Intrinsic Knotting of Partite Graphs

Chloe Collins, Ryan Hake, Cara Petonic, Laura Sardagna

Abstract: A graph is intrinsically knotted (IK) if for every embedding of the graph there exists a knotted cycle. Let G be a partite graph, and form the partite graph G’ by increasing the number of vertices in each of the parts except one and then deleting an edge. We show that if G is IK, then the resulting graph G’ is also IK. We use this idea to produce new examples of IK graphs. In particular we use the fact that $K_{5,5}\setminus 2e$ is IK to show that for a bipartite graph with 5 (resp. 6) vertices in one part and $E(G) \geq 4V(G)-17$ (resp. $E(G) \geq 5V(G)-27$) is IK. Our method can’t be improved since we also show that $K_{5,5}\setminus 3e$ is not IK in general.

1. Introduction

We bring together ideas from knot theory and graph theory. When a graph reaches a certain complexity it will contain a knot. We have found new ways of characterizing when this must happen and new ways to generate graphs that contain knots. To explain, we need to give some definitions. We’ll start by explaining graphs.

A graph is a collection of edges and vertices. Edges connect pairs of vertices and any given pair of vertices share either one edge or none. Any particular depiction of a graph in space (with the vertices represented as points and the edges as curves connecting those points) is referred to as an embedding. The degree of a vertex is the number of edges incident to that vertex. For instance, in figure 1 A1 has degree 2 while B1 has degree 3. Two vertices are adjacent, or neighbors, if they share an edge. For an example of adjacent vertices, consider A1,B2. Similarly, two edges are adjacent if they share a vertex.

A graph is n-partite if the vertices of the graph can be partitioned into n disjoint sets (or parts) such that any two vertices in the same part do not share an edge. For example, figure 1 is a bipartite graph (i.e. n=2) with 3 vertices in one part and 2 vertices in the other. We denote it $K_{3,2}$. K refers to the fact that it’s complete. It includes all possible edges between any two vertices taken from different parts. Additionally we may specify that some number of edges have been removed from a graph. For example $K_{3,2}\setminus 3e$ means 3 edges have been removed from the graph $K_{3,2}$. The notation $K_{3,2}\setminus 3e$ represents a set of graphs as there are many ways to remove 3 edges.

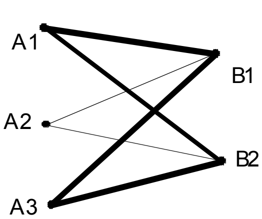


Figure 1

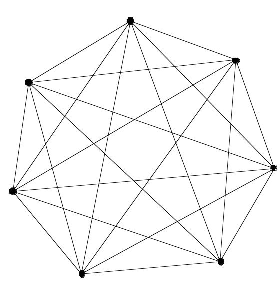


Figure 2

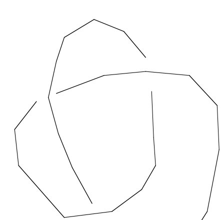


Figure 3

Funded by NSF REU Award 0354174 and supported by the MAA's NREUP program with funding from the NSF, NSA, and Moody's.

A cycle is a sequence of adjacent vertices such that the sequence begins and ends with the same vertex. For example the cycle {A1,B2,A3,B1,A1} in figure 1. A cycle is either trivial or knotted. For an example of a knotted cycle, see figure 3. A trivial cycle can be deformed into a circle in the plane. In other words, it bounds a disc. If not, we say the cycle is knotted. A graph is intrinsically knotted(IK) if for every embedding of the graph, there exists at least one knotted cycle.

In 1983 Conway & Gordon [CG] showed that K_7 is intrinsically knotted. Figure 2 is a particular embedding of K_7, the complete graph on 7 vertices. This graph is the simplest example of an IK graph. In other words, K_7 is the only IK graph on 7 or fewer vertices. It is known that if H is a subgraph of G, and G is not IK, then H is also not IK. Additionally, if H is IK and a subgraph of G, then G must also be IK. More recently, work done by [BBFFML] and [CMOPRW] has completely characterized IK graphs up to 8 vertices. There are 20 IK graphs on 8 vertices.

In our work, we have improved upon a 5n-14 bound on bipartite graphs. The bound 5n-14 tells us that graphs on n vertices with 5n-14 edges or more are IK [CMOPRW]. This is a powerful result. For example, although very little is known about graphs on 9 or more vertices, this bound immediately tells us that all 45 of the 9 vertex graphs with 31 or more edges are IK. In this paper we use a similar technique to improve the bound and therefore determine a new, large class of IK graphs. Our argument is based on the fact that $K_{5,5} \setminus 2e$ is IK [CMOPRW] and presented in Section 2. It follows that we cannot generalize this argument since we also show (in Section 3) that $K_{5,5} \setminus 3e$ is in general not IK.

