A Multi-objective Approach to Portfolio Optimization

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Abstract: Optimization models play a critical role in determining portfolio strategies for investors. The traditional mean variance optimization approach has only one objective, which fails to meet the demand of investors who have multiple investment objectives. This paper presents a multi-objective approach to portfolio optimization problems. The proposed optimization model simultaneously optimizes portfolio risk and returns for investors and integrates various portfolio optimization models. Optimal portfolio strategy is produced for investors of various risk appetite. Detailed analysis based on convex optimization and application of the model are provided and compared to the mean variance approach.

1. Introduction to Portfolio Optimization

Portfolio optimization plays a critical role in determining portfolio strategies for investors. What investors hope to achieve from portfolio optimization is to maximize portfolio returns and minimize portfolio risk. Since return is compensated based on risk, investors have to balance the risk-return tradeoff for their investments. Therefore, there is no a single optimized portfolio that can satisfy all investors. An optimal portfolio is determined by an investor's risk-return preference.

There are a few key concepts in portfolio optimization. First, reward and risk are measured by expected return and variance of a portfolio. Expected return is calculated based on historical performance of an asset, and variance is a measure of the dispersion of returns. Second, investors are exposed to two types of risk: systematic risk and unsystematic risk. Systematic risk is the risk generally associated with the market which cannot be eliminated. Unsystematic risk is an asset's intrinsic risk which can be diversified away by owning a large number of well-diversified assets. Portfolio risk refers to the systematic risk because the unsystematic risks of each asset in a portfolio alone do not present enough information about the overall risk of the entire portfolio. Third, variability or risk of a portfolio is given by covariance between different asset returns. Therefore, a well-diversified portfolio contains assets that have little or negative correlations [1].

The key to achieve investors' objectives is to provide an optimal portfolio strategy which shows investors how much to invest in each asset in a given portfolio. Therefore, the decision variable of portfolio optimization problems is the asset weight vector $\bar{x} = \left[x_1\, x_2 \cdots x_n\right]^T$ with

x_i as the weight of asset i in the portfolio. The expected return for each asset in the portfolio is expressed in the vector form $\vec{p} = [p_1 p_2 \cdots p_n]^T$ with p_i as the mean return of asset i. The portfolio expected return is the weighted average of individual asset return $x_p = \vec{p}^T \vec{x} = \sum_{i=1}^{n} x_i p_i$. Variance and covariance of individual asset are characterized by a variance-covariance matrix $V = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}$, where $\sigma_{i,i}$ is the variance of asset i and $\sigma_{i,j}$ is the covariance between asset i and asset j. The portfolio variance is $\sigma_p{}^2 = \vec{x}^T V \vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{i,j}$ [2].

1.1 Problem Formulations

Modern portfolio theory assumes that for a given level of risk, a rational investor wants the maximal return, and for a given level of expected return, the investor wants the minimal risk. There are also extreme investors who only care about maximizing return (disregard risk) or minimizing risk (disregard expected return). There are generally five different formulations that serve investors of different investment objectives:

Model 1: Maximize expected return (disregard risk)

Maximize: $x_p = \vec{p}^T \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$

where $\vec{1}^T = [1 \cdots 1]$. The constraint $\vec{1}^T \vec{x} = 1$ requires the sum of all asset weights to be equal to 1.

Model 2: Minimize risk (disregard expected return)

Minimize: $\sigma_p{}^2 = \vec{x}^T V \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$

Model 3: Minimize risk for a given level of expected return $p*$

Minimize: $\sigma_p{}^2 = \vec{x}^T V \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$ and $\vec{p}^T \vec{x} = p*$

Model 4: Maximize return for a given level of risk σ^{2*}

Maximize: $x_p = \vec{p}^T \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$ and $\vec{x}^T V \vec{x} = \sigma^{2*}$

Model 5: Maximize return and Minimize risk

Maximize: $x_p = \vec{p}^T \vec{x}$ and Minimize: $\sigma_p{}^2 = \vec{x}^T V \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$

The five models above include both rational and extreme investors with different investment objective. Model 3 and 4 are extensions of Model 1 and 2 with fixed constraints. The classic solution to portfolio optimization is the mean variance optimization proposed by Nobel Prize winner Harry Markowitz in 1990 [2]. The mean variance method aims at minimizing variance of a portfolio for any given level of expected return, which shares the same formulation of model 3. Since the mean variance method assumes all investors' objectives are to minimize risk, it may not be the best model for those who are extremely risk seeking. Also, the formulation does not allow investors to simultaneously minimize risk and maximize expected return.

