On Pólya's Orchard Problem

A. Hening* and M. Kelly†

Abstract

In 1918 Pólya formulated the following problem: *"How thick must the trunks of the trees in a regularly spaced circular orchard grow if they are to block completely the view from the center?"*(Pólya and Szegő [2]). We study a more general orchard model, namely any domain that is compact and convex, and find an expression for the minimal radius of the trees. As examples, solutions for rhombus-shaped and circular orchards are given. Finally, we give some estimates for the minimal radius of the trees if we see the orchard as being 3-dimensional.

1 Introduction

Let $\Lambda := \mathbb{Z}^2 \backslash O$ where $O := (0,0)$ is the origin of the $\mathbb{R}^2$ plane. A tree will be represented by a closed disk centered at some point $P \in \Lambda$. We will assume all of the disks have the same radius r. Let D be a compact, convex domain in $\mathbb{R}^2$ such that $O \in D$. One can see the boundary ∂D as being the fence surrounding the orchard D. Suppose there are disks at all the integer points which lie inside the orchard $D \backslash O$, or in other words, at all points from $D' := D \cap \Lambda$. A point $P = (\xi, \eta) \in \mathbb{R}^2 \backslash int(D)$ is said to be visible if the ray from O through P does not intersect any disk (where $int(D)$ is the interior of D).
The problem is to find the minimal radius ρ of the trees such that no point of ∂D is visible. G. Pólya ([2] , [3]) and R. Honsberger([4]) found the following estimates for ρ when D is a disk of radius $R \in \mathbb{N}$:

$$\frac{1}{\sqrt{(R^2+1)}} \leq \rho \leq \frac{1}{R} \tag{1}$$

In [1] T.T. Allen solves the orchard problem for disks of arbitrary real radius. He shows that $\rho = \frac{1}{d}$, where d is the distance from O to the closest point $P \in \Lambda \backslash D'$ which has coprime coordinates. In the following we give a different proof to Pólya's problem and generalize Allen's result to orchards D satisfying the following two conditions:

(i) $D \subset \mathbb{R}^2$ is a compact, convex domain.

(ii) The consecutive rays that pass through integer points of D form acute angles.

*International University Bremen, Germany(a.hening@iu-bremen.de)
†Oklahoma State University(mbk181@aol.com)

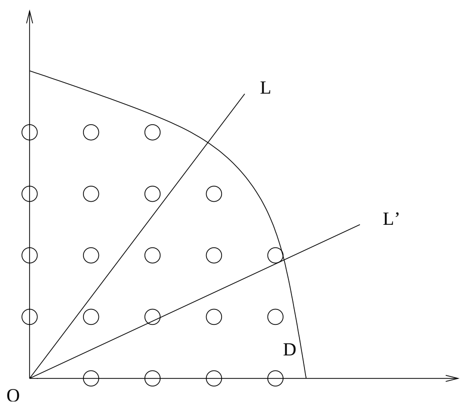

Figure 1: Example of an orchard. Only the first quadrant is exhibited.

2 The Orchard Problem

Theorem 2.1. *Let D be a domain satisfying conditions (i) and (ii) above. The minimal radius ρ of the disks from $D \cap \Lambda$ such that no part of ∂D is visible is $\rho = \frac{1}{d}$ where d is the distance from O to the closest lattice point which lies outside of D and has coprime coordinates.*

Proof. Without loss of generality we restrict our reasoning to the first quadrant. We will make use of the following well-known result:

Theorem 2.2. *(Pick's Theorem) Let F be a polygon whose vertices are in $\mathbb{Z}^2$. Let b be the number of integer points that are on ∂F and let i be the number of integer points that are in $int(F)$. Then:*

$$area(F) = i + \frac{b}{2} - 1 \tag{2}$$

We break up the proof of Theorem 2.1 into the two following propositions:

Proposition 2.3. *The minimal radius satisfies the inequality $\rho \leq \frac{1}{d}$.*

Proof. Take any ray OC. Let $A, B \in D'$ be the two integer points from D which are closest to OC and which lie on different sides with respect to OC (Figure 2).

