# Some Geometry of the $p$-adic Rationals

Catherine Crompton

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**Abstract**

Given a prime $p$, we introduce the $p$-adic absolute value on the rational numbers. We define triangles and angles using this absolute value and investigate their behavior in $p$-adic geometry and how it differs from Euclidean geometry. Finally, we consider the existence of other polygons using the new absolute value.

## 1 Introduction

Absolute value on the integers is often defined by $|a| = \sqrt{a^2}$. This represents the "usual" distance from $a$ to zero, and underlies our notion of how Euclidean geometry behaves. However, there are other ways of defining absolute value. In this paper we will be working with the $p$-adic absolute value.

Recall that any positive integer except 1 has a unique prime factorization. Suppose we are interested in only one prime, $p$ (whence the $p$ in $p$-adic). Then we can write any positive integer $a$ uniquely as $a = p^n \cdot a'$, where $n$ is the highest power of $p$ that divides $a$ and $a' \in \mathbb{Z}^+$. If we allow $a'$ to be a rational number in lowest terms and $n$ to be any integer, we can write any rational number this way. This idea, defined more carefully, leads to the $p$-adic valuation, which we then use to define the $p$-adic absolute value.

We start with a review of $p$-adic numbers-see [1] for more details. Throughout this paper $p$ will denote a prime.

### 1.1 The $p$-adic Valuation and Absolute Value

Let $a \in \mathbb{Q}$, $a \neq 0$. Write $a = p^n \frac{x}{y}$, with $\gcd(x, y) = 1$, $n \in \mathbb{Z}$, and $p \nmid xy$. The integer $n$ and the rational number $x/y$ are well-defined by the Fundamental Theorem of Arithmetic. The $p$-adic valuation is the function $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ with

$$v_p(a) = n$$

$$v_p(0) = \infty.$$

Hereafter we will write $v$ for $v_p$ when the prime is understood.

**Example 1.** *Suppose* $p = 3$. *Then* $v_3(3) = v_3(3^1 \cdot 1) = 1$, $v_3(5) = v_3(5^1 \cdot 3^0) = 0$, *and* $v_3(1/9) = v_3(3^{-2} \cdot 1) = -2$.

The $p$-adic valuation has two important properties that we will use constantly throughout this paper.

**Proposition 2.** *For all $a, b \in \mathbb{Q}$ we have*

1. $v(ab) = v(a) + v(b)$
2. $v(a-b) \geq \min\{v(a), v(b)\}$, *with equality when* $v(a) \neq v(b)$.

*Proof.* Let $a, b \in \mathbb{Q}$. Write $a' = p^n a'$ and $b = p^m b'$, with $v(a') = v(b') = 0$ and $a'$ and $b'$ are rational numbers. Then

$$v(ab) = v(p^{n+m}a'b') = n + m = v(a) + v(b).$$

This proves $(1)$. For $(2)$, first consider what happens if $v(a) = v(b) = n$. We have

$$v(a-b) = v(p^n(a'-b')) = v(p^n) + v(a'-b') = n + v(a'-b').$$

Write $a' = \frac{x}{y}$ and $b' = \frac{w}{z}$ with $\gcd(x, y) = \gcd(w, z) = 1$. Then

$$v(a'-b') = v(\frac{x}{y} - \frac{w}{z}) = v(\frac{xz - yw}{yz})$$

and $v(a') = v(b') = 0$ implies $p \nmid y$ and $p \nmid z$, so $p \nmid yz$. So $v(a'-b') = k \geq 0$. Therefore

$$v(a-b) = n + v(a'-b') = n + k \geq \min\{v(a), v(b)\}.$$

Now suppose $v(a) \neq v(b)$. Since

$$v(b-a) = v(-1(a-b)) = v(-1) + v(a-b) = 0 + v(a-b) = v(a-b)$$

it suffices to consider the case where $v(a) > v(b)$. So $n > m$. Then

$$v(a-b) = v(p^n a' - p^m b') = v(p^m(p^{n-m}a' - b')) = v(p^m) + v(p^{n-m}a' - b').$$

Since $p$ divides $p^{n-m}a'$ but $p$ does not divide $b'$ we know $p$ does not divide their difference, hence $v(p^{n-m}a' - b') = 0$ and so

$$v(p^m) + v(p^{n-m}a' - b') = m + 0 = v(b) = \min\{v(a), v(b)\}.$$

$\square$

Now we can use the valuation function to define an absolute value function on $\mathbb{Q}$.