1.2 Introduction to Multi-objective optimization

An alternative approach to the portfolio optimization problem is multi-objective optimization. The multi-objective optimization principle is first developed by a French-Italian economist V. Pareto for use in economics [3]. In multi-objective optimization problems, one seeks a set of values for the design variable that optimizes a set of objective functions. The set of variables that produce the optimal outcome is referred to as the Pareto optimal set. A point x is called Pareto optimal if $f(x)$ is a minimal element in the Pareto optimal set. In other words, there exists no other feasible point that gives a better optimal value than that of the Pareto optimal point. The formulation combines multiple objectives into one objective function by assigning a weighting coefficient to each objective. The standard solution technique is to minimize a positively weighted convex sum of the objectives using single-objective method [4].

The portfolio optimization problem is a typical multi-objective problem because investors want to maximize return and minimize risk at the same time (Model 5). Since the multi-objective formulation of the portfolio optimization problem belongs to the category of convex vector optimization, it can be solved using techniques of convex vector optimization. Because of the special property of convex functions, convex optimization guarantees that any local optimum is necessarily a global optimum [5]. This paper focuses on the analysis and application of the multi-objective approach to portfolio optimization based on convex vector optimization technique.

2. Methodology

Before going into details about multi-objective optimization, it is essential to introduce the concept of convex vector optimization.

2.1 Convex Vector Optimization

As shown in Figure 1, a set $S \subseteq \mathrm{R}^n$ is a convex set if it contains all line segments joining any pair of points in S, that is,

$$x, y \in S,\ \ \theta \geq 0 \Rightarrow \theta x + (1-\theta) y \in S$$

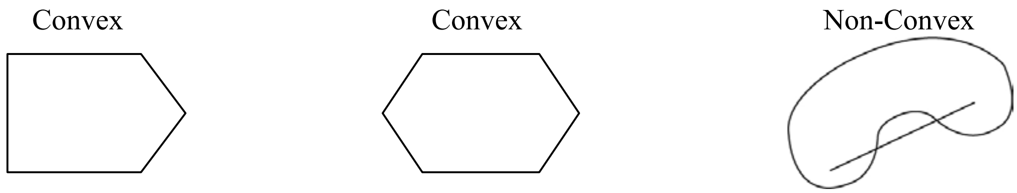

Figure 1 (a) **Figure 1 (b)** **Figure 1 (c)**

A function $f : \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}$ is convex if its domain $dom\, f$ is convex and for all $x, y \in dom\, f\ , \theta \in [0,1]$

$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$$

A function f is concave if $-f$ is convex. Geometrically, one can think of the curve of a convex function as always lying below the line segment of any two points. Here is an example of a convex function:

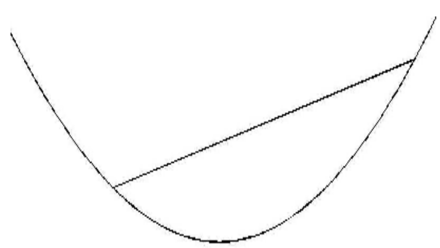

Figure 2 Convex Function

A vector optimization in standard form is a convex optimization if $f_0, f_1, \ldots, f_m$ are all convex and h_i are all affine, meaning that h_i has the form linear plus a constant $Ax + b$

Minimize (w.r.t. x) $\quad f_0(x)$

Subject to $\quad f_i(x) \leq 0,\ \ i = 1, \ldots, m$

$$h_i(x) = 0, \; i = 1, \dots, p$$

An important property of convex optimization is that the feasible region must be a convex region, which guarantees that any local optimum is necessarily a global optimum [5].

2.2 Multi-objective Formulation

The two optimization objectives are minimizing portfolio risk ${\sigma_p}^2 = \vec{x}^T V \vec{x}$ and maximizing portfolio expected return $x_p = \vec{p}^T \vec{x}$, which is the same as minimizing negative $x_p = \vec{p}^T \vec{x}$. The new formulation of the vector optimization problem is given as:

$$\text{Minimize (w.r.t } x\text{): } (F_1(x), F_2(x)) = (-\vec{p}^T \vec{x}, \vec{x}^T V \vec{x}) \qquad \text{(Model 5)}$$

$$\text{Subject to: } \quad \vec{1}^T \vec{x} = 1$$

This multi-objective optimization can be solved using scalarization, a standard technique for finding Pareto optimal points for any vector optimization problem by solving the ordinary scalar optimization [4]. Choose any $\lambda_1, \lambda_2 > 0$ for the two objectives. The vectors λ_1 and λ_2 are weighting coefficients assigned to each objective. By varying λ_1 and λ_2 , one can obtain different Pareto optimal solutions of the vector optimization problem. Without loss of generality, one can take $\lambda_1 = 1$ and $\lambda_2 = \mu > 0$:

