Some Geometry of the p-adic Rationals

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June 8, 2006

Abstract

Given a prime p, we introduce the p-adic absolute value on the rational numbers. We define triangles and angles using this absolute value and investigate their behavior under the p-adic absolute value and how it differs from Euclidean geometry. Finally, we consider the existence of other polygons using the new absolute value.

1 Introduction

Absolute value on the integers is often defined by $|a| = \sqrt{a^2}$. This represents the "usual" distance from a to zero, and underlies our notion of how Euclidean geometry behaves. However, there are other ways of defining absolute value. In this paper we will be using the p-adic absolute value.

Recall that any positive integer except 1 has a unique prime factorization. Suppose we are interested in only one prime, p (whence the "p" in p-adic). Then we can write any positive integer a uniquely as $a = p^n \cdot a'$, where n is the highest power of p that divides a and $a' \in \mathbb{Z}^+$. If we allow a' to be a rational number in lowest terms and n to be any integer, we can write any rational number this way. This idea, defined more carefully, leads to the p-adic valuation, which we then use to define the p-adic absolute value.

Throughout this paper p will denote a prime.

1.1 The p-adic Valuation and Absolute Value

Let $a \in \mathbb{Q}$, $a \neq 0$, and write $a = p^n a'$, $n \in \mathbb{Z}$ and p does not divide the numerator nor the denominator of a' when a' is expressed as a reduced fraction. (Clearly a' is well-defined.) The p-adic valuation is the function $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ with

$$v_p(a) = n$$

$$v_p(0) = \infty.$$

Hereafter we will write v for v_p when the prime is understood.

Example 1. *Suppose* $p = 3$. *Then* $v_3(3) = v_3(3^1 \cdot 1) = 1$, $v_3(5) = v_3(5^1 \cdot 3^0) = 0$, *and* $v_3(1/9) = v_3(3^{-2} \cdot 1) = -2$.

The p-adic valuation has two important properties that we will use constantly throughout this paper.

Proposition 2. *For all $a, b \in \mathbb{Q}$ we have*

1. $v(ab) = v(a) + v(b)$
2. $v(a-b) \geq \min\{v(a), v(b)\}$, *with equality when* $v(a) \neq v(b)$.

Proof. Let $a, b \in \mathbb{Q}$. Write $a = p^n a'$ and $b = p^m b'$, m, $n \in \mathbb{Z}$ and p does not divide the numerator nor the denominator of a' and b'. Then $v(a) = n$ and $v(b) = m$ and

$$v(ab) = v(p^{n+m}a'b') = n + m = v(a) + v(b).$$

This proves (1). For (2), assume first that $v(a) > v(b)$. So $n > m$. Then

$$v(a-b) = v(p^n a' - p^m b') = v(p^m(p^{n-m}a' - b')) = v(p^m) + v(p^{n-m}a' - b') = m + 0 = m = v(b).$$

If $v(a) = v(b)$,

$$v(a-b) = v(p^n(a' - b')) = v(p^n) + v(a' - b') = n + k,$$

where $k > 0$ if p divides $a' - b'$ and $k = 0$ otherwise. □

Definition 3. *The p-*adic absolute value *is the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ given by*

$$|a|_p = p^{-v_p(a)}$$

$$|0|_p = 0.$$

Example 4. *Consider 24 and 1/24, and suppose $p = 2$. Since $24 = 2^3 \cdot 3$, $v_2(24) = 3$, and $|24|_2 = 2^{-3} = 1/8$. Since $1/24 = 2^{-3} \cdot 3^{-1}$, $v_2(1/24) = -3$, and $|1/24|_2 = 2^{-(-3)} = 8$. Similarly, if $p = 3$ then $|24|_3 = 3^{-1}$ and $|1/24|_3 = 3^{-(-1)} = 3$. Finally, if $p = 5$, $|24|_5 = |2^3 \cdot 3 \cdot 5^0|_5 = 5^0 = 1$ and $|1/24|_5 = 5^{-0} = 1$.*

The next proposition will be useful later. The proof, which is omitted, is an easy application of the definition of the p-adic absolute value.

