Intrinsic Knotting of Multipartite Graphs

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Abstract: A graph is intrinsically knotted (IK) if for every embedding of the graph there exists a knotted cycle. Let *G* be a multipartite graph, and form the multipartite graph *G'* by increasing the number of vertices in each of the parts except one and then deleting an edge. We show that if *G* is IK, then the resulting graph *G'* is also IK. We use this idea to describe large families of IK multipartite graphs. In particular we use the fact that $K_{5,5}\backslash$ 2e is IK to show that a bipartite graph with 10 or more vertices (respectively 12 or more vertices) with exactly 5 (resp. 6) in one part and $E(G) \geq 4V(G)-17$ (resp. $E(G) \geq 5V(G)-27$) is IK. Our method can't be improved since we also show that $K_{5,5}\backslash$ 3e is not IK in general.

1. Introduction

We bring together ideas from knot theory and graph theory. When a graph reaches a certain complexity it will contain a knot in every spatial embedding. We have found new ways of characterizing when this must happen and new ways to generate graphs that contain knots in every spatial embedding. To explain, we need to give some definitions. We'll start by explaining graphs.

A graph is a collection of edges and vertices. Edges connect pairs of vertices and any given pair of vertices share either one edge or none. A spatial embedding is a depiction of a graph in three dimensional space($\mathbb{R}^3$). In this depiction vertices are represented by points in $\mathbb{R}^3$ and edges by curves joining pairs of these points. In a spatial embedding edges are not allowed to intersect (except at their ends). Throughout this paper we will use the term embedding to mean spatial embedding. The degree of a vertex is the number of edges incident to that vertex. For instance, in figure 1 *a1* has degree 2 while *b1* has degree 3. Two vertices are adjacent, or neighbors, if they share an edge. For an example of adjacent vertices, consider *a1*,*b2*. Similarly, two edges are adjacent if they share a vertex.

A graph is *n*-partite if the vertices of the graph can be partitioned into *n* disjoint sets (or parts) such that any two vertices in the same part do not share an edge. For example, figure 1 is a bipartite graph (i.e. n=2) with 3 vertices in one part and 2 vertices in the other. We denote it $K_{3,2}$. The symbol *K* refers to the fact that it is complete, that is to say it includes all possible edges. Additionally we may specify that some number of edges have been removed from a graph. For example $K_{3,2}\backslash$ 3e means 3 edges have been removed from the graph $K_{3,2}$. The notation $K_{3,2}\backslash$ 3e represents a set of graphs as there are many ways to remove 3 edges.

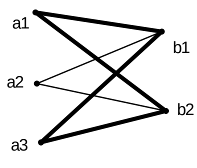

Figure 1

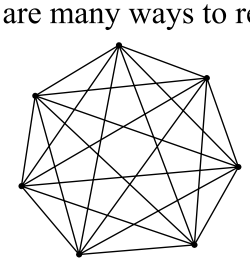

Figure 2

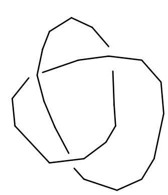

Figure 3

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A cycle is a sequence of adjacent vertices such that the sequence begins and ends with the same vertex and no vertex is included twice other than the vertex which begins and ends the cycle; for example the cycle *(a1, b2, a3, b1, a1)* in figure 1. A cycle is either trivial or knotted. For an example of a knotted cycle, see figure 3. A trivial cycle can be deformed into a circle in the plane. In other words, it bounds a disc. If not, we say the cycle is knotted. A graph is intrinsically knotted (IK) if for every embedding of the graph, there exists at least one knotted cycle.

In 1983 Conway & Gordon [CG] showed that K_7 is intrinsically knotted. Figure 2 is a particular projection of an embedding of K_7, the complete graph on 7 vertices. It is well-known that this graph is the only IK graph on 7 or fewer vertices. It is known that if H is a subgraph of G, and G is not IK, then H is also not IK. Additionally, if H is IK and a subgraph of G, then G must also be IK. More recently, work done by [BBFFML] and [CMOPRW] has completely characterized IK graphs up to 8 vertices. There are 20 IK graphs on 8 vertices.