$$\text{Minimize: } -\vec{p}^T \vec{x} + \mu \vec{x}^T V \vec{x} \; [1] \qquad \text{(Modified Model 5)}$$

$$\text{Subject to: } \vec{1}^T \vec{x} = 1$$

The weighting coefficient μ represents how much an investor weights risk over expected return. One can consider μ as a risk aversion index that measures the risk appetite of an investor. A smaller value of μ indicates that the investor is more risk seeking, and a larger value of μ indicates that the investor is more risk-averse. All Pareto optimal portfolios can be obtained by varying μ except for two extreme cases where $\mu \to 0$ and $\mu \to \infty$. As $\mu \to 0$, the variance term $\mu \vec{x}^T V \vec{x} \to 0$ and the objective function is dominated by the expected return term $-\vec{p}^T \vec{x}$. This

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[1] The objective function in Modified Model 5 is convex because V is positive semi-definite. A twice differentiable function f is convex if and only if the second derivative of f is positive semi-definite for all $x \in dom\, f$ [5].

replicates Model 1 where investors only want to maximize return and disregard risk. In this case, the investor is being extremely risk seeking. The optimal strategy for this extreme case is to concentrate the portfolio entirely on the asset that gives the highest expected return. As $\mu \to \infty$, $\mu \vec{x}^T V \vec{x} \to \infty$. The objective function is dominated by the variance term $\mu \vec{x}^T V \vec{x}$. This replicates Model 2 where the investor only wants to minimize risk without regard to expected return. In this case, the investor is being extremely risk averse. The optimal strategy for such type of investor is to invest all resources on the asset that has the minimal variance. By varying μ, one can generate various optimization models that serve investors of any risk appetite.

2.3 Solving Multi-objective optimization

The multi-objective optimization can be solved using Lagrangian multiplier:

$$L(\vec{x}) = -\vec{p}^T \vec{x} + \mu \vec{x}^T V \vec{x} + \lambda(\vec{1}^T \vec{x} - 1)$$

Set $\frac{\delta L}{\delta \vec{x}} = 0$, it follows that

$$\vec{x} = \frac{1}{2\mu}(V^{-1})(\vec{p} - \lambda \vec{1}) \tag{1.2}$$

To solve the Lagrangian multiplier λ, substitute equation 1.2 to the constraint $1^T x = 1$:

$$\lambda = \frac{\vec{1}^T V^{-1} \vec{p}}{\vec{1}^T V^{-1} \vec{1}} - \frac{2\mu}{\vec{1}^T V^{-1} \vec{1}} \tag{1.3}$$

Let $a_1 = \vec{1}^T V^{-1} \vec{1}$ and $a_2 = \vec{1}^T V^{-1} \vec{p}$, both of which are scalars, equation 1.3 can be written as:

$$\lambda = \frac{a_2}{a_1} - \frac{2\mu}{a_1}$$

The optimized solution for the portfolio weight vector x is:

$$\vec{x}^* = \frac{1}{2\mu} V^{-1} \vec{p} - \frac{V^{-1}}{2\mu}\left(\frac{a_2}{a_1} - \frac{2\mu}{a_1}\right)\vec{1} \tag{1.4}$$

Detailed derivation of the optimal solution is provided in Appendix A.

3 Applications of Multi-objective Portfolio Optimization

The mathematical results from the multi-objective portfolio optimization (1.4) can be applied to portfolios consisting of any number of assets. As a specific example, assume that an investor is interested in owning a portfolio that contains five of his favorite stocks: IBM (IBM), Microsoft (MSFT), Apple (AAPL), Quest Diagnostics (DGX), and Bank of America (BAC). Cases involving of short selling are excluded in this example assuming that the investor is not sophisticated in finance. The expected return and variance of each stock in the portfolio is calculated based on historical stock price and dividend payment from February 1, 2002 to February 1, 2007. (Appendix C)

Stock	Exp. Return	Variance
IBM	0.400%	0.006461
MSFT	0.513%	0.0039
AAPL	4.085%	0.012678
DGX	1.006%	0.005598361
BAC	1.236%	0.001622897

Table 1 Expected Return and Variances of Selected Stocks

Using Matlab to implement the multi-objective optimization on this portfolio, the investor can see the optimal asset allocation strategy for any value of risk aversion index μ. Figure 1 shows how much the investor should invest in each stock given different values of μ.