Proposition 5. *For all $a, b \in \mathbb{Q}$, $|ab|_p = |a|_p \cdot |b|_p$.*

We define the distance between two points by $d_p(a, b) = |a - b|_p$. Hereafter we will usually suppress the "sub-p" notation, understanding that we will be working with the p-adic absolute value for some fixed p.

Theorem 6. *The distance $d(a, b)$ is a metric, i.e., for all $a, b, c \in \mathbb{Q}$,*

1. $d(a, b) > 0$ *for* $a \neq b$ *and* $d(a, a) = 0$
2. $d(a, b) = d(b, a)$
3. $d(a, c) \leq d(a, b) + d(b, c)$ *(the Triangle Inequality).*

Proof. Let $a, b, c \in \mathbb{Q}$. Suppose $a \neq b$, $a - b = p^n d$, for $n \in \mathbb{Z}$, $d \in \mathbb{Q}$, and $v(d) = 0$. Clearly properties 1 and 2 of Theorem 6 hold.

For the third property we need to show

$$d(a,c) = |a-c| = p^{-v(a-c)} \leq p^{-v(a-b)} + p^{-v(b-c)} = d(a,b) + d(b,c). \quad (1)$$

Adding and subtracting b from $a - c$, we can rewrite $v(a-c)$ as $v\big((a-b)-(c-b)\big)$. This is greater than or equal to the minimum of $\{v(a-b), v(c-b)\}$ by part 2 of Proposition 2. Note that since by property 2 of Theorem 6 $d(b,c) = d(c,b)$, $v(b-c)$ must equal $v(c-b)$.

If $v(a-c)$ equals one of $v(a-b)$, $v(c-b)$, i.e., $v(a-b) \neq v(c-b)$, we can subtract $p^{-v(a-c)}$ from both sides of (1) to give $0 \leq p^{-v(a-b)}$ or $0 \leq p^{-v(b-c)}$, which is true by property 1 of Theorem 6. If $v(a-c)$ is greater than both $v(a-b)$ and $v(b-c)$, then clearly $p^{-v(a-c)}$ is less than $p^{-v(a-b)} + p^{-v(b-c)}$. □

Thus $\mathbb{Q}$ is what we call a *metric space* (see [3] for a full definition). We will refer to the elements of such a set, here $\mathbb{Q}$, as points. The terms line segment, side, and edge will all refer to pairs of distinct points. We will see later that no three points are collinear.

1.2 The p-adic Completion of $\mathbb{Q}$

Now that we have a metric space, we are interested in investigating how $\mathbb{Q}$ with the p-adic distance compares to the set of real numbers with its usual distance. For instance, one of the properties of $\mathbb{R}$ is that it is complete.

Definition 7. *A sequence (s_n) in a metric space X is said to be a* Cauchy sequence *if for every $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that whenever n, $m \in \mathbb{Z}$, n, $m \geq N$, $d(s_n, s_m) < \varepsilon$.*

A metric space in which every Cauchy sequence converges is said to be *complete*. As mentioned above, the real numbers with the usual distance are complete (see [1], p. 159). However, it is known that $\mathbb{Q}$ with the p-adic valuation is not complete; there exist sequences where the terms become arbitrarily close, yet the sequence does not converge in $\mathbb{Q}$.