The distances from the two points to the ray will be $d(A, OC)$ and $d(B, OC)$. Now, if $d(A, OC) > d(B, OC)$ then by increasing continuously the radii of the disks from 0, the disk from B will be the first to hit the ray OC. Rotate OC around O so that C becomes C', a position at which $d(A, OC') = d(B, OC')$. This process can only increase the minimal radius ρ of the disks. Consider without loss of generality that $d(A, OC) = d(B, OC) = h$. If we look at the points A and B as being vectors in $\mathbb{R}^2$ we can define their sum: $O' := A + B \in \Lambda$. Notice that O' lies on the ray OC. If $O' \in D$ then the ray OC is blocked by the disk from O' or if OC hits any disk from D' then $\rho = 0$ will do and we are done. Consider that $O' \notin D'$ and that the ray OC does not hit any disk from D'. By assumption:

$$OO' \geq d \tag{3}$$

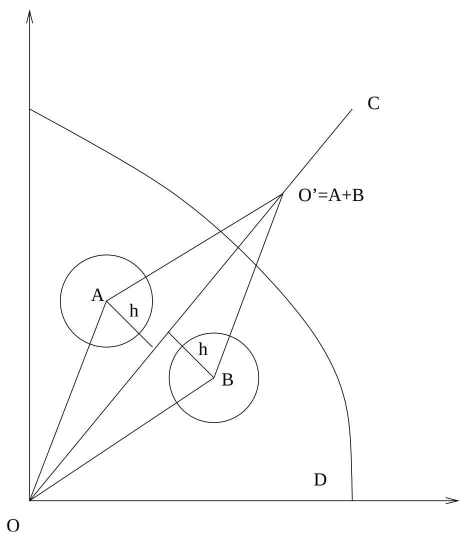

Figure 2: A and B are the closest lattice points to the ray OC.

By the construction of $OAO'B$ we have $OAO'B \cap \mathbb{Z}^2 = \{O, A, O', B\}$.
Pick's Theorem 2.2 thus yields:

$$area(OAO'B) = 0 + \frac{4}{2} - 1 = 1 \tag{4}$$

On the other hand, because $OAO'B$ is a parallelogram the area can be computed by the formula:

$$area(OAO'B) = h \cdot OO' \tag{5}$$

Suppose that $h > \frac{1}{d}$. Then by using the last two equations: $h \cdot OO' > \frac{OO'}{d} \Rightarrow d > OO'$ which contradicts equation (3).
Thus $h \leq \frac{1}{d}$ and since we want the trees from A, B to hit OC, we see that $h = r$, where r is the radius of the trees, will do.
In summation, if $r = \frac{1}{d}$ then any ray will hit one of the trees before it hits ∂D. This forces $\rho \leq \frac{1}{d}$.

□

Proposition 2.4. *The minimal radius satisfies the inequality $\rho \geq \frac{1}{d}$.*

Proof. It is enough to show that if the radius r of the disks is less than $\frac{1}{d}$, then there is a ray which does not hit any disk before it hits the boundary ∂D.
Let $P_* \in \Lambda \backslash D'$ be the lattice point that is closest to O and that has coprime coordinates. If $P_* := (\xi, \eta)$, $d(P_*, O) = d$ and $gcd(\xi, \eta) = 1$ then OP_* will not hit any points from D'. Note that $\xi \neq 0$ because D satisfies condition (ii) above. The line through O and P_* is given by $y = \frac{\eta}{\xi} \cdot x$ so $(a, b) \in OP_* \cap D'$ if and only if there exists $k \in \mathbb{N}$ such that $k \cdot (a, b) = (\xi, \eta)$. This means that $k \mid gcd(\xi, \eta)$ so $k = 1$.
Let $A \in D'$ be the integer point closest to OP_*. Suppose again that P_* and A are vectors in $\mathbb{R}^2$ and define $B := P_* - A$. OAP_*B will be a parallelogram so we can denote both distances from A, B to OP_* by h.