2. Partite Graphs with IK Subgraphs

In this section we will prove Theorem 2.1 and then deduce several corollaries. In particular, Corollaries 2.6 and 2.7 give a new sufficient condition for IK bipartite graphs that improves on the 5n-14 bound of [CMOPRW].

Theorem 2.1: *A graph of the form $K_{a,(a+1)n} \setminus e$ has $K_{a,(a)n}$ as a subgraph.*

By $K_{a,(a+1)n} \setminus e$ we mean an (n+1)-partite graph, with one part having a vertices and the remaining n parts having a+1 vertices each, with one edge removed.

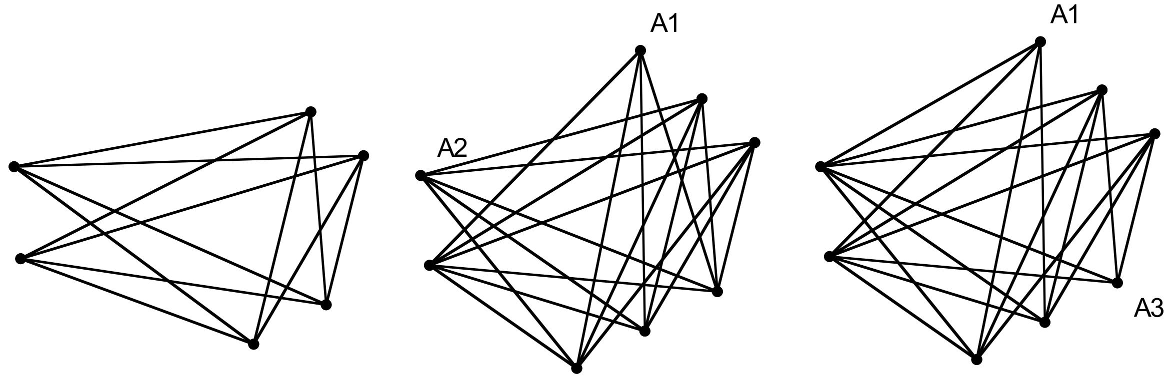
Proof: We will think of $K_{a,(a+1)n} \setminus e$ as obtained from $K_{a,(a)n}$ by adding a vertex to all but one part and then removing an edge. The edge removed must be taken from either parts with a and a+1 vertices (1) or parts with a+1 and a+1 vertices (2).

Case 1: The edge is removed between parts with a and a+1 vertices. By choosing the a vertices with no edge removed in the a+1 part in question and any a vertices in the other parts, we construct $K_{a,(a)n}$.

Case 2: The edge is between a+1 and a+1. Again we can choose a vertices from those two parts and every other part to form $K_{a,(a)n}$ as a subgraph.

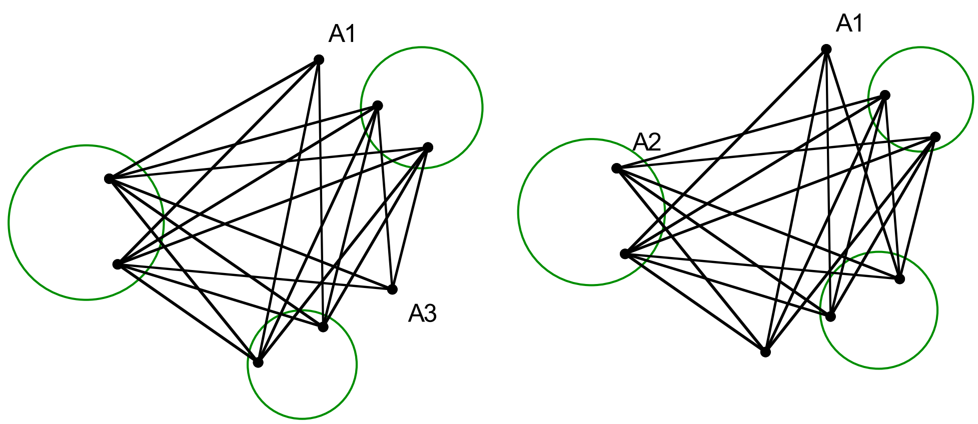
Therefore $K_{a,(a+1)n} \setminus e$ has a $K_{a,(a)n}$ as a subgraph. ▯

As an example of this process, we offer the following specific example, with a=2, n=2. Here $K_{a,(a+1)n}\backslash$ e is either of the form G \(A1-A2) or G \(A1-A3) where G = $K_{3,3,2}$. We will argue that $K_{2,2,2}$ is contained in both G \ (A1-A2) and G \ (A1-A3).