Example 8. *Let $a \in \mathbb{Z}, a > 0$. We can expand a uniquely as*

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n$$

with $0 \leq a_i \leq p-1$ and $a_n \neq 0$. This is called the p-adic expansion *of a. Note that $v(a) = \min\{i | a_i \neq 0\}$. Consider the sequence*

$$r_0 = 1,\ r_1 = 1+p,\ r_2 = 1+p+p^2,\ r_3 = 1+p+p^2+p^3,\ \cdots .$$

For $n, m \in \mathbb{Z}$, $n < m$, we get

$$d(r_n, r_m) = \left|p^{n+1} + \cdots + p^m\right| = p^{-(n+1)}.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $N > \log_p(1/\varepsilon) - 1$. Then for all $n < m \in \mathbb{Z}$, $n, m \geq N$ we have $d(r_n, r_m) = p^{-(n+1)} < \varepsilon$. Therefore the sequence $\{r_i\}$ is Cauchy. However, the sequence does not converge: clearly

$$\lim_{n\to\infty} r_n = \sum_{i=0}^{\infty} p^i$$

which is certainly not a rational. Therefore $\mathbb{Q}$ is not complete under the p-adic metric. This example also shows that $\mathbb{Z}$ is not complete under the p-adic metric.

Jacobson in [2] gives an explicit construction for the completion of any field with a real valuation. Here we will only give the definition of the completion of $\mathbb{Q}$.

Definition 9. *The p-*adic completion *of $\mathbb{Q}$, denoted $\mathbb{Q}_p$, is defined to be*

$$\mathbb{Q}_p = \left\{ \sum_{i>-\infty}^{\infty} a_i p^i \,|\, 0 \leq a_i \leq p-1 \right\}.$$

It can be shown that $\mathbb{Q}_p$ is a field. The *p-adic completion* of $\mathbb{Z}$, denoted $\mathbb{Z}_p$, is the same as for $\mathbb{Q}_p$ except the sum runs from $i = 0$ to ∞. Notice that $v(a)$ for $a \in \mathbb{Q}_p$ is the exponent of the power of p with the least non-zero coefficient.

Example 10. *Suppose $p = 5$. Then $76/625 = 1 \cdot 5^{-4} + 0 \cdot 5^{-3} + 3 \cdot 5^{-2} + 0 \cdot 5^{-1} + 0 \cdot 5^0 + 0 \cdot 5^1 + \cdots$, and $|76/625| = |5^{-4} \cdot 76| = 5^{-(-4)} = 625$.*

2 Triangles and Angles in $\mathbb{Q}_p$

2.1 Triangles in $\mathbb{Q}_p$

We often envision $\mathbb{R}$ as a line. In $\mathbb{Q}_p$, however, property 3 of Theorem 6, the "Triangle Inequality", turns out to be strict inequality, i.e., no three points in $\mathbb{Q}_p$ are collinear. So instead we define p-adic triangles and investigate their behavior using the p-adic distance. Our assertion that no three points in $\mathbb{Q}_p$ are collinear will be easier to prove later on in this section.

Definition 11. *A* triangle *is three distinct points a, b, and c in $\mathbb{Q}_p$. We will denote a triangle by $\triangle abc$.*

From the proof of Theorem 6 we obtain several immediate properties of triangles under the p-adic valuation.

Theorem 12. *Using the p-adic absolute value, all triangles are isosceles.*

Proof. Let $\triangle abc$ be a triangle with vertices a, b, c and sides of length $p^{-v(a-c)}$, $p^{-v(a-b)}$, and $p^{-v(b-c)}$. If any two of $v(a-c)$, $v(a-b)$, and $v(b-c)$ are equal, we are done.

Without loss of generality, suppose $v(a-b) \neq v(b-c)$. Then from Proposition 2, $v(a-c) = \min\{v(a-b), v(b-c)\}$. Therefore at least two of the sides must be of equal length. $\square$

Theorem 13. *If a triangle is not equilateral, the unequal side has the largest valuation and hence the shortest length.*

Proof. Let $\triangle abc$ be a triangle with $v(a-b) = v(b-c) \neq v(a-c)$. Then

$$v(a-c) = v((a-b)-(c-b)) \geq \min\{v(a-b), v(b-c)\},$$

and since $v(a-c) \neq v(a-b) = v(b-c)$ we have $v(a-c) > v(a-b)$. $\square$

Corollary 14. *Given three distinct points $a, b, c \in \mathbb{Q}_p$, $d(a,c) < d(a,b) + d(b,c)$. In other words, no three points in $\mathbb{Q}_p$ are collinear.*

The proof follows easily from Theorem 13 and is left to the reader.