By Pick's Theorem 2.2 we have $area(OAP_*B) \geq 1$ and by standard planar geometry $area(OAP_*B) = OP_* \cdot h$. But $OP_* = d$ so:

$$d \cdot h \geq 1 \tag{6}$$

As a result $h \geq \frac{1}{d}$, so the ray OP_* does not hit any disk from D' if $r < \frac{1}{d}$. For a disk to hit this ray it is necessary that $r \geq \frac{1}{d} \Rightarrow \rho \geq \frac{1}{d}$.

□

By combining Propositions 2.3 and 2.4 we get the desired result $\rho = \frac{1}{d}$, thus completing the proof of Theorem 2.1.

□

3 Some Number Theory

After having proven Theorem 2.1 it is natural to ask the following question: When can a natural number n be written as the sum of squares of two coprime integers? This would be of help if one would like to compute d numerically for complicated domains D. We will give a complete classification of all $n \in \mathbb{N}$ which can be written in this way. It is helpful to study this problem in an extension of the ring $\mathbb{Z}$, namely in the ring of Gaussian integers $\mathbb{Z}[i]$.

Definition 3.1. *$\mathbb{Z}[i] := \{a + bi \,|\, a, b \in \mathbb{Z}\}$ is the ring of Gaussian integers. The norm $N : \mathbb{Z}[i] \to \mathbb{Z}_+$ of a Gaussian integer $a + bi$ is defined to be $N(a + bi) := a^2 + b^2$.*

Elements of the ring $\mathbb{Z}[i]$ will be called Gaussian integers while numbers from $\mathbb{Z}$ will be called rational integers. First, we need to know how rational and Gaussian primes relate to one another.

Theorem 3.2. *If $p \in \mathbb{N}$ is a rational prime, then p factors as a Gaussian integer according to the following:*

a. If $p = 2$, then $p = -i(1 + i)^2 = i(1 - i)^2$ where $1 + i, 1 - i$ are associate Gaussian primes and $N(1 + i) = N(1 - i) = 2$.

b. If $p \equiv 3 \,(mod\, 4)$, then $p = \pi$ is a Gaussian prime with $N(\pi) = p^2$.

c. If $p \equiv 1 \,(mod\, 4)$, then $p = \pi\overline{\pi}$ where $\pi, \overline{\pi}$ are Gaussian primes that are not associate and $\overline{\pi}$ is the complex conjugate of π.

Second, we are interested as to when a rational integer can be written as the sum of two squares.

Theorem 3.3. *A rational integer $n \in \mathbb{N}$ can be written as the sum of two squares if and only if the prime factorization of n is of the following form: $n = 2^m p_1^{e_1} p_2^{e_2} ... p_s^{e_s} q_1^{2 \cdot f_1} q_2^{2 \cdot f_2} ... q_l^{2 \cdot f_l}$ where $m, e_1, e_2, ...e_s, f_1, f_2, ...f_l \in \mathbb{Z}_+$, $p_1, p_2 ... p_s$ are odd rational primes congruent to 1 modulo 4 and $q_1, q_2, ...q_l$ are rational primes congruent to 3 modulo 4.*

For the proofs of theorems 3.2 and 3.3 one can consult any number theory book(for example [5]).
We are now ready to state and prove the main result of this section:

Theorem 3.4. *Let $n \in \mathbb{N}$. Then there exist natural numbers x, y with $gcd(x,y) = 1$ such that $n = x^2 + y^2$ if and only if n's prime factorization is of the form $n = 2^m p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$ where $m \in \{0,1\}$, $e_1, e_2, \dots, e_s \in \mathbb{Z}_+$ and $p_1, p_2, \dots, p_s$ are odd rational primes congruent to 1 modulo 4.*

