/2} ds}.$$

We use sine and cosine functions instead of exponential function, it leads to

$$\phi(x) \approx C_1 \sigma^{-\frac{1}{4}} \cos(\lambda^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds) + C_2 \sigma^{-\frac{1}{4}} \sin(\lambda^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds)$$

where C_1 and C_2 are arbitrary constants. Substitute $x = 0$ in the above expression and by the boundary condition in (4), $\phi(0) = 0$, it gives

$$C_1 \sigma^{-\frac{1}{4}} = 0$$

which implies $C_1 = 0$. Therefore,

$$\phi(x) \approx C_2 \sigma^{-\frac{1}{4}} \sin(\lambda^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds).$$

Substitute $x = L$ in the above expression and by another boundary condition in (4), $\phi(L) = 0$, we have

$$C_2 \sigma^{-\frac{1}{4}} \sin(\lambda^{\frac{1}{2}} \int_0^L \sigma^{\frac{1}{2}} ds) = 0.$$

As $C_2 \neq 0$, otherwise $\phi(x) \equiv 0$, we obtain

$$\lambda^{\frac{1}{2}} \int_0^L \sigma^{\frac{1}{2}} ds = n\pi \text{ for } n = 1, 2, 3.... \tag{15}$$

This expression is equivalent to

$$\lambda_n = \left(\frac{n\pi}{\int_0^L \sigma^{\frac{1}{2}} ds} \right)^2. \tag{16}$$

Therefore,

$$\phi_n(x) \approx C_{2_n} \sigma^{-\frac{1}{4}} \sin(\lambda_n^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds).$$

We choose C_{2_n} such that $\{\phi_n\}$ forms an orthonormal set, that is,

$$\int_0^L \phi_n \phi_m \sigma dx = \begin{cases} 0 \text{ if } n \neq m \\ 1 \text{ if } n = m \end{cases}.$$

Thus, if $n = m$, then

$$(C_{2_n})^2 \int_0^L \sigma^{\frac{1}{2}} \sin^2 \left(\lambda_n^{\frac{1}{2}} \int_0^x \sigma^{\frac{1}{2}} ds \right) dx \approx 1.$$

Let $u = \lambda_n^{1/2} \int_0^x \sigma^{1/2} ds$, then $du = \lambda_n^{1/2} \sigma^{1/2} dx$. By (15), we have

$$(C_{2_n})^2 \int_0^{n\pi} \lambda_n^{-\frac{1}{2}} \sin^2 u du \approx 1.$$

It implies

$$\begin{aligned} (C_{2_n})^2 &\approx \frac{\lambda_n^{\frac{1}{2}}}{\int_0^{n\pi} \sin^2 u du} \\ &= \frac{2}{\lambda_n^{-\frac{1}{2}} n\pi}. \end{aligned}$$

From (16), we obtain

$$(C_{2_n})^2 \approx \frac{2}{\int_0^L \sigma^{\frac{1}{2}} ds}.$$

This gives our approximation to the solution of (7)

$$\phi_n(x) \approx \left(\frac{2}{\int_0^L \sigma^{\frac{1}{2}} ds} \right)^{\frac{1}{2}} \sigma^{-\frac{1}{4}} \sin \left(n\pi \frac{\int_0^x \sigma^{\frac{1}{2}} ds}{\int_0^L \sigma^{\frac{1}{2}} ds} \right). \tag{17}$$

3. An Example

In this section, we let $L = 1$. We want to look at the behavior of the approximated solution to the problem (1)-(2) for some $\sigma(x)$ and $f(x)$.

Let $\sigma(x) = 1 + 50x$ and $f(x) = x(1 - x)$. From (4),

$$\frac{d^2\phi}{dx^2} + \lambda (1 + 50x) \phi = 0, \ \phi(0) = 0 = \phi(1).$$

Using (16), we have

$$\lambda_n = \left(\frac{75n\pi}{51^{3/2} - 1} \right)^2. \tag{18}$$

We note that $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence. That is,