We know that graphs on n vertices with $5n-14$ edges or more are IK [CMOPRW]. This is a powerful result. For example, although very little is known about IK graphs on 9 or more vertices, this bound immediately tells us that all 45 of the 9 vertex graphs with 31 or more edges are IK. In this paper we use a similar technique to improve the bound and therefore describe a large class of IK graphs. Our argument is based on the fact that $K_{5,5} \setminus 2e$ is IK [CMOPRW] and presented in Section 2. It follows that we cannot generalize this argument since we also show (in Section 3) that $K_{5,5} \setminus 3e$ is in general not IK.

2. Multipartite Graphs with IK Subgraphs

In this section we will prove Theorem 2.1 and then deduce several corollaries. In particular, Corollaries 2.6 and 2.7 give a new sufficient condition for IK bipartite graphs that improves on the $5n-14$ bound of [CMOPRW]. Theorem 2.1 refers to an induced subgraph which is a subgraph formed from a subset of the vertices of a graph G together with any edges whose endpoints are both in this subset.

Theorem 2.1: *A graph of the form $K_{a,(a+1)n} \setminus e$ has $K_{a,(a)n}$ as an induced subgraph.*

By $K_{a,(a+1)n} \setminus e$ we mean a complete (n+1)-partite graph, with one part having a vertices and the remaining n parts having a+1 vertices each, with one edge removed. The edge removed must be taken from either parts with a and a+1 vertices or parts with a+1 and a+1 vertices.

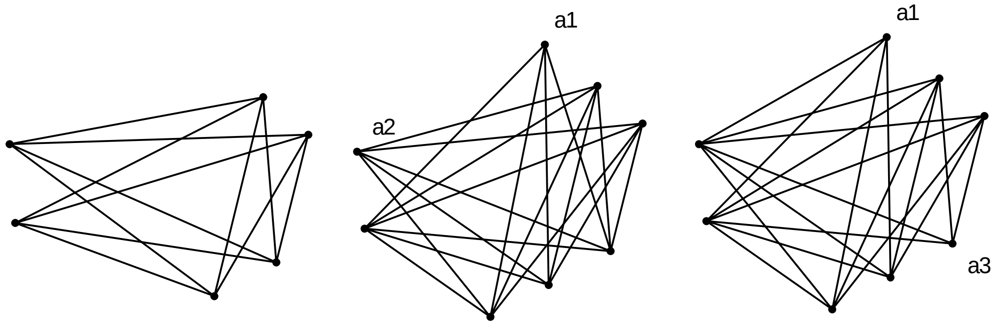
Proof:

Case 1: The edge is removed between parts with a and a+1 vertices. Consider the part with a+1 vertices; by choosing the a vertices with no edge removed in this part, and any a vertices in the other parts, we will find an induced $K_{a,(a)n}$.

Case 2: The edge is between parts with a+1 vertices. Again we can choose a vertices from those two parts and every other part to form $K_{a,(a)n}$ as an induced subgraph.

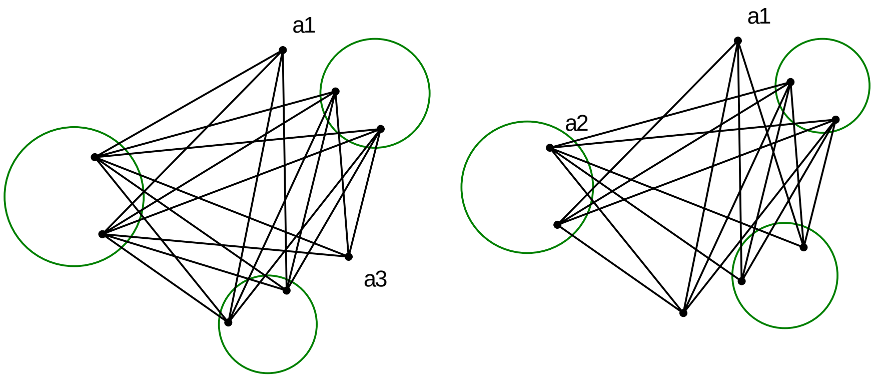
Therefore $K_{a,(a+1)n} \setminus e$ has $K_{a,(a)n}$ as an induced subgraph.