Proof. If $n = x^2 + y^2 \equiv 0 \pmod 4$ then since squares mod 4 are 0 or 1, one has $x^2 \equiv y^2 \equiv 0 \pmod 4$ which implies $x \equiv y \equiv 0 \pmod 2$ yielding $gcd(x,y) \neq 1$. Thus those n which are divisible by 4 cannot be written in the desired way.
Note that $n = x^2 + y^2$ for $x, y \in \mathbb{N}$ if and only if $n = (x+iy)(x-iy)$ for $x, y \in \mathbb{N}$. We want $gcd(x,y) = 1$ so for all rational primes $p \,|\, n$ we must have $p \nmid (x+iy)$. By theorems (3.2) and (3.3) we observe that if a rational integer $n \in \mathbb{N}$ can be written as the sum of two squares then its decomposition into Gaussian primes is of the form: $n = \pi_1 \overline{\pi}_1 \pi_2 \overline{\pi}_2 \dots \pi_m \overline{\pi}_m$ for $m \in \mathbb{N}$ and $\pi_1, \overline{\pi}_1, \dots, \pi_m, \overline{\pi}_m$ Gaussian primes, not all necessarily distinct. Then

$$x + iy = \prod_{i \in I} \pi_i \cdot \prod_{j \in J} \overline{\pi}_j \tag{7}$$

for some sets $I, J \subset \{1, \dots, m\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, m\}$ (a partition of the set$\{1, \dots, m\}$). Thus in order for n to be written as the sum of two squares which are coprime it is enough to see if there exist sets I, J such that for any prime p dividing n: $p \nmid (x+iy) = \prod_{i \in I} \pi_i \cdot \prod_{j \in J} \overline{\pi}_j$.

a. Suppose $p \,|\, n$, $p \equiv 3 \pmod 4$ and suppose without loss of generality that $\pi_1 = \overline{\pi}_1 = p$. Then for any choice of sets I, J one has $p \in I$ or $p \in J$. This forces $p \,|\, \prod_{i \in I} \pi_i \cdot \prod_{j \in J} \overline{\pi}_j$.

b. Suppose $p \,|\, n$, $p \equiv 1 \pmod 4$ and suppose without loss of generality that $\pi_1 \overline{\pi}_1 = p$. Then let $I = \{1, \dots, m\}$ and $J = \emptyset$ i.e. $x + iy = \prod_{i=1}^n \pi_i$. Then $p = \pi_1 \overline{\pi}_1 \,|\, x + iy = \prod_{i=1}^n \pi_i \Rightarrow \overline{\pi}_1 \,|\, \prod_{i=2}^n \pi_i$ or there exists $k \in \{1, \dots, m\}$ such that $\overline{\pi}_1 \,|\, \pi_k$. This is impossible since we know by theorem 3.3 that if $p = \pi_k \overline{\pi}_k$ then π_k and $\overline{\pi}_k$ are *not* associate and if $p \neq p' = \pi_k \overline{\pi}_k$ then $N(\pi_k) = N(\overline{\pi}_k) = p'^2$. Thus $p \nmid x + iy = \prod_{i=1}^n \pi_i$. Also, notice that the choice of the sets I, J does not depend on the rational prime p.

c. Suppose $2 \,|\, n$, then we can use the same reasoning as above by picking sets I, J: $I = \{1, \dots, m\}$ and $J = \emptyset$ such that $\pi_m = i(1-i)$. This partition can clearly be made compatible with the one from *b*.

By *a.*, *b.*, *c.* above and by the first result, namely $4 \nmid n$, we can conclude the proof.