We have seen that all triangles in $\mathbb{Q}_p$ are isosceles. Equilateral triangles are usually easy to construct: for instance, suppose $p = 5$ and take the points 10, 15, and 20. More generally, for $p \geq 3$ the points $2p$, $3p$ and $4p$ form an equilateral triangle since $d(2p, 3p) = |p(3-2)| = p^{-1}$, $d(3p, 4p) = |p(4-3)| = p^{-1}$, and $d(2p, 4p) = |p(4-2)| = p^{-1}$. In $\mathbb{Q}_2$, however, equilateral triangles do not exist.

Theorem 15. *Given a prime p, any subset of $\mathbb{Q}_p$ has at most p equidistant points.*

Proof. Suppose on the contrary that there is a set of $p+1$ distinct equidistant points $a_1, a_2, ..., a_{p+1}$ with $a_i = \sum_{k=j_i}^{\infty} a_{ik} p^k$, $a_{ij_i} \neq 0$. Since the a_i's are all equidistant, there exists $m \in \mathbb{Z}$ such that $v(a_i - a_j) = m$ for all i, j. Thus

$$\begin{aligned} a_i - a_j &= p^m \sum_{k=0}^{\infty} (a_{ik} - a_{jk}) p^k \\ &= \left(p^m \sum_{k=0}^{\infty} a_{ik} p^k\right) - \left(p^m \sum_{k=0}^{\infty} a_{jk} p^k\right) \\ &= p^m a'_i - p^m a'_j \\ &= p^m (a'_i - a'_j) \end{aligned}$$

where $a'_i = \sum_{k=0}^{\infty} a_{ik} p^k$, $a'_j = \sum_{k=0}^{\infty} a_{jk} p^k$, and $v(a'_i - a'_j) = 0$. Therefore $v(a_i - a_j) = m + v(a'_i - a'_j) = m + 0$, $m \in \mathbb{Z}$. However, because $0 \leq a'_{i0} \leq p-1$ for all i and because there are $p+1$ points, there exist distinct i and j so that $a'_{i0} = a'_{j0}$. Therefore $v(a'_i - a'_j) > 0$ for some i and j. $\square$

Corollary 16. *Equilateral triangles do not exist under the 2-adic metric.*

On the theme of nonexistence, we next show that right triangles in $\mathbb{Q}_p$ do not exist. We will say a *right triangle* in $\mathbb{Q}_p$ is a triangle whose side lengths satisfy the Pythagorean Theorem.

Theorem 17. *For any $a, b, c \in \mathbb{Q}_p$ we have $d(a,c)^2 \neq d(a,b)^2 + d(b,c)^2$. In other words, right triangles in $\mathbb{Q}_p$ do not exist.*

Proof. Suppose $\triangle abc$ is a so-called right triangle with longest side ab. Then $d(b,c) = d(a,c)$. Since $\triangle abc$ is a right triangle we have

$$d(a,b)^2 = d(b,c)^2 + d(a,c)^2 = 2d(b,c)^2.$$

Thus

$$\begin{aligned}
\left(p^{-v(a-b)}\right)^2 &= 2\left(p^{-v(b-c)}\right)^2 \\
\frac{1}{2} &= \frac{p^{2v(a-b)}}{p^{2v(b-c)}} \\
\frac{1}{2} &= p^{2v(a-b)-2v(b-c)} \\
\frac{1}{2} &= p^{2v(\frac{a-b}{b-c})}. \qquad (2)
\end{aligned}$$

Then (2) implies $p = 2$ and $v(\frac{a-b}{b-c}) = \frac{-1}{2}$. But v is always an integer by definition, so this cannot occur. $\square$

2.2 Angles in $\mathbb{Q}_p$

We have shown that right triangles, and so presumably right angles, do not exist in $\mathbb{Q}_p$. But we have yet to properly define what we mean by angles in $\mathbb{Q}_p$. The p-adic absolute value tells us only about distances. Therefore we define angles using what would be the Law of Cosines if our distance were the Euclidean metric.