**Definition 3.** *The $p$-adic* absolute value *is the function* $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ *given by*

$$|a|_p = p^{-v_p(a)}$$

$$|0|_p = 0.$$

**Example 4.** *Consider* $24$ *and* $1/24$*, and suppose* $p = 2$*. Since* $24 = 2^3 \cdot 3$*,* $v_2(24) = 3$*, and* $|24|_2 = 2^{-3} = 1/8$*. Since* $1/24 = 2^{-3} \cdot 3^{-1}$*,* $v_2(1/24) = -3$*, and* $|1/24|_2 = 2^{-(-3)} = 8$*. Similarly, if* $p = 3$ *then* $|24|_3 = 3^{-1}$ *and* $|1/24|_3 = 3^{-(-1)} = 3$*. Finally, if* $p = 5$*,* $|24|_5 = |2^3 \cdot 3 \cdot 5^0|_5 = 5^0 = 1$ *and* $|1/24|_5 = 5^{-0} = 1$*.*

We define the distance between two points by $d_p(a, b) = |a - b|_p$. Hereafter we will usually suppress the "sub-$p$" notation, understanding that we will be working with the $p$-adic absolute value for some fixed $p$.

**Theorem 5.** *For all* $a, b, c \in \mathbb{Q}$*,*

1. $d(a, b) > 0$ *for* $a \neq b$ *and* $d(a, a) = 0$
2. $d(a, b) = d(b, a)$
3. $d(a, c) \leq d(a, b) + d(b, c)$ *(the Triangle Inequality)*
4. $d(a, c) \leq \max\{d(a, b), d(b, c)\}$ *(the Ultrametric Inequality).*

*Proof.* Let $a, b, c \in \mathbb{Q}$. Suppose $a \neq b$. Write $a - b = p^n d$, for $n \in \mathbb{Z}$, $d \in \mathbb{Q}$, and $v(d) = 0$. First, $d(a, b) = |p^n d| = p^{-n} > 0$. Also $d(a, a) = |a - a| = |0| = 0$ by definition. Second,

$$d(a, b) = |a - b| = p^{-v(a-b)} = p^{-v(b-a)} = |b - a| = d(b, a).$$

Finally we prove Property $(4)$, which implies Property $(3)$, as follows:

$$\begin{aligned} d(a, c) &= |a - c| = |(a - b) - (c - b)| \\ &= p^{-v\big((a-b)-(c-b)\big)} \\ &\leq p^{-\min\{v(a-b), v(c-b)\}} \qquad (1) \\ &= \max\{d(a, b), d(c, b)\} \\ &= \max\{d(a, b), d(b, c)\}. \qquad (2) \end{aligned}$$

We get (1) from Property (2) of Proposition 2 and (2) from $d(a, b) = d(b, a)$ as above. □

Any distance satisfying Properties 1-3 is a *metric*, and a set with such a distance is a *metric space*. If the distance also satisfies Property $4$ then the set with its distance is an *ultrametric space* [1].

## 1.2 The $p$-adic Completion of $\mathbb{Q}$

Most of our exploration of the $p$-adic distance could happily remain in $\mathbb{Q}$ without venturing further. But the more general setting for the $p$-adic distance will not be hard to work with and we will briefly explain it here. First we need two definitions from a first course on real analysis.

**Definition 6.** *1. A sequence* $(s_n)$ *in a metric space* $X$ *is said to be a* Cauchy sequence *if for every* $\varepsilon > 0$ *there exists* $N \in \mathbb{Z}$ *such that whenever* $n$, $m \in \mathbb{Z}$ *and* $n$, $m \geq N$, $d(s_n, s_m) < \varepsilon$.