□

4 Three Dimensional Estimates

One could extend the Orchard Problem by considering its 3-d generalization. Let $O := (0,0,0) \in \mathbb{R}^3$, $\Lambda := \mathbb{Z}^3 \backslash O$, $D' := \Lambda \cap D$ and let $D := \overline{B(0,R)} \subset \mathbb{R}^3$ be a closed sphere of radius R, centered at the origin of $\mathbb{R}^3$. Trees will be represented by closed spheres of radius r, centered at all the points of D'. What is the minimal radius ρ of the spheres such that every ray from the origin intersects at least one of the spheres before it hits ∂D? We adapt some of the techniques

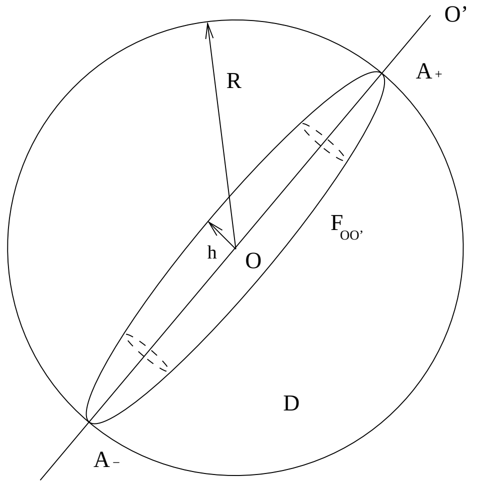

Figure 3: Picture depicting the use of Minkowski's theorem.

used by R. Honsberger([5]) and T.T. Allen([1]) in order to give some bounds for ρ. Minkowski's Theorem will be a very useful tool for giving ρ an upper bound.

Theorem 4.1. *(Minkowski's Theorem) Suppose $m \in \mathbb{Z}_+$ and $F \subset \mathbb{R}^n$ satisfy the following:*

i. F is symmetric with respect to the origin O of $\mathbb{R}^n$.

ii. F is convex.

iii. $vol(F) \geqslant m2^n$.

Then F contains at least m pairs of points $\pm A_i \in \mathbb{Z}^n \backslash O$ $(1 \leq i \leq m)$ which are distinct from each other.

We will also make use of the formula giving the distance between a line and a point in $\mathbb{R}^3$.

Proposition 4.2. *Let $\vec{x}_0, \vec{x}_1$ and $\vec{x}_2$ be points in $\mathbb{R}^3$. The distance between $\vec{x}_0$ and the line passing through $\vec{x}_1$ and $\vec{x}_2$ is given by: $d(\vec{x}_0, \overline{x_1 x_2}) = \frac{|(\vec{x}_2 - \vec{x}_1) \times (\vec{x}_1 - \vec{x}_0)|}{|\vec{x}_2 - \vec{x}_1|}$*

Proposition 4.3. *If the radius R of the Orchard satisfies $R^3 \geq \frac{6}{\pi}$ then $\rho \leq \sqrt{\frac{6}{\pi}} \cdot \frac{1}{\sqrt{R}}$.*

Proof. Suppose F is an ellipsoid with semi-axes of lengths R, h, and h. Also, say F is centered at O. By Minkowski's Theorem 4.1 we see that if $vol(F) \geq 2^3 = 8$ then $F \cap \Lambda' \neq \emptyset$. The volume of an ellipsoid is given by $vol(F) = \frac{4\pi}{3} abc$, where a, b and c are the lengths of the three semi-axes. So if $vol(F) = \frac{4\pi}{3} Rh^2 = 8$ then there exists some integer pair of points $\pm P \in F \cap \Lambda'$. This gives us $h = \sqrt{\frac{6}{\pi}} \frac{1}{\sqrt{R}}$. Now take any ray through O, say it is OO', passing through the point $O' \in \mathbb{R}^3 \backslash D$. The line defined by OO' will intersect the boundary of D, ∂D, in two symmetric points A_+, $A_- \in \partial D$. Now let $F_{OO'}$(Figure 3) be the ellipsoid, centered at O, with semi-axes of lengths R, h and h and whose semi-axis of length R is along the line OO'. If $h = \sqrt{\frac{6}{\pi}\frac{1}{\sqrt{R}}}$ we know by the above reasoning that $F_{OO'}$ contains at least two lattice points other than O. Also, the distance $d(x, OO')$ from any point $x \in F_{OO'}$ to the segment OO' will satisfy $d(x, OO') \leq h$. Thus, if $r = h$ the ray OO' will intersect one of the spheres from $\pm P$. Note that we want $\pm P \in D$ so for sufficiency we need to have: $F_{OO'} \subset D \Leftrightarrow h \leq R \Rightarrow R^3 \geq \frac{6}{\pi}$, which gives the reason why we suppose this condition in the statement of the proposition.