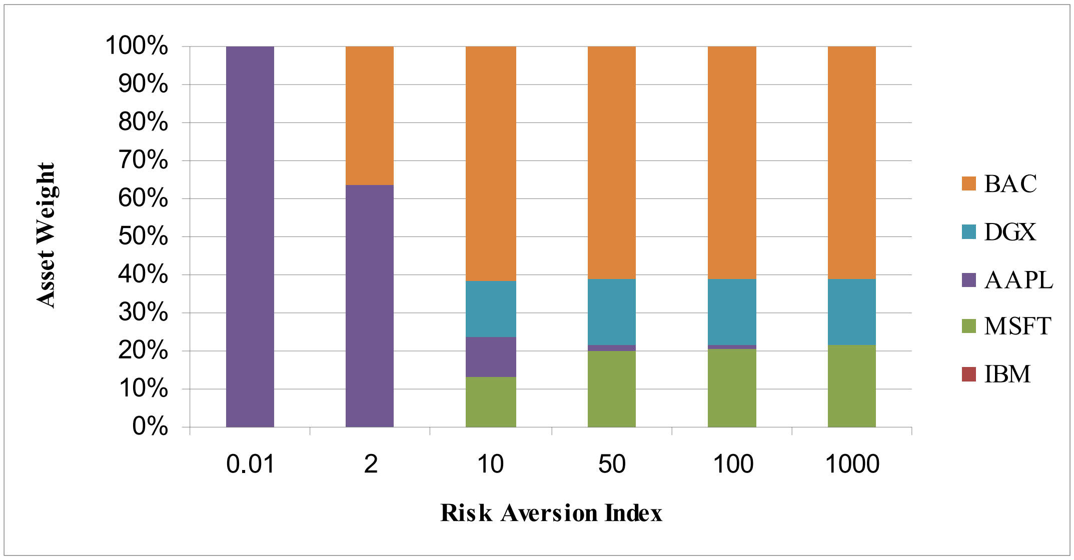

Figure 3 Risk Aversion Index vs. Optimal Asset Allocations

The optimized results of this simple example agree with investors' intuition. As μ equals to 0.01, the investor is being highly risk seeking. The optimal solution shows that the portfolio strategy for such type of investor is to concentrate the portfolio 100% on the highest expected return stock AAPL. As μ increases, the investor is becoming more sensitive to risk, and the composition of portfolio starts to show a mix of other lower return (lower variance) stocks. When μ equals to 50, the optimal portfolio strategy shows that the investor should invest 2.05% of total resources in AAPL stock, 61.22% in BAC stock, 19.77% in MSFT stock, and 16.96% in DGX stock. The allocation on AAPL stock has significantly decreased from 100% to 2.05% as μ increases from 0.01 to 50 because AAPL has the highest return variance. Note that none of the optimal portfolio strategies indicate any asset allocation in IBM stock. That is because IBM gives the lowest return but somewhat high variance compared to other four stocks in the portfolio.

Another important observation from Figure 1 is that there is no significant difference in asset allocation strategy as μ increases from 100 to 1000. The actual data suggests that $1 \leq \mu \leq 100$ is the meaningful range of risk index μ. Figure 2 illustrates that 1 and 100 are two thresholds of μ that are determinate to the investor's portfolio strategy. When $\mu < 1$, the optimal solutions indicate that the investor should invest all his resources on the highest return stock AAPL. For $1 \leq \mu \leq 100$, the optimal solutions indicate a variety of asset allocation strategies. As $\mu > 100$, the asset allocation strategy has little change.