2. *A metric space is* complete *if every Cauchy sequence converges.*

It is well known that $\mathbb{Q}$ with the usual distance $|a-b| = \sqrt{(a-b)^2}$ is not complete. For example, the alternating harmonic series $\sum_{i=1}^{\infty} \left(\frac{(-1)^{n+1}}{n}\right)$ is a Cauchy sequence of rational numbers that converges to an irrational. The real numbers $\mathbb{R}$ are complete ( [2], p. 159); they form the *completion of* $\mathbb{Q}$ with respect to the usual distance. (Being a completion of $\mathbb{Q}$ also involves the notion that $\mathbb{Q}$ is dense in $\mathbb{R}$; see [1]. We will not worry about denseness in this paper.) Our problem is that $\mathbb{Q}$ with the $p$-adic valuation is not complete.

**Example 7.** *Let* $a \in \mathbb{Z}, a > 0$. *We can expand* $a$ *uniquely as*

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n$$

*with* $0 \leq a_i \leq p-1$ *and* $a_n \neq 0$. *This is called the* $p$-adic expansion of $a$. *Note that* $v(a) = \min\{i | a_i \neq 0\}$. *Consider the sequence*

$$r_0 = 1,\ r_1 = 1+p,\ r_2 = 1+p+p^2,\ r_3 = 1+p+p^2+p^3,\ \cdots.$$

*For* $n, m \in \mathbb{Z}$, $n < m$, *we get*

$$d\left(r_n, r_m\right) = \left|p^{n+1} + \cdots + p^m\right| = p^{-(n+1)}.$$

*Let* $\varepsilon \in \mathbb{R}, \varepsilon > 0$. *Choose* $N \in \mathbb{N}$ *so that* $N > \log_p(1/\varepsilon) - 1$. *Then for all* $n < m \in \mathbb{Z}$, $n, m \geq N$ *we have* $d(r_n, r_m) = p^{-(n+1)} < \varepsilon$. *Therefore the sequence* $\{r_i\}$ *is Cauchy.*

*Now the sequence* $(r_i)$ *converges to* $a \in \mathbb{Q}$ *if and only if for every positive real number* $\delta$, *the set* $B(a, \delta) = \{q \in \mathbb{Q} | d(q, a) < \delta\}$ *contains all but a finite number of terms of* $(r_i)$.

*Let us suppose that* $(r_i)$ *converges to* $a \in \mathbb{Q}$ *and look for a contradiction. Write*

$$a = \sum_{i=k}^{N} a_i p^i,\ 0 \leq a_i \leq p-1.$$

*Choose* $\delta = p^{-(N+2)}$, *and let* $M \in \mathbb{Z}$, $M > N$. *Now* $v(r_M - a)$ *equals the greatest power of* $p$ *that we can factor out of the difference* $r_M - a$, *and since*

$$r_M - a = (1 + p + \cdots + p^N + \cdots + p^M) - (a_k p^k + a_{k+1} p^{k+1} + \cdots + a_N p^N),$$

$v(r_M - a) \leq N+1$. *Therefore* $d(r_M, a) \geq p^{-(N+1)} > \delta$, *and* $B(a, \delta)$ *contains at most* $N$ *terms of* $(r_i)$. *Therefore* $\mathbb{Q}$ *is not complete under the* $p$*-adic metric.*

Gouvea in [1] gives a nice construction for the completion of $\mathbb{Q}$. The construction takes some doing, however, and we will go straight to the end result.

