

# On Pólya's Orchard Problem

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## Abstract

In 1918 Pólya formulated the following problem: “*How thick must the trunks of the trees in a regularly spaced circular orchard grow if they are to block completely the view from the center?*” (Pólya and Szegő [2]). We study a more general orchard model, namely any domain that is compact and convex, and find an expression for the minimal radius of the trees. As an example, rhombus-shaped orchards are treated in some depth. Finally, we give some estimates for the minimal radius of the trees if we see the orchard as being 3-dimensional.

## 1 Introduction

Let  $\Lambda := \mathbb{Z}^2 \setminus O$  where  $O := (0, 0)$  is the origin of the  $\mathbb{R}^2$  plane. A tree will be represented by a closed disk centered at some point  $P \in \Lambda$ . We will assume all of the disks have the same radius  $r$ . Let  $D$  be a compact, convex domain in  $\mathbb{R}^2$  such that  $O \in D$ . One can see the boundary  $\partial D$  as being the fence surrounding the orchard  $D$ . Suppose there are disks at all the integer points which lie inside the orchard  $D \setminus O$ , or in other words, at all points from  $D' := D \cap \Lambda$ . A point  $P = (\xi, \eta) \in \mathbb{R}^2 \setminus \text{int}(D)$  is said to be visible if the ray from  $O$  through  $P$  does not intersect any disk (where  $\text{int}(D)$  is the interior of  $D$ ).

The problem is to find the minimal radius  $\rho$  of the trees such that no point of  $\partial D$  is visible. G. Pólya ([2], [3]) and R. Honsberger([4]) found the following estimates for  $\rho$  when  $D$  is a disk of radius  $R \in \mathbb{N}$ :

$$\frac{1}{\sqrt{(R^2 + 1)}} \leq \rho \leq \frac{1}{R} \quad (1)$$

In [1] T.T. Allen solves the orchard problem for disks of arbitrary real radius. He shows that  $\rho = \frac{1}{d}$ , where  $d$  is the distance from  $O$  to the closest point  $P \in \Lambda \setminus D'$  which has coprime coordinates. In the following we give a different proof to Pólya's problem and generalize Allen's result to orchards  $D$  satisfying the following two conditions:

- (i)  $D \subset \mathbb{R}^2$  is a compact, convex domain.
- (ii) The consecutive rays that pass through integer points of  $D$  form acute angles.

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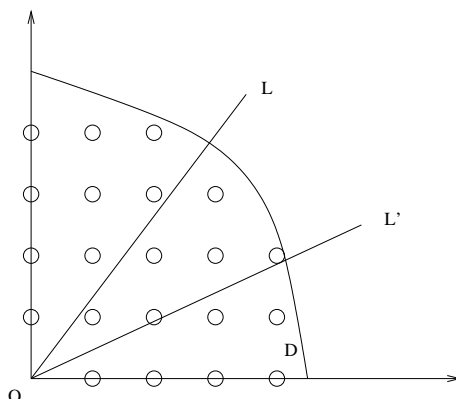


Figure 1: Example of an orchard. Only the first quadrant is exhibited.

## 2 The Orchard Problem

**Theorem 2.1.** *Let  $D$  be a domain satisfying conditions (i) and (ii) above. The minimal radius  $\rho$  of the disks from  $D \cap \Lambda$  such that no part of  $\partial D$  is visible is  $\rho = \frac{1}{d}$  where  $d$  is the distance from  $O$  to the closest lattice point which lies outside of  $D$  and has coprime coordinates.*

*Proof.* Without loss of generality we restrict our reasoning to the first quadrant. We will make use of the following well-known result:

**Theorem 2.2.** (*Pick's Theorem*) *Let  $F$  be a polygon whose vertices are in  $\mathbb{Z}^2$ . Let  $b$  be the number of integer points that are on  $\partial F$  and let  $i$  be the number of integer points that are in  $\text{int}(F)$ . Then:*

$$\text{area}(F) = i + \frac{b}{2} - 1 \quad (2)$$

We break up the proof of Theorem 2.1 into the two following propositions:

**Proposition 2.3.** *The minimal radius satisfies the inequality  $\rho \leq \frac{1}{d}$ .*

*Proof.* Take any ray  $OC$ . Let  $A, B \in D'$  be the two integer points from  $D$  which are closest to  $OC$  and which lie on different sides with respect to  $OC$  (Figure 2).