□

We can now use Proposition 4.2 to give a lower bound for ρ.

Proposition 4.4. *The minimal radius of the spheres satisfies the inequality $\rho \geq \frac{1}{d}$, where d is the distance from O to the closest integer point, having coprime coordinates, that lies outside the orchard.*

Proof. Take a ray OX through O and $X := (x, y, z) \in \Lambda \backslash D'$ and suppose this ray does not hit any integer points from D. The distance from any point $X' := (x', y', z') \in D'$ to the ray OX can be computed explicitly using the formula given in Proposition 4.2:

$$d^2(X', OX) = \frac{(z'y - y'z)^2 + (x'z - z'x)^2 + (y'x - x'y)^2}{x^2 + y^2 + z^2} \tag{8}$$

Now, by assumption $X \neq X'$ so not all $(z'y - y'z), (x'z - z'x), (y'x - x'y)$ are zero. This yields that for any $X' \in D'$: $d^2(X', OX) \geq \frac{1}{x^2+y^2+z^2}$. Thus, if the radii r of the spheres are smaller than $\frac{1}{\sqrt{x^2+y^2+z^2}}$, the ray OX does not intersect any sphere. So in order to be able to be hit by at least one sphere, the minimal radius ρ is bound to satisfy $\rho \geq \frac{1}{\sqrt{x^2+y^2+z^2}}$ for all $(x, y, z) \in \Lambda \backslash D'$ with $\gcd(x, y, z) = 1$. The point $X_* := (x_*, y_*, z_*) \in \Lambda \backslash D'$ that has coprime coordinates and is closest to the origin gives $d = \sqrt{x_*^2 + y_*^2 + z_*^2}$ and thus yields the desired result $\rho \geq \frac{1}{d}$.

□

5 Examples

In the following we will look at different planar shapes and find ρ in each case.

5.1 Circle

The first mathematicians concerned with the Orchard Problem considered circular domains D centered at the origin with radius $R \in \mathbb{Z}_+$. By Theorem 2.1 we see that $\rho = \frac{1}{\sqrt{1+R^2}}$ since $(R, 1)$ is the closest point to the origin that lies outside the orchard which has coprime coordinates. Now consider a circular orchard D of any radius $R \in \mathbb{R}_+$ with $R \geq 1$. We can let S be the set of integers that can be written as the sum of two squares, $x^2 + y^2$, where x and y are coprime integers. Theorem 3.4 describes these numbers. Clearly S is unbounded, so if we order S in the usual way we can find unique consecutive integers $a_1, a_2 \in S$ such that $a_1 \leq R^2 < a_2$. Since $a_2 \in S$, there exist $x, y \in \mathbb{Z}_{\geq 0}$, coprime, with $x^2 + y^2 = a_2$. Then (x, y) lies outside the circle. Suppose (x', y') is another lattice point with coprime coordinates closer to the origin than (x, y). Then $x'^2 + y'^2 \leq a_1$ since

$x'^2 + y'^2 \in S$ and S is ordered with a_2 following a_1. Then (x', y') is inside the orchard, so the closest distance to the origin of a point outside D is $d = \sqrt{a_2}$. Therefore $\rho = \frac{1}{\sqrt{a_2}}$. The above cases have already been studied by T.T. Allen in [1].