**Definition 8.** *The* $p$-adic completion *of* $\mathbb{Q}$*, denoted* $\mathbb{Q}_p$*, is defined to be*

$$\mathbb{Q}_p = \left\{ \sum_{i=m}^{\infty} a_i p^i \,|\, 0 \le a_i \le p-1,\, m \in \mathbb{Z} \right\}.$$

See [1] again for proofs that $\mathbb{Q}_p$ has the properties of a completion. Since we use integers in many of our examples, we mention that the *p-adic completion* of $\mathbb{Z}$, denoted $\mathbb{Z}_p$, is the same as for $\mathbb{Q}_p$ except the sum runs from $i = 0$ to $\infty$. Notice that $v(a)$ for $a \in \mathbb{Q}_p$ is the exponent of the power of $p$ with the least non-zero coefficient.

**Example 9.** *Suppose* $p = 5$*. Then* $76/625 = 1 \cdot 5^{-4} + 0 \cdot 5^{-3} + 3 \cdot 5^{-2} + 0 \cdot 5^{-1} + 0 \cdot 5^0 + 0 \cdot 5^1 + \cdots$*, and* $|76/625| = |5^{-4} \cdot 76| = 5^{-(-4)} = 625$.

# 2 Triangles and Angles in $\mathbb{Q}_p$

## 2.1 Triangles in $\mathbb{Q}_p$

We often envision $\mathbb{R}$ as a line. In $\mathbb{Q}_p$, things are not so simple. We start with three definitions, following Euclidean geometry as much as possible.

**Definition 10.**

1. *A* point *is an element of* $\mathbb{Q}_p$*.*
2. *A* triangle *is three distinct points* $a$*,* $b$*, and* $c$ *in* $\mathbb{Q}_p$*. We will denote a triangle by* $\triangle abc$*.*
3. *A triangle* $abc$ *has* side lengths *given by* $d(a, b)$*,* $d(b, c)$*, and* $d(c, a)$*.*

Note that our definition of a triangle appears to differ from the definition in Euclidean geometry, since ours allows for three collinear points to be triangles. However, we will see that we never have three collinear points in $p$-adic geometry.

A well-known characteristic of the $p$-adic absolute value is most striking when phrased in terms of triangles.

**Theorem 11.** *Using the* $p$*-adic absolute value, all triangles are isosceles.*

*Proof.* Let $\triangle abc$ be a triangle. It has sides of length $p^{-v(a-c)}$, $p^{-v(a-b)}$, and $p^{-v(b-c)}$. If any two of $v(a-c)$, $v(a-b)$, and $v(b-c)$ are equal, we are done.

Without loss of generality, suppose $v(a-b) \neq v(b-c)$. Then from Proposition 2, $v(a-c) = \min\{v(a-b), v(b-c)\}$. Therefore at least two of the sides must be of equal length. □

**Theorem 12.** *If a triangle is not equilateral, the unequal side has the largest valuation and hence the shortest length.*

*Proof.* Let $\triangle abc$ be a triangle with $v(a-b) = v(b-c) \neq v(a-c)$. Then

$$v(a-c) = v((a-b)-(c-b)) \ge \min\{v(a-b), v(b-c)\},$$

and since $v(a-c) \neq v(a-b) = v(b-c)$ we have $v(a-c) > v(a-b)$. □

In Euclidean geometry with the usual distance which we call "dist", three points are collinear if and only if $dist(a, c) = dist(a, b) + dist(b, c)$, assuming $dist(a, c) > dist(a, b)$ and $dist(a, c) > dist(b, c)$. If we use the $p$-adic distance on $\mathbb{Q}_p$ we see that collinearity is impossible with more than two points.

**Corollary 13.** *Given three distinct points $a, b, c \in \mathbb{Q}_p$, $d(a, c) < d(a, b) + d(b, c)$. In other words, no three points in $\mathbb{Q}_p$ are collinear.*

The proof follows easily from Theorem 12 and is left to the reader. Corollary 13 is what we had in mind when we said that we often envision $\mathbb{R}$ as a line, but things in $\mathbb{Q}_p$ are not so simple.