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ....$$

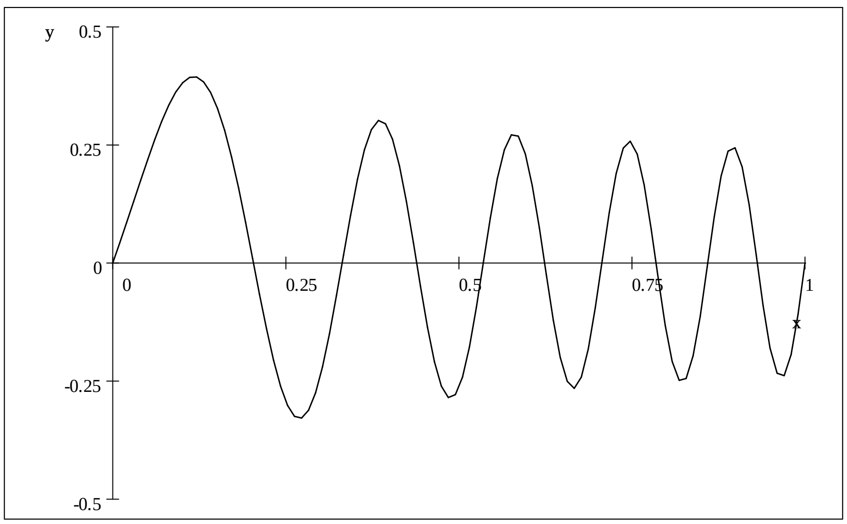
From (17),

$$\phi_n(x) \approx \left(\frac{150}{51^{3/2}-1}\right)^{1/2} (1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right). \quad (19)$$

Let $n = 10$, then $\lambda_{10} = \left(\frac{750\pi}{51^{3/2}-1}\right)^2$ and

$$\phi_{10}(x) \approx \left(\frac{150}{51^{3/2}-1}\right)^{1/2} (1+50x)^{-\frac{1}{4}} \sin\left(\frac{750\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right).$$

The graph of this approximated eigenfunction is:



In the above graph, there are exactly 9 roots in the interval $(0, 1)$. This matches to the result of the following theorem (cf. Haberman [2, p. 135]).

Theorem 3. *The eigenfunction $\phi_n(x)$ has exactly $n-1$ roots in the interval $0 < x < L$.*

Substitute (18) and (19) into (5), we get the approximated solution to the problem (1)-(2)

$$u(x,t) \approx \sum_{n=1}^{\infty} a_n \left(\frac{150}{51^{3/2}-1}\right)^{1/2} (1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right) e^{-\left(\frac{75n\pi}{51^{3/2}-1}\right)^2 t}.$$

Rewrite the above expression, it yields

$$u(x,t) \approx \sum_{n=1}^{\infty} \tilde{a}_n (1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right) e^{-\left(\frac{75n\pi}{51^{3/2}-1}\right)^2 t}$$

where $\tilde{a}_n = a_n \left(\frac{150}{51^{3/2}-1}\right)^{1/2}$. To solve for $\tilde{a}_n$, we use the initial condition of (2). When $t = 0$,

$$\sum_{n=1}^{\infty} \tilde{a}_n (1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right) \approx u(x,0)$$
$$= f(x)$$
$$= x(1-x).$$

From (6), $\tilde{a}_n$ is given by

$$\tilde{a}_n \approx \frac{\int_0^1 (1+50x)\, x(1-x)(1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right) dx}{\int_0^1 (1+50x)\left[(1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right)\right]^2 dx}.$$

Simplify the above expression, we have

$$\tilde{a}_n \approx \frac{\int_0^1 (1+50x)^{\frac{3}{4}}\, x(1-x) \sin\left(\frac{\left[(1+50x)^{3/2}-1\right]n\pi}{51^{3/2}-1}\right) dx}{\int_0^1 (1+50x)^{\frac{1}{2}} \sin^2\left(\frac{\left[(1+50x)^{3/2}-1\right]n\pi}{51^{3/2}-1}\right) dx}.$$

Since the analytic solution of the right-hand side of the above expression cannot be determined, we use numerical method to compute the approximated solution. Adaptive Gaussian Quadrature is used to compute the value of the numerator and denominator, this numerical integration method is built in `Maple`®[2] version 9.03. We use the finite sum

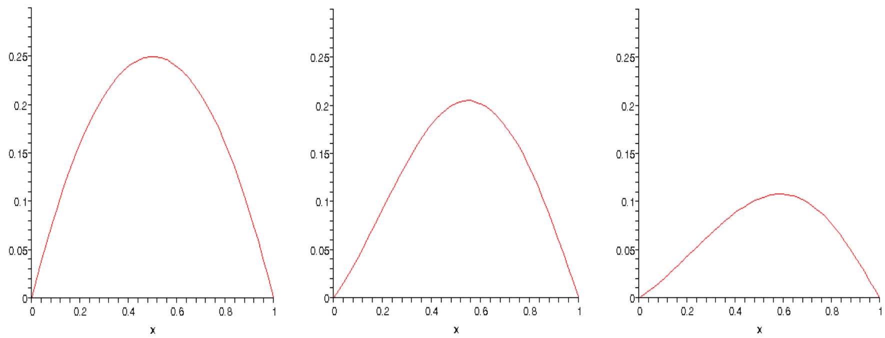
$$\tilde{u}(x,t) = \sum_{n=1}^{100} \tilde{a}_n (1+50x)^{-\frac{1}{4}} \sin\left(\frac{75n\pi}{51^{3/2}-1}\int_0^x (1+50s)^{\frac{1}{2}}\,ds\right) e^{-\left(\frac{75n\pi}{51^{3/2}-1}\right)^2 t}$$

to approximate $u(x,t)$. The table below shows that the values of $f(x) = x(1-x)$ and $\tilde{u}(x,0)$ are close to each other at $x = 0,\ 0.1,\ 0.2,\ 0.3, ..., 1$.