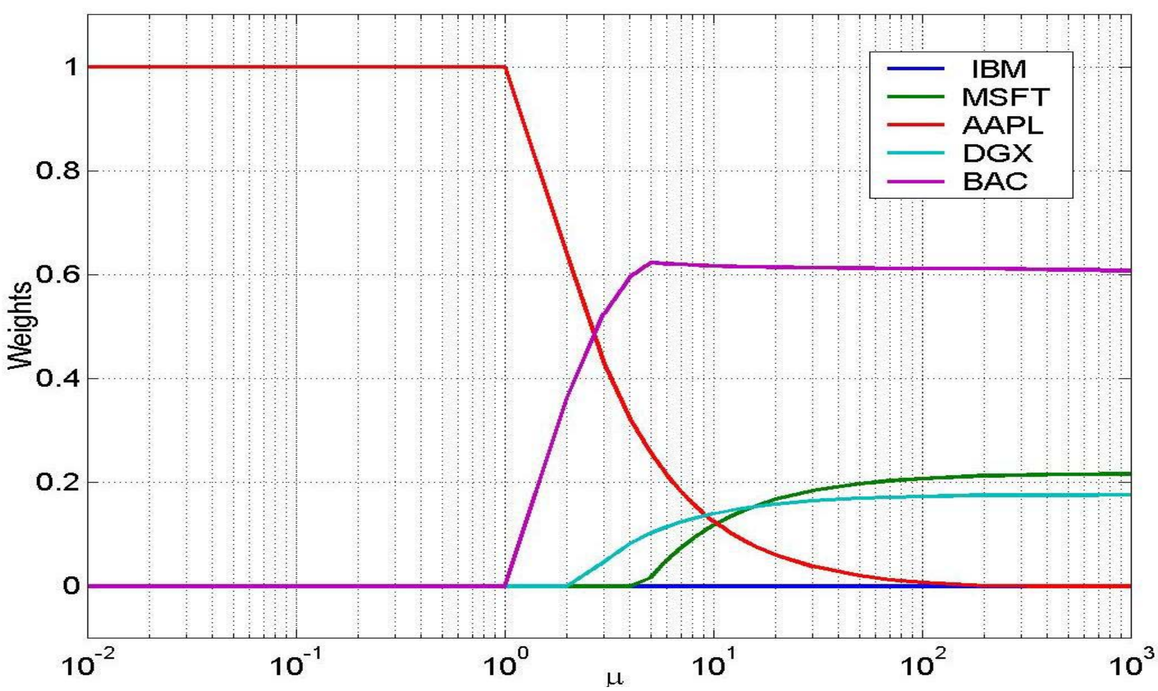

Figure 4 Risk Aversion Index vs. Asset Weight

4 Multi-objective optimization vs. Mean variance optimization

The mathematical analysis and numerical experiment of the multi-objective optimization and mean variance optimization prove that both methods provide the same optimal solution despite their different formulations. From the analytic point of view, the reason why both methods produce the same analytic solution is because their Lagrangian equations share similar forms. When taking the first derivative $\frac{\delta L}{\delta \bar{x}}$, the constants terms vanish and thus the optimal solutions share equivalent forms. Details are provided in Appendix B.

The numerical experiments of both methods have also shown that they share equivalent optimal solutions. By running both the multi-objective and mean variance optimization on the same example portfolio, one can produce the efficient frontiers that show investors the set of all possible optimized portfolio points on a risk-return tradeoff curve. Therefore, efficient frontiers are the graphical representations of all optimal solutions for investors of various risk appetite. Figure 3 shows that the efficient frontiers generated from both the multi-objective and mean variance methods share the same optimal solutions. Since the lower part of the efficient frontier (the part below the minimum variance portfolio point) gives higher risk and lower return, investors disregard the lower part of efficient frontier.