Definition 18. *Given distinct $a, b, c \in \mathbb{Q}_p$, we define the angle θ between sides ac and bc by*

$$\theta = \arccos\left(\frac{-d(a,b)^2 + d(b,c)^2 + d(a,c)^2}{2d(b,c)d(a,c)}\right)$$

There are two possibilities for θ, depending on whether $d(a,b) = d(b,c)$ or $d(a,c) = d(b,c)$.

Suppose $d(a,b) = d(a,c)$ (Figure 1).

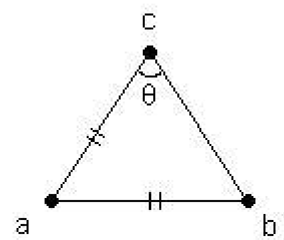

Figure 1.

Then we have

$$\begin{aligned}\cos\theta &= \frac{d(b,c)^2}{2d(b,c)d(a,c)} \\ &= \frac{d(b,c)}{2d(a,c)} \\ &= \frac{p^{v(a-c)}}{2p^{v(b-c)}} \\ &= \frac{1}{2}p^{v(a-c)-v(b-c)}.\end{aligned}$$

By Corollary 13 $v(a-c) \le v(b-c)$. So $0 < \cos\theta \le 1/2$. Thus $\frac{\pi}{3} \le \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{1}{2}p^k$ where $k = v(a-c) - v(b-c)$.

Suppose $d(a,c) = d(b,c)$ (Figure 2).

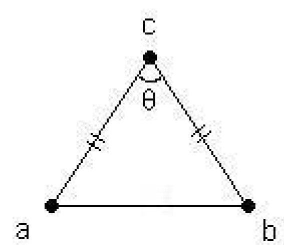

Figure 2.

Then

$$\begin{aligned}
\cos\theta &= \frac{-d(a,b)^2 + 2d(b,c)^2}{2d(b,c)^2} \\
&= -\Big(\frac{d(a,b)}{2d(b,c)}\Big)^2 + 1 \\
&= \frac{-p^{2v(b-c)}}{2p^{2v(a-b)}} + 1 \\
&= 1 - \frac{1}{2}p^{2(v(b-c)-v(a-b))} \\
&= 1 - \frac{1}{2p^k} > 0 \\
&= \frac{2p^k - 1}{2p^k}
\end{aligned}$$

where $k = 2(v(b-c)-v(a-b))$. Thus $\frac{\pi}{3} < \cos\theta < 1$. Also, given any positive integer k there exist points $a, b, c \in \mathbb{Q}_p$ so that $\cos\theta = \frac{2p^k-1}{2p^k}$. Therefore all such possible angles exist. One possible construction would use the following proposition, which shows that any two points can be viewed as one of the long sides of some triangle.

Proposition 19. *Given $a, c \in \mathbb{Q}_p$, $a \neq c$, we can choose $b \in \mathbb{Q}_p$ such that $\triangle abc$ has $d(a,c) = d(b,c)$ and $d(a,b) < d(b,c)$.*

Proof. Let $a, c \in \mathbb{Q}_p$ with $a - c = p^\alpha d$, $v(d) = 0$ and $\alpha \in \mathbb{Z}$. Choose $\beta \in \mathbb{Z}$ so that $\alpha < \beta$. Let $b = p^\beta + a$. Then

$$v(b-a) = v(p^\beta) = \beta.$$

Since $\beta > \alpha$ we have $p^{-\beta} < p^{-\alpha}$, hence $d(a,b) < d(a,c)$.