We offer the following specific example of this process with a=2 and n=2. Here $K_{a, (a+1)n} \setminus e$ is either of the form $G \setminus (a1\text{-}a2)$ or $G \setminus (a1\text{-}a3)$ where $G = K_{3,3,2}$. We will argue that $K_{2,2,2}$ is contained in both $G \setminus (a1\text{-}a2)$ and $G \setminus (a1\text{-}a3)$.



K2,2,2 G \ a1-a2 G \ a1-a3

In identifying our induced subgraph we need to pick groups of vertices that correspond to a $K_{2,2,2}$ graph.



G \ a1-a3 G \ a1-a2

In each case we see that the missing edge can be avoided and thus we can find a $K_{2,2,2}$ induced subgraph, as desired.

Corollary 2.2: *If $K_{a, (a+1)n} \setminus e$ is not IK, then $K_{a, (a)n}$ is not IK*

Corollary 2.3: *If $K_{a, (a)n}$ is IK, then $K_{a, (a+1)n} \setminus e$ is IK.*

We can use the proof of the theorem to describe an infinite family of IK graphs.

Corollary 2.4: *A graph of the form $K_{a,(a+n)k} \setminus (n+a-2)e$ is IK for $a \geq 3$, $n \geq 0$, and $k \geq 2$.*

Proof (by Induction on n):

Let n=0 and $a \geq 3$, $k \geq 2$.
A graph of the form $K_{a,a,a,\ldots,a} \setminus (a-2)e$ always has a $K_{3,3,3} \setminus e$ induced subgraph. Since $K_{3,3,3} \setminus e$ is IK [CMOPRW], $K_{a,(a+n)k} \setminus (n+a-2)e$ is also IK for n=0.

Assume that every graph of the form $K_{a,(a+n)k} \setminus (n+a-2)e$ is IK, with $a \geq 3$, $n \geq 0$, $k \geq 2$. For the induction step, assume n is increased by 1 to form a graph $G` = K_{a,(a+n+1)k} \setminus (n+a-1)$. Compared to

G, the number of edges removed from $G`$ is also increased by 1, from $n+a$-2 to $n+a$-1. We view $G`$ as formed from G by adding vertices and then removing an edge. The extra edge that is removed is either removed between parts with a and $a+(n+1)$ vertices or between parts with $a+(n+1)$ and $a+(n+1)$ vertices.

Case 1: The edge is removed between parts with a and $a+(n+1)$ vertices. We still have $a+n$ choices of vertices from the $a+(n+1)$ part so that we can choose our induced subgraph and avoid including the edge that has been removed. Therefore we will have a $K_{a,(a+n)k}\backslash(n+a-2)e$ induced subgraph which is IK (by inductive hypothesis).

Case 2: The edge is removed between two $a+(n+1)$ parts. Again there are still $a+n$ choices of vertices from which we can choose our induced subgraph and still avoid including that edge. Therefore we will have a $K_{a,(a+n)k}\backslash(n+a-2)e$ induced subgraph which is IK (by inductive hypothesis).

Therefore when every $K_{a,(a+n)k}\backslash(n+a-2)e$ is IK, then every $K_{a,(a+(n+1))k}\backslash((n+1)+a-2)e$ is also IK. By induction, $K_{a,(a+n)k}\backslash(n+a-2)e$ is IK.

We can use the theorem to improve the sufficient condition for IK of [CMOPRW]: $E(G) \geq 5V(G)-14$, where $E(G)$ is the number of edges and $V(G)$ is the number of vertices of the graph G.

Corollary 2.5: *A graph of the form* $K_{a,\, a+n}\backslash\ (a+n-3)$ *edges is IK for* $a \geq 5,\ n \geq 0$.

We omit the proof, since it is similar in nature to Corollary 2.4, using the graph $K_{5,\,5}\backslash$ 2e to start the induction. Note that $K_{5,\,5}\backslash$ 2e is IK [CMOPRW].

Corollary 2.6: *A bipartite graph with 5 vertices in one part, at least 5 vertices in the other part and* $E(G) \geq 4V(G)-17$ *is IK.*

Proof:

If $a=5$, $K_{5,\,n+5}\backslash\ (5+n-3)$e = $K_{5,\,n+5}\backslash\ (n+2)$e has $5n+25-(n+2) = 4n+23 = 4(n+10)-17$ edges and $5+(n+5) = n+10$ vertices.