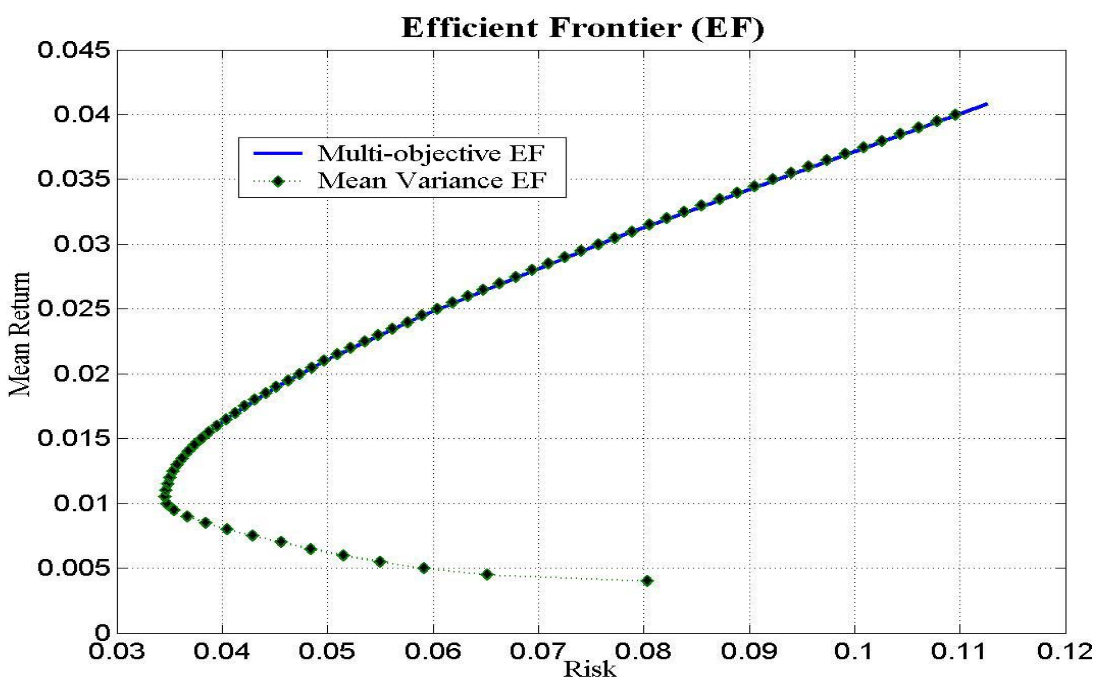

Figure 5 Efficient Frontiers of Multi-objective and Mean variance optimization

The main difference between the mean variance and multi-objective approach is their problem formulations. The multi-objective approach puts two optimization objectives (minimizing risk and maximizing expected return) into one objective function where as the mean variance approach has only one objective of minimizing risk. The mean variance method places the expected value as a constraint in the formulation, which forces the optimization model to provide the minimal risk for each specified level of expected return. There are two comparative advantages for the multi-objective formulation over the mean variance formulation. First, since the mean variance approach assumes that the investor's sole objective is to minimize risk, it may not be a good fit for investors who are extremely risk seeking. The multi-objective formulation is applicable for investors of any risk appetite. Second, the mean variance method requires investors to place an expected value constraint, but there are when investors do not want to place any constraints on their investment or do not know what kind of return to expect from his investment. The multi-objective optimization provides the entire picture of optimal risk-return trade off.

Another key difference between these two methods lies in their approach to producing efficient frontiers. The efficient frontier of the multi-objective optimization is determined by the risk aversion index μ because different values of μ determine different values of risk and expected return. The efficient frontier of the mean variance method is produced by varying the proportion of two optimal portfolios because the Two Fund Separation Theorem guarantees that any optimized portfolio can be duplicated by a combination of two optimal portfolios [6]. In order to have the two optimal portfolios to trace out the entire efficient frontier, one can use the minimum variance portfolio and use it to replicate a secondary portfolio with the given expected return vector $\bar{p}$. In general, the process of producing the efficient frontier using the mean variance method is more cumbersome than the multi-objective approach.

5 Concluding Remarks

The multi-objective optimization is an alternative solution to portfolio optimization problem. Both the mean variance and multi-objective optimization produce the same optimal solutions. The traditional approach to portfolio optimization solves the problem by having one of the optimization objectives in the objective function and fixes the other objective as a constraint. Consequently, investors choose have to pick the optimal solutions based on given expected return or risk. The multi-objective approach satisfies the demand of investors who want to simultaneously optimize risk and return without setting additional constraints. In addition, the multi-objective optimization is applicable to investors of any risk appetite, including those who are extremely risk-seeking or risk-averse. The risk-aversion index measures how much an investor weights risk over expected return. Given any specified value of risk-aversion index, the multi-objective optimization provides investors with optimal asset allocation strategy that can simultaneously maximize expected return and minimize risk.

Appendix A: Derivation of Analytic Solution to Multi-objective optimization

Objective function: Minimize (w.r.t. x) $-p^T \vec{x} + \mu \vec{x}^T V \vec{x}$

Subject to: $\vec{1}^T \vec{x} = 1$

Solution:

Using Lagrangian Multiplier to solve the multi-objective optimization problem:

$$L(\vec{x}) = -\vec{p}^T \vec{x} + \mu \vec{x}^T V \vec{x} + \lambda(\vec{1}^T \vec{x} - 1)$$

Set $\frac{\delta L}{\delta \vec{x}} = 0$. Then we have

$$2\mu V \vec{x} = \vec{p} - \lambda \vec{1}$$

thus $\vec{x} = \frac{1}{2\mu}(V^{-1})(\vec{p} - \lambda \vec{1})$ (1).

To solve the Lagrangian multiplier λ , substitute equation (1) to the constraint $1^T \vec{x} = 1$:

$$\vec{1}^T [\frac{1}{2\mu}(V^{-1})(\vec{p} - \lambda \vec{1})] = 1$$

thus $\frac{1}{2\mu} \vec{1}^T V^{-1} \vec{p} - \frac{\lambda}{2\mu} \vec{1}^T V^{-1} \vec{1} = 1$

thus $\lambda = \frac{\vec{1}^T V^{-1} \vec{p}}{\vec{1}^T V^{-1} \vec{1}} - \frac{2\mu}{\vec{1}^T V^{-1} \vec{1}}$ (2)

Set $a_1 = \vec{1}^T V^{-1} \vec{1}$ and $a_2 = \vec{1}^T V^{-1} \vec{p}$. Both a_1 and a_2 are scalars. Substitute a_1 and a_2 into equation (2):

$$\lambda = \frac{a_2}{a_1} - \frac{2\mu}{a_1}$$

The optimized solution for the asset weight vector:

$$\vec{x}^* = \frac{1}{2\mu} V^{-1} \vec{p} - \frac{V^{-1}}{2\mu} (\frac{a_2}{a_1} - \frac{2\mu}{a_1}) \vec{1}$$