Solving $a - c = p^\alpha d$ for c we have

$$v(b-c) = v(p^\beta + a - (a - p^\alpha d)) = v(p^\beta + p^\alpha d) \geq \min\{\alpha, \beta\} = \alpha$$

since $\alpha < \beta$. Then $v(b-c) = v(a-c)$. Therefore $d(a,c) = d(b,c)$. □

We showed in the previous section that distances in $\mathbb{Q}_p$ are not additive (Corollary 14). The reader might verify that angles in $\mathbb{Q}_p$ are not additive.

3 Polygons in $\mathbb{Q}_p$

Having looked (far from exhaustively) at triangles in $\mathbb{Q}_p$, curiosity leads us naturally to consider other n-gons, $n > 3$.

Definition 20. *A* p-adic regular n-gon *is a set of n points $a_1, a_2, ..., a_n$ which we will call vertices such that the side length $|a_i a_{i+1}| = |a_{i+1} a_{i+2}|$ for all i, $1 \leq i \leq n$, with the understanding that $a_{n+1} = a_1$.*

For $p \geq 3$, we can construct p-adic regular n-gons for any $k \in \mathbb{N}$, $k \geq 3$. For example, take the points $0, 1, 2, 3, 4$ in order and suppose $p = 3$. Then as in the figure, $d(0,1) = d(1,2) = d(2,3) = d(3,4) = d(4,1) = 1$.

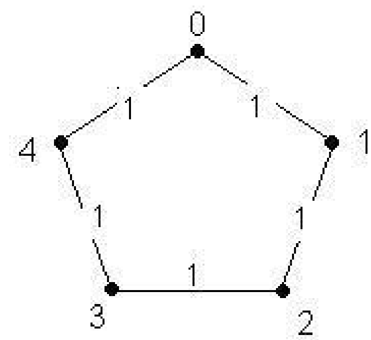

Figure 3. A regular 3-adic polygon

This construction will work for any n-gon unless the point $n-1$ is a multiple of p, making the side $|a_n a_1| > |a_1 a_2| = 1$. In this case arrange the vertices as $0, 1, ..., n-3, n-1, n-2$. Then since $p \geq 3$ all the side lengths are 1 as before.
For $p = 2$ this construction works only half of the time. We already knew that 3-gons do not exist for $p = 2$. Now we show that this is part of a more general behavior. To do this we will need the following lemma.

Lemma 21. *For all $a, b, c \in \mathbb{Q}_p$*

1. $d(a,b) = d((a+c),(b+c))$
2. $d(ca, cb) = |c| d(a,b)$.

Proof. Let $a, b, c \in \mathbb{Q}_p$. To show property 1 of the lemma, recall

$$|a - b| = |(a+c) - (b+c)|.$$

So we can add the same value to any points without changing the distance between them.

For property 2 of the lemma, notice

$$d(ca, cb) = |ca - cb| = |c(a-b)| = |c||a-b|$$

by Proposition 5. Therefore multiplying two points by some constant value scales the distance between them by the constant factor $|c| = p^{-v(c)}$. $\square$

Theorem 22. *2-adic regular n-gons exist if and only if n is even.*

Proof. Let $p = 2$, and suppose there exists a regular 2-adic n-gon, where n is odd. Let $a_1, a_2, ..., a_n$ be the vertices of the n-gon.