The distances from the two points to the ray will be  $d(A, OC)$  and  $d(B, OC)$ . Now, if  $d(A, OC) > d(B, OC)$  then by increasing continuously the radii of the disks from 0, the disk from  $B$  will be the first to hit the ray  $OC$ . Rotate  $OC$  around  $O$  so that  $C$  becomes  $C'$ , a position at which  $d(A, OC') = d(B, OC')$ . This process can only increase the minimal radius  $\rho$  of the disks. Consider without loss of generality that  $d(A, OC) = d(B, OC) = h$ . If we look at the points  $A$  and  $B$  as being vectors in  $\mathbb{R}^2$  we can define their sum:  $O' := A + B \in \Lambda$ . Notice that  $O'$  lies on the ray  $OC$ . If  $O' \in D$  then the ray  $OC$  is blocked by the disk from  $O'$  or if  $OC$  hits any disk from  $D'$  then  $\rho = 0$  will do and we are done. Consider that  $O' \notin D$  and that the ray  $OC$  does not hit any disk from  $D'$ . By assumption:

$$OO' \geq d \quad (3)$$

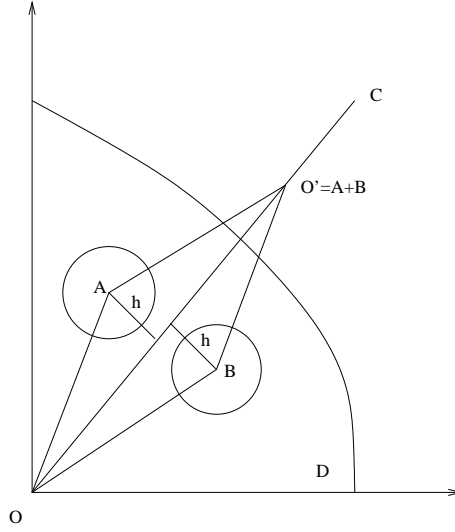


Figure 2: A and B are the closest lattice points to the ray OC.

By the construction of  $OAO'B$  we have  $OAO'B \cap \mathbb{Z}^2 = \{O, A, O', B\}$ . Pick's Theorem 2.2 thus yields:

$$\text{area}(OAO'B) = 0 + \frac{4}{2} - 1 = 1 \quad (4)$$

On the other hand, because  $OAO'B$  is a parallelogram the area can be computed by the formula:

$$\text{area}(OAO'B) = h \cdot OO' \quad (5)$$

Suppose that  $h > \frac{1}{d}$ . Then by using the last two equations:  $h \cdot OO' > \frac{OO'}{d} \Rightarrow d > OO'$  which contradicts equation (3).

Thus  $h \leq \frac{1}{d}$  and since we want the trees from A, B to hit OC, we see that  $h = r$ , where  $r$  is the radius of the trees, will do.

In summation, if  $r = \frac{1}{d}$  then any ray will hit one of the trees before it hits  $\partial D$ . This forces  $\rho \leq \frac{1}{d}$ . □

**Proposition 2.4.** *The minimal radius satisfies the inequality  $\rho \geq \frac{1}{d}$ .*

*Proof.* It is enough to show that if the radius  $r$  of the disks is less than  $\frac{1}{d}$ , then there is a ray which does not hit any disk before it hits the boundary  $\partial D$ . Let  $P_* \in \Lambda \setminus D'$  be the lattice point that is closest to  $O$  and that has coprime coordinates. If  $P_* := (\xi, \eta)$ ,  $d(P_*, O) = d$  and  $\gcd(\xi, \eta) = 1$  then  $OP_*$  will not hit any points from  $D'$ . Note that  $\xi \neq 0$  because  $D$  satisfies condition (ii) above. The line through  $O$  and  $P_*$  is given by  $y = \frac{\eta}{\xi} \cdot x$  so  $(a, b) \in OP_* \cap D'$  if and only if there exists  $k \in \mathbb{N}$  such that  $k \cdot (a, b) = (\xi, \eta)$ . This means that  $k \mid \gcd(\xi, \eta)$  so  $k = 1$ .