Therefore, if $E(G) \geq 4V(G)-17$, the graph has an induced $K_{5,\,n+5}\backslash\ (n+2)$e subgraph and is IK, using Corollary 2.5.

Corollary 2.7: *A bipartite graph with 6 vertices in one part, at least 6 vertices in the other part, and* $E(G) \geq 5V(G)-27$ *is IK.*

Proof:

If $a=6$, $K_{6,\,n+6}\backslash\ (6+n-3)$e = $K_{6,\,n+6}\backslash\ (n+3)$e has $6n+36-(n+3) = 5n+33 = 5(n+12)-27$edges and $6+(n+6) = n+12$ vertices.

Therefore, if $E(G) \geq 5V(G)-27$, the graph has an induced $K_{6,\,n+6}\backslash\ (n+3)$e subgraph and is IK, using Corollary 2.5.

It follows from [CMOPRW] that a bipartite graph with 4 or fewer vertices in one part is not IK. If a bipartite graph has 7 or more vertices in both parts, we do get a bound of the form given in Corollary 2.5 and 2.6, but it is weaker than the bound $E(G) \geq 5V(G)-14$.

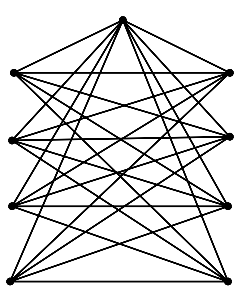
3. Not an IK Graph

Corollary 2.5 is based on showing graphs of form $K_{a,\ a+n}$\ *(a+n-3)* edges have a $K_{5,5}$\ 2e IK subgraph. Here we show that this method cannot be improved because, in general, $K_{5,5}$\ 3e is not IK.

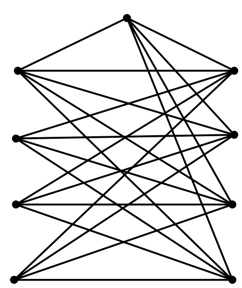
Following [CMOPRW], we denote the vertices in $K_{l,m,n}$ by *{a1, a2,…,al}*, *{b1, b2,…,bm}*, *{c1, c2,…,cn}*. Thus, $K_{5,5}$ \ *{a5-b1, a4-b1, a3-b1}* denotes $K_{5,5}$ with 3 edges removed all of them incident to the same vertex *b1* in the second part.

<u>Theorem 3.1:</u> *$K_{5,5}$ \ {a5-b1, a4-b1, a3-b1} is not IK.*

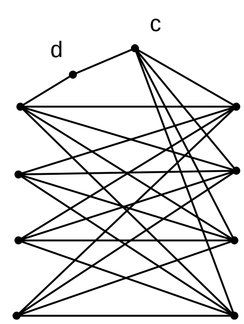
<u>Proof:</u> Let *G* = $K_{4,4,1}$ \ *{a5-c, a4-c}*. It is known that *G* is not IK. [CMOPRW]
Let *H* be $K_{4,4,1}$ \ *{a5-c, a4-c, a3-c}*. *H* is also not IK, since *H* is a subgraph of *G*.
In a knotless embedding of *H*, add a vertex *d* on the edge *a2-c* and label this graph *H`*. As is clear from the figure below, the graph is bipartite with partitions *{d, b2, b3, b4, b5}* and *{c, a2, a3, a4, a5}*.
Therefore $K_{5,5}$ \ *{a5-b1, a4-b1, a3-b1}* has at least one knotless embedding.



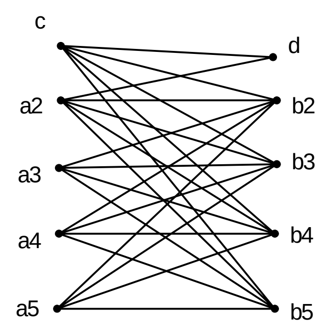
G = K4,4,1



H = K4,4,1 \ 3e



H` = K5,5 \ 3e



H` = K5,5 \ 3e
(Another view)

<u>4. References</u>

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