Case 1. Suppose one of the vertices a_i has value $|a_i| = 1$ and another vertex a_j has $|a_j| \neq 1$. By Lemma 21 if $|a_j| > 1$, we can multiply all the vertices by $p^{-v(a_j)}$ without changing the relationships of the distances among the vertices. Then $|a_j|$ would be 1 and $|a_i|$ would be less than 1. Therefore we can assume that $|a_j| < 1$. Also we may assume that a_i and a_j are adjacent. So $j = i+1$. Write $a_i = 1+2a'_i$ and $a_{i+1} = 2a'_{i+1}$. Then

$$d(a_i, a_{i+1}) = |1 - 2(a'_i - a'_{i+1})| = 1.$$

Therefore all the side lengths must be 1. Therefore the vertex a_{i-1} has $|a_{i-1}| < 1$ and the vertex a_{j+1} has $|a_{j+1}| = 1$. (To see this, apply Proposition 2, part 2.) Further, no two vertices both with absolute value equal to 1 or both with absolute value not equal to 1 can be adjacent, for then we would have one side length not equal to 1. Therefore supposing k vertices of valuation 1, there must be k vertices of valuation $\neq 1$, one between every two vertices of valuation 1. Therefore the total number of vertices must be even. But we assumed n was odd; contradiction.

Case 2. Suppose that all the vertices have absolute value 1 or that all the vertices have absolute value $\neq 1$. Then we can transform the n-gon to another that has the same relationships among edges, although scaled by some factor of 2, as follows. Choose a vertex a_g so that $|a_g| \geq |a_j|$ for all $j \neq g, 1 \leq g, j \leq n$. Subtract a_g from all the vertices. Now at least one vertex, the vertex 0, has a valuation not equal to 1.

Then identify the the vertex (vertices) having the smallest valuation v. Suppose this term is $a_m \cdot 2^m$. Multiply all the vertices by 2^{-m}. Now at least one vertex has valuation equal to 1. So we are back to case 1.

Finally, we can construct regular 2-adic m-gons where m is even, using the points $0, 1, 2, ..., m-1$, as shown previously. Since $m-1$ is odd, $d_2(0, m-1) = 1$, and all the other side lengths are also 1. □

We see we can usually construct regular p-adic polygons. However, given an arbitrary collection of k points in $\mathbb{Q}_p$, it is possible that no subset of the points forms a regular p-adic polygon.

Example 23. *Suppose we are given the set of k points $S = \{1, 2, p+2, p^2+2, ..., p^{k-2}+2\}$. Then since $v(a-b) = min\{v(a), v(b)\}$ for $v(a) \neq v(b)$, it is not too difficult to check the values of all possible sides between points in S. For $s \in S$ we have*

$$\begin{aligned}
|1 - s| &= 1, s \neq 1 \\
|p + 2 - s| &= 1/p, s \neq 1, p+2 \\
|p^2 + 2 - s| &= 1/p^2, s \neq 1, p+2, p^2+2 \\
&\vdots \\
|p^{k-2} + 2 - 2| &= 1/p^{k-2}.
\end{aligned}$$

Therefore there are no p-adic polygons with 1 as a vertex, since no pair of vertices not including 1 has a side of the same length as the sides with 1 as as a vertex. We can argue similarly for all the other vertices. Therefore there are no regular p-adic polygons formed by any subset of S. This is illustrated for $p = 2$ and $k = 5$.

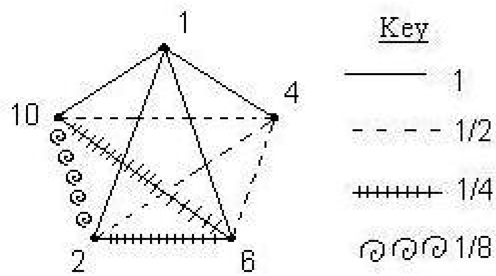

Figure 4. No subset of points forms a regular p-adic polygon.

4 Conclusion

Although there are many similarities between Euclidean geometry and geometry in $\mathbb{Q}_p$, the p-adic metric puts strong restrictions on the distances among points in $\mathbb{Q}_p$. In this paper we have considered some aspects of triangles, angles, and regular polygons in $\mathbb{Q}_p$. It might be interesting to define regular polyhedra in $\mathbb{Q}_p$ and see if they have similar restrictions.

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