We have seen that all triangles in $\mathbb{Q}_p$ are isosceles. Equilateral triangles are usually easy to construct: for instance, suppose $p = 5$ and take the points 10, 15, and 20. More generally, for $p \geq 3$ the points $2p$, $3p$ and $4p$ form an equilateral triangle since $d(2p, 3p) = |p(3 - 2)| = p^{-1}$, $d(3p, 4p) = |p(4 - 3)| = p^{-1}$, and $d(2p, 4p) = |p(4 - 2)| = p^{-1}$. In $\mathbb{Q}_2$, however, equilateral triangles do not exist.

**Theorem 14.** *Given a prime $p$, any subset of $\mathbb{Q}_p$ has at most $p$ equidistant points.*

*Proof.* Suppose on the contrary that there is a set of $p + 1$ distinct equidistant points $a_1, a_2, ..., a_{p+1}$ with $a_i = \sum_{k=j_i}^{\infty} a_{ik} p^k$, $a_{ij_i} \neq 0$. Since the $a_i$'s are all equidistant, there exists $m \in \mathbb{Z}$ such that $v(a_i - a_j) = m$ for all $i, j$ and $a_{ik} = a_{jk}$ for all $k < m$. Thus

$$\begin{aligned} a_i - a_j &= \sum_{k=m}^{\infty} (a_{ik} - a_{jk}) p^k \\ &= \left( p^m \sum_{k=0}^{\infty} a_{i(k+m)} p^k \right) - \left( p^m \sum_{k=0}^{\infty} a_{j(k+m)} p^k \right) \\ &= p^m a'_i - p^m a'_j \\ &= p^m (a'_i - a'_j) \end{aligned}$$

where $a'_i = \sum_{k=0}^{\infty} a_{i(k+m)} p^k$, $a'_j = \sum_{k=0}^{\infty} a_{j(k+m)} p^k$. Therefore $v(a_i - a_j) = m + v(a'_i - a'_j) = m + 0$, $m \in \mathbb{Z}$, because we assumed that $v(a_i - a_j) = m$ for all the $p + 1$ points and hence $v(a'_i - a'_j) = 0$. However, because $0 \leq a'_{i0} \leq p - 1$ for all $i$ and because there are $p + 1$ points, there exist distinct $i$ and $j$ so that $a'_{i0} = a'_{j0}$. Therefore $v(a'_i - a'_j) > 0$ for some $i$ and $j$. □

**Corollary 15.** *Equilateral triangles do not exist under the 2-adic metric.*

On the theme of nonexistence, we next show that right triangles in $\mathbb{Q}_p$ do not exist. We will say a *right triangle* in $\mathbb{Q}_p$ is a triangle whose side lengths satisfy the Pythagorean Theorem.

**Theorem 16.** *For any $a, b, c \in \mathbb{Q}_p$ we have $d(a, c)^2 \neq d(a, b)^2 + d(b, c)^2$. In other words, right triangles in $\mathbb{Q}_p$ do not exist.*

*Proof.* Suppose $\triangle abc$ is a right triangle with longest side $ab$. Then $d(b,c) = d(a,c)$. Since $\triangle abc$ is a right triangle we have

$$d(a,b)^2 = d(b,c)^2 + d(a,c)^2 = 2d(b,c)^2.$$

Thus

$$\begin{aligned} \left(p^{-v(a-b)}\right)^2 &= 2\left(p^{-v(b-c)}\right)^2 \\ \frac{1}{2} &= \frac{p^{2v(a-b)}}{p^{2v(b-c)}} \\ \frac{1}{2} &= p^{2v(a-b)-2v(b-c)} \\ \frac{1}{2} &= p^{2v(\frac{a-b}{b-c})}. \end{aligned} \tag{3}$$

Then (3) implies $p = 2$ and $v(\frac{a-b}{b-c}) = \frac{-1}{2}$. But $v$ is always an integer by definition, so this cannot occur. □

## 2.2 Angles in $\mathbb{Q}_p$

Still trying to create an analogy with Euclidean geometry, we define angles using what would be the Law of Cosines if our distance were the Euclidean metric.