Appendix B: Proof of Equivalent Analytic Solutions for Multi-objective and Mean variance optimization

Lagriangian Equation for Multi-objective optimization:

$$L(\vec{x})^{Multi-Objective} = -\vec{p}^{T}\vec{x} + \mu\vec{x}^{T}V\vec{x} + \lambda(\vec{1}^{T}\vec{x} - 1)$$

Since μ can be assigned to either $-\vec{p}^{T}\vec{x}$ or $\vec{x}^{T}V\vec{x}$,

The Lagrangian Equation for Multi-objective optimization can be rewritten as:

$$L(\vec{x})^{Multi-Objective} = -\mu\vec{p}^{T}\vec{x} + \vec{x}^{T}V\vec{x} + \lambda(\vec{1}^{T}\vec{x} - 1) \quad (1)$$

The Lagriangian Equation for Mean variance optimization:

$$L(\vec{x})^{Mean\ Variance} = \vec{x}^{T}V\vec{x} + \lambda_1(p^{*} - \vec{p}^{T}\vec{x}) + \lambda_2(1 - \vec{1}^{T}\vec{x}) \quad (2)$$

Now compare equation (1) and (2), let $\mu = \lambda_1$ and $\lambda = -\lambda_2$,

Solving for $\frac{\delta L(\vec{x})}{\delta \vec{x}}$=0 for both the Mean variance and Multi-objective Lagrangian Eqs:

(Multi-Objective) $2V\vec{x} - \lambda_1\vec{p} + \lambda_2\vec{1}^{T} = 0 \Rightarrow \vec{x}^{*Multi-Objective} = \frac{1}{2}V^{-1}(\lambda_1\vec{p} - \lambda_2\vec{1}^{T})$

(Mean variance) $2V\vec{x} - \lambda_1\vec{p} + \lambda_2\vec{1}^{T} = 0 \Rightarrow \vec{x}^{*Mean\ Variance} = \frac{1}{2}V^{-1}(\lambda_1\vec{p} - \lambda_2\vec{1}^{T})$

Therefore, $\vec{x}^{*Multi-Objective} = \vec{x}^{*Mean\ Variance}$

Appendix C: Expected Return of Five Selected Assets

Date	IBM	MSFT	AAPL	DGX	BAC
2/1/2007	-0.152%	-0.972%	-1.155%	1.105%	0.494%
1/3/2007	2.055%	3.349%	1.049%	-0.794%	-1.517%
12/1/2006	5.696%	1.703%	-7.441%	-0.320%	-0.854%
11/1/2006	-0.120%	2.621%	13.049%	6.910%	1.013%
10/2/2006	12.673%	4.952%	5.326%	-18.543%	0.547%
9/1/2006	1.193%	6.443%	13.456%	-4.856%	4.083%
8/1/2006	5.025%	7.200%	-0.162%	6.928%	0.971%
7/3/2006	0.763%	3.241%	18.666%	0.486%	7.135%
6/1/2006	-3.856%	2.890%	-4.183%	7.502%	-0.633%
5/1/2006	-2.611%	-5.860%	-15.087%	0.018%	-2.026%
4/3/2006	-0.160%	-11.223%	12.229%	8.855%	9.608%
3/1/2006	2.780%	1.242%	-8.425%	-2.972%	0.410%
2/1/2006	-1.050%	-4.216%	-9.297%	6.948%	3.680%
1/3/2006	-1.101%	7.642%	5.035%	-3.802%	-4.160%
12/1/2005	-7.535%	-5.533%	6.001%	2.780%	0.568%
11/1/2005	8.835%	8.036%	17.764%	7.237%	6.027%
10/3/2005	2.070%	-0.119%	7.424%	-7.420%	3.908%
9/1/2005	-0.493%	-6.020%	14.331%	1.112%	-2.157%
8/1/2005	-3.170%	7.211%	9.941%	-2.638%	-0.147%
7/1/2005	12.471%	3.122%	15.865%	-3.459%	-4.421%
6/1/2005	-1.785%	-3.718%	-7.420%	1.466%	-0.558%
5/2/2005	-0.818%	2.307%	10.261%	-0.766%	2.847%
4/1/2005	-16.418%	4.659%	-13.463%	0.810%	2.126%
3/1/2005	-1.294%	-3.946%	-7.111%	5.776%	-4.548%
2/1/2005	-0.714%	-3.947%	16.671%	4.300%	0.586%
1/3/2005	-5.235%	-1.653%	19.410%	-0.106%	-1.319%
12/1/2004	4.605%	-0.345%	-3.967%	1.929%	2.564%
11/1/2004	5.212%	6.833%	27.977%	7.078%	3.311%
10/1/2004	4.676%	1.159%	35.191%	-0.600%	3.372%
9/1/2004	1.238%	1.300%	12.348%	3.067%	-2.713%
8/2/2004	-2.532%	-3.908%	6.679%	4.290%	5.821%
7/1/2004	-1.227%	-0.241%	-0.615%	-3.216%	0.472%
6/1/2004	-0.488%	8.884%	15.966%	-1.396%	2.776%
5/3/2004	0.679%	0.395%	8.844%	2.151%	3.285%
4/1/2004	-4.001%	4.788%	-4.660%	2.022%	-0.609%
3/1/2004	-4.825%	-6.015%	13.043%	-0.049%	-0.166%