K2,2,2 G \ A1-A2 G \ A1-A3

In identifying our subgraph we need to pick groups of vertices that correspond to a $K_{2,2,2}$ graph.



G \ A1-A3 G \ A1-A2

In each case we see that the missing edge can be avoided and thus we can find a $K_{2,2,2}$ subgraph, as desired.

<u>Corollary 2.2:</u> *If $K_{a,(a+1)n}\backslash$ e is not IK, then $K_{a,(a)n}$ is not IK*

<u>Corollary 2.3:</u> *If $K_{a,(a)n}$ is IK, then $K_{a,(a+1)n}\backslash$ e is IK.*

We can use the proof of the Theorem to create an infinite family of graphs by beginning with a graph that is IK.

<u>Corollary 2.4</u>: *A graph of the form $K_{a,(a+n)k}\backslash(n+a-2)e$ is intrinsically knotted for $a\geq 3$, $n\geq 0$, and $k\geq 2$.*

<u>Proof (by Induction on n):</u>

Let n=0 and $a\geq 3$, $k\geq 2$.

A graph of the form $K_{a,a,a,\ldots,a}\backslash(a-2)e$ always has a $K_{3,3,3}\backslash e$ subgraph. Since $K_{3,3,3}\backslash e$ is intrinsically knotted [CMOPRW], $K_{a,(a+n)k}\backslash(n+a-2)e$ is also intrinsically knotted for n=0.

Assume that every graph of the form $K_{a,(a+n)k}\backslash(n+a-2)e$ is intrinsically knotted, with a≥3, n≥0, k≥2. For the induction step, assume n is increased by 1 to form a graph G\`= $K_{a,(a+n+1)k}\backslash(n+a-1)$. Compared to G, the number of edges removed from G\` is also increased by 1, from n+a-2 to n+a-1. We view G\` as formed from G by adding vertices and then removing an edge. The extra edge that is removed is either removed between parts with a and a+(n+1) vertices or between parts with a+(n+1) and a+(n+1) vertices.

Case 1: The edge is removed between parts with a and a+(n+1). We still have a+n choices of vertices from the a+(n+1) part so that we can choose our subgraph and avoid including the edge that has been removed. Therefore we will have a $K_{a,(a+n)k}\backslash(n+a-2)e$ subgraph which is intrinsically knotted(by inductive hypothesis).

Case 2: The edge is removed between two a+(n+1) parts. Again there are still a+n choices of vertices from which we can choose our subgraph and still avoid including that edge. Therefore we will have a $K_{a,(a+n)k}\backslash(n+a-2)e$ subgraph which is intrinsically knotted(by inductive hypothesis).

Therefore when every $K_{a,(a+n)k}\backslash(n+a-2)e$ is intrinsically knotted, then every $K_{a,(a+(n+1))k}\backslash((n+1)+a-2)e$ is also intrinsically knotted. By induction, $K_{a,(a+n)k}\backslash(n+a-2)e$ is IK.□

We can use the theorem to improve the sufficient condition for IK of [CMOPRW]:
$E(G) \geq 5V(G)-14$, where E(G)= the number of edges and V(G) is the number of vertices of the graph G.

Corollary 2.5: *A graph of the form $K_{a,\, a+n}\backslash$ (a+n-3) edges is IK for a≥5, n≥0.*

We omit proof, since it is similar in nature to Corollary 2.4, using the graph $K_{5,\,5}\backslash 2e$ to start the induction. Note that $K_{5,\,5}\backslash 2e$ is IK [CMOPRW].

Corollary 2.6: *A bipartite graph with 5 vertices in one part and $E(G) \geq 4V(G)-17$ is IK.*
Proof:
Using corollary 2.5, if a= 5, $K_{5,\, n+5}\backslash (5+n-3)e = K_{5,\, n+5}\backslash (n+2)e$ has 5n+25-(n+2) = 4n+23 = 4(n+10)-17 edges and 5+(n+5) = n+10 vertices.
Therefore, if $E(G) \geq 4V(G)-17$, the graph has a $K_{5,\, n+5}\backslash (n+2)e$ minor and is IK. □

Corollary 2.7: *A bipartite graph with 6 vertices in one part and $E(G) \geq 5V(G) -27$ is IK.*
Proof:
Using corollary 2.5, if a=6, $K_{6,\, n+6}\backslash (6+n-3)e = K_{6,\, n+6}\backslash (n+3)e$ has 6n+36-(n+3) = 5n+33 = 5(n+12)-27edges and 6+(n+6) = n+12 vertices.
Therefore, if $E(G) \geq 5V(G) -27$, the graph has a $K_{6,\, n+6}\backslash (n+3)e$ minor and is IK. □

It follows from [CMOPRW] that a bipartite graph with 4 or fewer vertices in one part is not IK. If there is a part of 7 or more vertices, we do get a bound of the form given in Corollary 2.5 and 2.6, but it is weaker than the bound $E(G) \geq 5V(G)-14$.