5.2 Square

Another interesting shape to consider is the square $D := \{(x, y) \in \mathbb{R}^2 \mid |x|+|y| \leq m\}$ for some fixed $m \in \mathbb{Z}_+$. Note that by solving this problem for $m \in \mathbb{Z}_+$ we have solved this problem for all $m \in \mathbb{R}_+$ because the lattice points inside the countour $|x| + |y| = m$ for $m \in \mathbb{R}_+$ are the same lattice points as the ones inside $|x| + |y| = \lfloor m \rfloor$.

Proposition 5.1. *If the orchard D is the square whose boundary is given by $|x| + |y| = m$ where m is a positive integer, then*

$$\rho = \begin{cases} 1/\sqrt{2} & \text{if } m = 1, \\ 1/\sqrt{2k^2 + 2k + 1} & \text{if } m = 2k, \\ 1/\sqrt{2k^2 + 4k + 4} & \text{if } m = 2k + 1 \text{ for } k \text{ odd}, \\ 1/\sqrt{2k^2 + 4k + 10} & \text{if } m = 2k + 1 \text{ for } k \geq 2,\ k \text{ even}. \end{cases}$$

Proof. By symmetry we only need to consider the part of the square that lies in the first quadrant, namely $O_m := \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } y \leq -x + m\}$. The line through the origin O that is perpendicular to $y = -x + m$ is given by $y = x$. By taking the next square O_{m+1} and intersecting the line $y = x$ with its boundary we see that the point that is on the line $y = -x + m + 1$ and is closest to O is $(\frac{m+1}{2}, \frac{m+1}{2})$. If m is of the form $m = 2k$ for some $k \in \mathbb{Z}_+$ then by plugging in we find that the two closest lattice points outside the square O_m are $(k + 1, k)$ and $(k, k + 1)$. These points always have coprime coordinates so for m even we get $\rho = \frac{1}{\sqrt{2k^2+2k+1}}$. If m is odd, let $m = 2k + 1$ for some $k \in \mathbb{Z}_+$, then the closest point is $(k + 1, k + 1)$. This point never has coprime coordinates unless $k = 0$ for which we get the special case $m = 1$ and $\rho = \frac{1}{\sqrt{2}}$. The next closest points are $(k, k+2)$ and $(k+2, k)$. These have relatively prime coordinates if k is odd. Thus if $m = 2k + 1$ for some odd positive integer k then $\rho = \frac{1}{\sqrt{2k^2+4k+4}}$. Now, by taking the next points out we have $(k - 1, k + 3)$ and $(k + 3, k - 1)$. These points have relatively prime coordinates for k even, so if $m = 2k + 1$ for some even positive integer k then $\rho = \frac{1}{\sqrt{2k^2+4k+10}}$. This completes the proof.

□

Remark*: A simple computation yields that the integer points from the second closest square O_{m+2} are farther from the origin O than the closest integer points from O_{m+1} with coprime coordinates.*

5.3 Rhombus

A generalization of the square is the rhombus. Consider the domain $D := \{(x, y) \in \mathbb{R}^2 \mid n|x| + m|y| \leq nmk\}$ for some positive integers n, m and k. An easier type of rhombus that we have solved for some specific cases is $D := \{(x, y) \in \mathbb{R}^2 | \ n|x| + |y| \leq m\}$ for fixed positive integers n, m satisfying $n \mid m$.

Again, by taking the line $y = -nx + m$ and its reciprocal through O, namely $y = \frac{1}{n}x$, we see that their point of intersection is $(\frac{nm}{n^2+1}, \frac{m}{n^2+1})$. For a specific value of n we must consider cases $m \pmod{n^2+1}$.

The following two propositions give the results for n=2, 3.