Date	IBM	MSFT	AAPL	DGX	BAC
1/2/2004	7.062%	1.049%	5.519%	16.517%	1.294%
12/1/2003	2.364%	6.429%	2.297%	0.196%	7.730%
11/3/2003	1.378%	-1.625%	-8.654%	7.867%	-0.393%
10/1/2003	1.302%	-5.440%	10.425%	11.542%	-2.959%
9/2/2003	7.705%	4.787%	-8.400%	1.057%	-0.525%
8/1/2003	1.137%	0.437%	7.306%	0.411%	-4.028%
7/1/2003	-1.522%	3.017%	10.598%	-6.320%	4.502%
6/2/2003	-6.284%	4.174%	6.125%	0.678%	7.410%
5/1/2003	3.882%	-3.747%	26.301%	6.064%	0.189%
4/1/2003	8.246%	5.627%	0.566%	0.103%	10.805%
3/3/2003	0.614%	2.143%	-5.859%	13.111%	-2.548%
2/3/2003	-0.120%	0.195%	4.596%	-1.903%	-1.175%
1/2/2003	0.901%	-8.199%	0.279%	-5.468%	0.710%
12/2/2002	-10.835%	-10.397%	-7.613%	1.981%	0.203%
11/1/2002	10.313%	7.882%	-3.487%	-12.600%	0.374%
10/1/2002	35.368%	22.234%	10.759%	3.759%	9.412%
9/3/2002	-22.639%	-10.854%	-1.762%	9.748%	-8.134%
8/1/2002	7.311%	2.268%	-3.277%	-7.184%	5.366%
7/1/2002	-2.223%	-12.278%	-13.883%	-29.805%	-5.480%
6/3/2002	-10.506%	7.461%	-23.948%	-1.569%	-6.433%
5/1/2002	-3.754%	-2.571%	-4.036%	-4.878%	4.597%
4/1/2002	-19.470%	-13.364%	2.534%	10.919%	6.567%
3/1/2002	5.992%	3.374%	9.124%	16.825%	6.342%
Exp. Returns	**0.400%**	**0.513%**	**4.085%**	**1.006%**	**1.236%**

Appendix D: Variance-Covariance Matrix of Five Selected Assets

	IBM	MSFT	AAPL	DGX	BAC
IBM	0.006461	0.002983	0.00235487	0.00235487	0.00096889
MSFT	0.002983	0.0039	0.00095937	-0.0001987	0.00063459
AAPL	0.002355	0.000959	0.01267778	0.00135712	0.00134481
DGX	0.002355	-0.0002	0.00135712	0.00559836	0.00041942
BAC	0.000969	0.000635	0.00134481	0.00041942	0.0016229

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Reference:

[1] Malkiel, Burton G., *A Random Walk Down Wall Street*. W.W.Norton & Company, New York, 2003

[2] Roman, Steven., *Introduction to the Mathematics of Finance: From Risk Management to Options Pricing.* Springer 1 Edition, 2004

[3] Cerbone, Duong., and Noe, Tomayko. *Multi-objective Optimum Design*. Azarm, 1996. http://www.glue.umd.edu/~azarm/optimum_notes/multi/multi.html

[4] Boyd, Stephen., and Vandemberghe, Lieven., *Convex Optimization*. Cambridge University Press, 2003. Material available at www.stanford.edu/~boyd

[5] Hindi, Haitham. *A Tutorial on Convex Optimization.* American Control Conference, 2004. Proceedings of the 2004, Volume 4, Issue , 30 June-2 July 2004 Page(s): 3252 – 3265

[6] Bodie, Zvi., Kane, Alex., and Marcus, Alan J., *Essentials of Investments*. McGraw-Hill/Irwin, 6 Edition, 2005