3. Not Intrinsically Knotted Graph

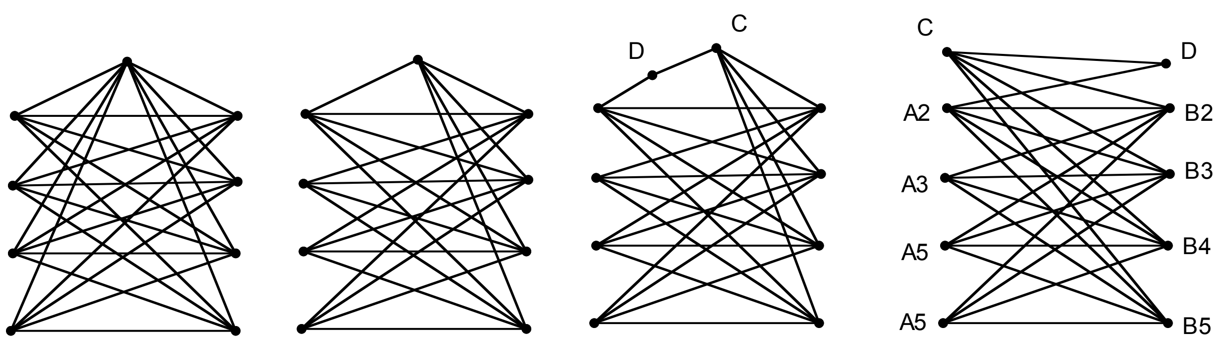
Theorem 2.1 is based on showing graphs of form $K_{a, a+n}$\ (a+n-3) edges have a $K_{5,5}$\ 2e IK subgraph. Here we show that this method cannot be improved because, in general, $K_{5,5}$\ 3e is not IK.

Following [CMOPRW], we denote the vertices in $K_{l,m,n}$ by {A1, A2,…,AL}, {B1, B2,…,BM}, {C1, C2,…,CN}. Thus, $K_{5,5}$ \ {A5-B1, A4-B1, A3-B1} denotes $K_{5,5}$ with 3 edges removed all of them incident to the same vertex B1 in the second part.

Theorem 3.1: *$K_{5,5}$ \ {A5-B1, A4-B1, A3-B1} is not IK.*

Proof: Let G = $K_{4,4,1}$ \ {A5-C, A4-C}. It is known that G is not IK. [CMOPRW]
Let H be $K_{4,4,1}$ \ {A5-C, A4-C, A3-C}. H is also not IK, since H is a subgraph of G.
In a knotless embedding of H, add a vertex D on the edge A2-C and label this graph H`. The graph is now bipartite as we will show. In adding D, the vertex C is no longer connected by an edge to any vertex in the A part. C is connected to {B2, B3, B4, B5, D}. Thus, we can consider C to be a vertex in the A part. D is connected to {A2, C}. Therefore we can consider D to be a vertex in the B part. But then we have 10 vertices split into 2 sets of 5, so H` is a bipartite graph. Since deg(C)=5,deg(D)=2,deg(A3)= deg(A4)= deg(A5)= 4 and deg(A2)= deg(B2)= deg(B3)= deg(B4)= deg(B5)= 5, we have a bipartite graph on 10 vertices. This graph is precisely $K_{5,5}$ \ 3e with a vertex D of degree two.

Therefore $K_{5,5}$ \ {A5-B1, A4-B1, A3-B1} has at least one knotless embedding. ▯



K4,4,1 H = K4,4,1 \ 3e H`= K5,5 \ 3e H` = K5,5 \ 3e (Another view)

4. References

[BBFFHL] P. Blain, G. Bowlin, T. Fleming, J. Foisy, J. Hendricks, and J. LaCombe, 'Some Results on IK Graphs', (preprint).

[CG] J.H. Conway and C. McA. Gordon, 'Knots and Links in Spatial Graphs', *Journal of Graph Theory*, **Vol 7,** (1983), 445-453.

[CMOPRW] J. Campbell, T. Mattman, R. Ottman, J. Pyzer, M. Rodrigues, and S. Williams, 'Intrinsic Knotting and Linking of Almost Complete Graphs' (preprint available at www.arXiv.org).