Proposition 5.2. *If the orchard D is the rhombus whose boundary is given by $2|x| + |y| = m$ where m is a positive integer, then*

$$\rho = \begin{cases} 1/\sqrt{5k^2+2k+1} & \text{if } m = 5k, \\ 1/\sqrt{5k^2+4k+1} & \text{if } m = 5k+1, \\ 1/\sqrt{5k^2+6k+2} & \text{if } m = 5k+2, \\ 1/\sqrt{5k^2+8k+4} & \text{if } m = 5k+3, \\ 1/\sqrt{5k^2+10k+10} & \text{if } m = 5k+4. \end{cases}$$

Proposition 5.3. *If the orchard D is the rhombus whose boundary is given by $3|x| + |y| = m$ where m is a positive integer, then*

$$\rho = \begin{cases} 1/\sqrt{10k^2+2k+1} & \text{if } m = 10k, \\ 1/\sqrt{10k^2+4k+4} & \text{if } m = 10k+1 \text{ and } k \text{ even}, \\ 1/\sqrt{10k^2+4k+2} & \text{if } m = 10k+1 \text{ and } k \text{ odd}, \\ 1/\sqrt{10k^2+6k+2} & \text{if } m = 10k+2, \\ 1/\sqrt{10k^2+8k+8} & \text{if } m = 10k+3 \text{ and } k \text{ odd}, \\ 1/\sqrt{10k^2+8k+2} & \text{if } m = 10k+3 \text{ and } k \text{ even}, \\ 1/\sqrt{10k^2+10k+5} & \text{if } m = 10k+4, \\ 1/\sqrt{10k^2+12k+4} & \text{if } m = 10k+5 \text{ and } k \text{ odd}, \\ 1/\sqrt{10k^2+12k+10} & \text{if } m = 10k+5 \text{ and } k \text{ even}, \\ 1/\sqrt{10k^2+14k+5} & \text{if } m = 10k+6, \\ 1/\sqrt{10k^2+16k+8} & \text{if } m = 10k+7 \text{ and } k \text{ odd}, \\ 1/\sqrt{10k^2+16k+10} & \text{if } m = 10k+7 \text{ and } k \text{ even}, \\ 1/\sqrt{10k^2+18k+9} & \text{if } m = 10k+8, \\ 1/\sqrt{10k^2+20k+20} & \text{if } m = 10k+9. \end{cases}$$

The proofs of Propositions 5.2 and 5.3 were omitted as they are mainly computations.

Conclusion

Starting from Pick's Formula and basic Euclidean geometry we have given a different proof to *Pólya's Orchard Problem*. Moreover, we have generalized T.T. Allen's result in a natural way to arbitrary compact, convex orchards D. By looking at the three dimensional equivalent of the *Orchard Problem* we were able to give bounds for the minimal radius of the trees ρ. This problem needs a complete solution and we are still working on finding a formula for ρ in this case. At the end of our paper we give some in-depth examples of various types of orchards so that one can see how to apply the abstract machinery that was developed throughout the article.

Acknowledgment

The results of this article were obtained during the REU program at Central Michigan University. The research was supported by the NSF-REU grant 05-

52594. We would also like to thank our supervisor prof. Boris Bekker whose advice was essential towards finishing this article.

References

[1] T.T. Allen, Pólya's Orchard Problem, Amer. Math. Monthly(93), 1986, pp. 98-104.

[2] G. Pólya and G. Szegő, Problems and Theorems in Analysis, vol.2, Chap.5, Problem 239, Springer Verlag, New York, 1976.

[3] G. Pólya, Zahlentheoretisches und wahrscheinlichkeits-theoretisches über die Sichtweite im Walde, Arch.Math. und Phys.,27, Series 2 (1918) 135-142.

[4] R. Honsberger, Mathematical Gems I, Dolciani Mathematical Expositions, no.1, Chap. 4, Mathematical Association of America, Washington, DC, 1973.

[5] K. Rosen, Elementary Number Theory and its Applications, Addison Wesley, 2004.