**Definition 17.** *Given distinct $a, b, c \in \mathbb{Q}_p$, we define the angle $\theta$ between sides $ac$ and $bc$ by*

$$\theta = \arccos\left(\frac{-d(a,b)^2 + d(b,c)^2 + d(a,c)^2}{2d(b,c)d(a,c)}\right)$$

If we do not have right triangles in $\mathbb{Q}_p$, presumably we do not have right angles either. This bears investigating, using our definition from the Law of Cosines.

There are two possibilities for $\theta$, depending on whether $d(a,b) = d(b,c)$ or $d(a,c) = d(b,c)$.

Suppose $d(a,b) = d(a,c)$ (Figure 1).

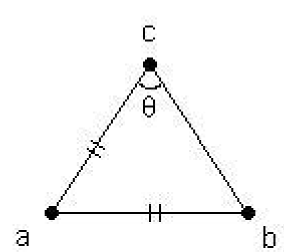


Figure 1.

Then we have

$$
\begin{aligned}
\cos\theta &= \frac{d(b,c)^2}{2d(b,c)d(a,c)} \\
&= \frac{d(b,c)}{2d(a,c)} \\
&= \frac{p^{v(a-c)}}{2p^{v(b-c)}} \\
&= \frac{1}{2}p^{v(a-c)-v(b-c)}.
\end{aligned}
$$

By Theorem 12 $v(a-c) \le v(b-c)$. So $0 < \cos\theta \le 1/2$. Thus $\frac{\pi}{3} \le \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{1}{2}p^k$ where $k = v(a-c) - v(b-c)$.

Suppose $d(a,c) = d(b,c)$ (Figure 2).

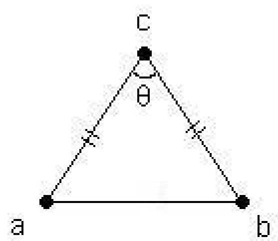


Figure 2.

Then

$$
\begin{aligned}
\cos\theta &= \frac{-d(a,b)^2 + 2d(b,c)^2}{2d(b,c)^2} \\
&= -\Big(\frac{d(a,b)}{2d(b,c)}\Big)^2 + 1 \\
&= \frac{-p^{2v(b-c)}}{2p^{2v(a-b)}} + 1 \\
&= 1 - \frac{1}{2}p^{2(v(b-c)-v(a-b))}. \qquad (4)
\end{aligned}
$$

By Theorem 12 $v(b-c) - v(a-b) < 0$, so we can write

$$\cos\theta = 1 - \frac{1}{2p^k} > 0$$

where $k = -2(v(b-c) - v(a-b)) > 0$. Thus $1/2 < \cos\theta < 1$.

Note some of the following special cases. If $k = 0$ we have $\theta = \pi/3$ from both Case 1 and Case 2. As $k$ increases we have in Case 1 that $\theta$ approaches $\pi/2$ but never equals $\pi/2$ since $\cos(1/(2p^k)) > 0$ for all $k$. In Case 2 as $k$ increases $\theta$ approaches 0 but never equals 0 since $1 - 1/(2p^{2k}) < 1$ for all $k$.

The next proposition shows that given any positive integer $j$ and any two points $a$ and $c$ in $\mathbb{Q}_p$, we can find a third point $b \in \mathbb{Q}_p$ such that the difference $v(b-c) - v(a-b) = -j$. Thus for any positive integer $j$ there exist points $a, b, c \in \mathbb{Q}_p$ such that $\cos\theta = 1 - \frac{1}{2p^{2j}}$. Therefore all such possible angles exist.

**Proposition 18.** *Given a positive integer $j$ and $a, c \in \mathbb{Q}_p$, $a \neq c$, we can choose $b \in \mathbb{Q}_p$ such that $\triangle abc$ has $d(a,b) < d(a,c)$, $d(a,c) = d(b,c)$, and* $\cos(\angle acb) = 1 - \frac{1}{2p^{2j}}$.