x	$f(x)$	$\tilde{u}(x,0)$
0	0	0
0.1	0.09	0.08997841586
0.2	0.16	0.1599906913
0.3	0.21	0.2100016569
0.4	0.24	0.2399981029
0.5	0.25	0.2499991097
0.6	0.24	0.2400001991
0.7	0.21	0.2100001255
0.8	0.16	0.1599993234
0.9	0.09	0.08999867687
1	0	$-9.377258464 \times 10^{-12}$

[2] `Maple`® is a registered trademark of Waterloo Maple Inc., Waterloo, Ontario, Canada

The graphs of $\tilde{u}(x,t)$ when $t = 0$, 0.5, and 2 are:



$t = 0$ $\quad$ $t = 0.5$ $\quad$ $t = 2$

The graphs above show that $\tilde{u}(x,t)$ skews to the right-hand side when t is increasing. For each $x \in [0,1]$, $\tilde{u}(x,t)$ is a non-increasing function of t.

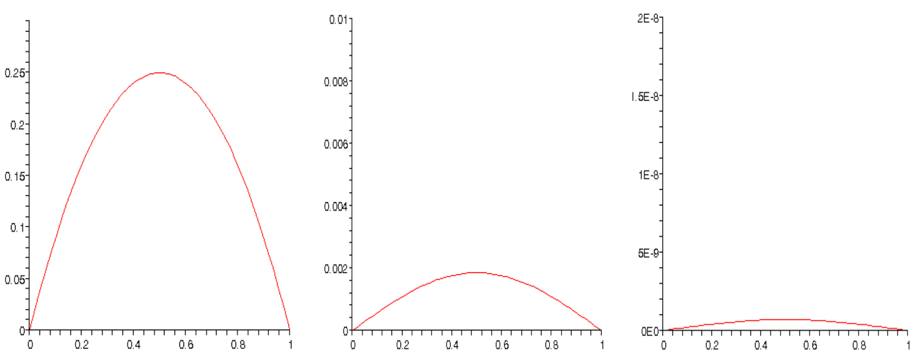
If $\sigma(x) = 1$ and $f(x) = x(1-x)$, the exact solution to problem (1)-(2) is

$$u(x,t) = 2\sum_{n=1}^{\infty}\left[\int_0^1 x(1-x)\sin(n\pi x)\,dx\right]\sin(n\pi x)\,e^{-(n\pi)^2 t} \tag{20}$$

(cf. Haberman [2, pp. 42-43]). We use the finite sum

$$\hat{u}(x,t) = 2\sum_{n=1}^{100}\left[\int_0^1 x(1-x)\sin(n\pi x)\,dx\right]\sin(n\pi x)\,e^{-(n\pi)^2 t}$$

to approximate the solution in (20). Use `Maple`® to graph $\hat{u}(x,t)$ for $t = 0$, 0.5, and 2. The graphs are:



$t = 0$ $\quad$ $t = 0.5$ $\quad$ $t = 2$

The graphs above show that $\hat{u}(x,t)$ is symmetric with respect to the line $x = 0.5$ for all t. For each $x \in [0,1]$, $\hat{u}(x,t)$ is a non-increasing function of t. Compare the graphs of $\tilde{u}(x,t)$ and $\hat{u}(x,t)$, the rate of decrease of $\hat{u}(x,t)$ is faster than $\tilde{u}(x,t)$.

References

[1] K. Gustafson, *Introduction to Partial Differential Equations and Hilbert Space Methods*, Mineola, NY, Dover, 1999, p. 175.

[2] R. Haberman, *Elementary Applied Partial Differential Equations*, Englewood Cliffs, NJ, Prentice-Hall, 1987, pp. 42-43, 135, 142, and 184.

[3] R. Nagle, E. Saff, and A. Snider, *Fundamentals of Differential Equations and Boundary Value Problems*, New York, NY, Addison Wesley, 2004, p. 578.

[4] F. Olver, *Asymptotics and Special Functions*, New York, NY, Academic Press, 1974, pp. 190-191.