*Proof.* Let $j$ be a positive integer. Let $a, c \in \mathbb{Q}_p$ with $a - c = p^{\alpha} d$, $v(d) = 0$ and $\alpha \in \mathbb{Z}$. Choose $\beta = \alpha + j$, and let $b = p^{\beta} + a$. Then

$$v(b-a) = v(p^{\beta}) = \beta.$$

Since $\beta > \alpha$ we have $p^{-\beta} < p^{-\alpha}$, hence $d(a,b) < d(a,c)$.

Solving $a - c = p^{\alpha} d$ for $c$ we have

$$v(b-c) = v(p^{\beta} + a - (a - p^{\alpha} d)) = v(p^{\beta} + p^{\alpha} d) \geq \min\{\alpha, \beta\} = \alpha$$

since $\alpha < \beta$. Then $v(b-c) = v(a-c)$. Therefore $d(a,c) = d(b,c)$.

Lastly, $v(b-c) - v(a-b) = \alpha - \beta = \alpha - (\alpha + j) = -j$, and therefore from (4) $\cos(\angle acb) = 1 - \frac{1}{2p^{2j}}$. □

We showed in the previous section that distances in $\mathbb{Q}_p$ are not additive (Corollary 13). The reader may verify that angles in $\mathbb{Q}_p$ are not additive.

# 3 Polygons in $\mathbb{Q}_p$

Having looked (far from exhaustively) at triangles in $\mathbb{Q}_p$, curiosity leads us naturally to consider other $n$-gons, $n > 3$.

**Definition 19.** *A* $p$-adic regular $n$-gon *is a set of $n$ points $a_1, a_2, ..., a_n$ which we will call vertices such that the side length $d(a_i, a_{i+1}) = d(a_{i+1}, a_{i+2})$ for all $i$, $1 \leq i \leq n$, with the understanding that $a_{n+1} = a_1$.*

For $p \geq 3$, we can construct $p$-adic regular $n$-gons for any $n \in \mathbb{N}$, $n \geq 3$. For example, take the points $0, 1, 2, 3, 4$ in order and suppose $p = 3$. Then as in the figure, $d(0,1) = d(1,2) = d(2,3) = d(3,4) = d(4,1) = 1$.

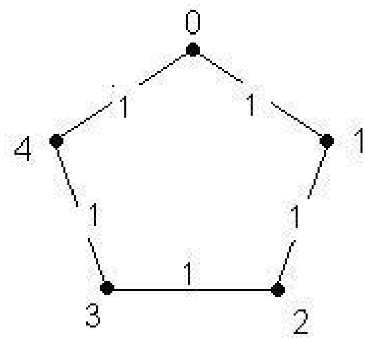

Figure 3. A regular 3-adic polygon

This construction will work for any $n$-gon unless the point $n-1$ is a multiple of $p$, making $d(a_n, a_1) > d(a_1, a_2) = 1$. In this case arrange the vertices as $0, 1, ..., n-3, n-1, n-2$. Then since $p \geq 3$ all the side lengths are 1 as before. For $p = 2$ this construction works only half of the time. We already knew that 3-gons do not exist for $p = 2$. Now we show that this is part of a more general behavior.

**Theorem 20.** *2-adic regular $n$-gons exist if and only if $n$ is even.*

*Proof.* Let $p = 2$. Let $G$ be a regular 2-adic $n$-gon with side length $2^{-N}$, where $N \in \mathbb{Z}$. Let $a_1, a_2, \cdots, a_n$ be the vertices of $G$. Write the $a_i$ as binary expansions, so that

$$a_i = a_{ix_i}2^{x_i} + a_{i(x_i+1)}2^{x_i+1} + a_{i(x_i+2)}2^{x_i+2} + ...$$

for some $x_i \in \mathbb{Z}$; note $x_i$ may be negative. Now by assumption $|a_i - a_{i\pm1}| = 2^{-N}$ for $1 \leq i \leq n$. Therefore the first term in the binary expansion of $a_i$ that differs from the expansion of $a_{i\pm1}$ is the $N^{th}$ term. Possibly their expansions are the same for a finite number of terms, but these will drop out when we take the difference $|a_i - a_{i\pm1}|$, since the only possible coefficients of the powers of 2 are 0 and 1.

Further, $a_{i\pm1}$ *must* differ from $a_i$ in the $N^{th}$ term, otherwise $|a_i - a_{i\pm1}| < 2^{-N}$. Therefore $a_{im} = a_{jm}$ for all $i, j$ when $m < N$, and $a_{iN} \neq a_{(i\pm1)N}$.

If we list the $a_i$, writing the sum of the first $N-1$ terms as $A$, we see the pattern more clearly. Assume $a_{iN} = 1$.

$$\begin{aligned} a_i &= A + 1 \cdot 2^{-N} + \cdots \\ a_{i+1} &= A + 0 \cdot 2^{-N} + \cdots \\ a_{i+2} &= A + 1 \cdot 2^{-N} + \cdots \\ a_{i+3} &= A + 0 \cdot 2^{-N} + \cdots \\ &\vdots \end{aligned}$$

So $a_{i+(n-1)} = A + 0 \cdot 2^{-N} + \cdots$ if $n$ is even and $a_{i+(n-1)} = A + 1 \cdot 2^{-N} + \cdots$ if $n$ is odd. But if $n$ is odd

$$|a_i - a_{i+(n-1)}| = |0 \cdot 2^{-N} + b \cdot 2^{-N+1} + b \cdot 2^{-N+2} + \cdots| < 2^{-N},$$

where $b \in \{0, 1\}$. So $G$ would not be a regular $p$-adic $n$-gon. Therefore $n$ must be even.

Conversely, we can construct regular 2-adic $n$-gons where $n$ is even, using the points $0, 1, 2, ..., m-1$, as shown previously. Since $m-1$ is odd, $d_2(0, m-1) = 1$, and all the other side lengths are also 1. $\square$

We see we can usually construct regular $p$-adic polygons. However, given an arbitrary collection of k points in $\mathbb{Q}_p$, it is possible that no subset of the points forms a regular $p$-adic polygon.

**Example 21.** *Suppose we are given the set of k points $S = \{1, 2, p+2, p^2+2, ..., p^{k-2}+2\}$. Then since $v(a-b) = min\{v(a), v(b)\}$ for $v(a) \neq v(b)$, it is not too difficult to check the values of all possible sides between points in S. For $s \in S$ we have*

$$\begin{aligned} |1 - s| &= 1, s \neq 1 \\ |p + 2 - s| &= 1/p, s \neq 1, p+2 \\ |p^2 + 2 - s| &= 1/p^2, s \neq 1, p+2, p^2+2 \\ &\vdots \\ |p^{k-2} + 2 - 2| &= 1/p^{k-2}. \end{aligned}$$

*Therefore there are no regular $p$-adic polygons with 1 as a vertex, since no pair of vertices not including 1 has a side of the same length as the sides with 1 as as a vertex. We can argue similarly for all the other vertices. Therefore there are no regular $p$-adic polygons formed by any subset of S. This is illustrated for $p = 2$ and $k = 5$.*

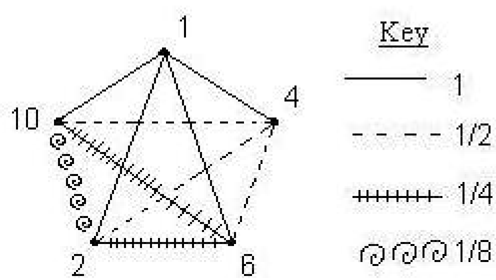


*Figure 4. No subset of points forms a regular $p$-adic polygon.*

# 4 Conclusion

Although there are many similarities between Euclidean geometry and geometry in $\mathbb{Q}_p$, the $p$-adic metric puts strong restrictions on the distances among points in $\mathbb{Q}_p$. In this paper we have considered some aspects of triangles, angles, and regular polygons in $\mathbb{Q}_p$. It might be interesting to define regular polyhedra in $\mathbb{Q}_p$ and see if they have similar restrictions.

# References


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