Let  $A \in D'$  be the integer point closest to  $OP_*$ . Suppose again that  $P_*$  and  $A$  are vectors in  $\mathbb{R}^2$  and define  $B := P_* - A$ .  $OAP_*B$  will be a parallelogram so we can denote both distances from A, B to  $OP_*$  by  $h$ .

By Pick's Theorem 2.2 we have  $\text{area}(OAP_*B) \geq 1$  and by standard planar geometry  $\text{area}(OAP_*B) = OP_* \cdot h$ . But  $OP_* = d$  so:

$$d \cdot h \geq 1 \quad (6)$$

As a result  $h \geq \frac{1}{d}$ , so the ray  $OP_*$  does not hit any disk from  $D'$  if  $r < \frac{1}{d}$ . For a disk to hit this ray it is necessary that  $r \geq \frac{1}{d} \Rightarrow \rho \geq \frac{1}{d}$ .  $\square$

By combining Propositions 2.3 and 2.4 we get the desired result  $\rho = \frac{1}{d}$ , thus completing the proof of Theorem 2.1.  $\square$

### 3 Some Number Theory

After having proven Theorem 2.1 it is natural to ask the following question: When can a natural number  $n$  be written as the sum of squares of two coprime integers? This would be of help if one would like to compute  $d$  numerically for complicated domains  $D$ . We will give a complete classification of all  $n \in \mathbb{N}$  which can be written in this way. It is helpful to study this problem in an extension of the ring  $\mathbb{Z}$ , namely in the ring of Gaussian integers  $\mathbb{Z}[i]$ .

**Definition 3.1.**  $\mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\}$  is the ring of Gaussian integers. The norm  $N : \mathbb{Z}[i] \rightarrow \mathbb{Z}_+$  of a Gaussian integer  $a + bi$  is defined to be  $N(a + bi) := a^2 + b^2$ .

Elements of the ring  $\mathbb{Z}[i]$  will be called Gaussian integers while numbers from  $\mathbb{Z}$  will be called rational integers. First, we need to know how rational and Gaussian primes relate to one another.

**Theorem 3.2.** If  $p \in \mathbb{N}$  is a rational prime, then  $p$  factors as a Gaussian integer according to the following:

- If  $p = 2$ , then  $p = -i(1 + i)^2 = i(1 - i)^2$  where  $1 + i, 1 - i$  are associate Gaussian primes and  $N(1 + i) = N(1 - i) = 2$ .
- If  $p \equiv 3 \pmod{4}$ , then  $p = \pi$  is a Gaussian prime with  $N(\pi) = p^2$ .
- If  $p \equiv 1 \pmod{4}$ , then  $p = \pi\bar{\pi}$  where  $\pi, \bar{\pi}$  are Gaussian primes that are not associate and  $\bar{\pi}$  is the complex conjugate of  $\pi$ .

Second, we are interested as to when a rational integer can be written as the sum of two squares.

**Theorem 3.3.** A rational integer  $n \in \mathbb{N}$  can be written as the sum of two squares if and only if the prime factorization of  $n$  is of the following form:

$n = 2^m p_1^{e_1} p_2^{e_2} \dots p_s^{e_s} q_1^{2 \cdot f_1} q_2^{2 \cdot f_2} \dots q_l^{2 \cdot f_l}$  where  $m, e_1, e_2, \dots, e_s, f_1, f_2, \dots, f_l \in \mathbb{Z}_+$ ,  $p_1, p_2, \dots, p_s$  are odd rational primes congruent to 1 modulo 4 and  $q_1, q_2, \dots, q_l$  are rational primes congruent to 3 modulo 4.

For the proofs of theorems 3.2 and 3.3 one can consult any number theory book (for example [5]).

We are now ready to state and prove the main result of this section:

**Theorem 3.4.** *Let  $n \in \mathbb{N}$ . Then there exist natural numbers  $x, y$  with  $\gcd(x, y) = 1$  such that  $n = x^2 + y^2$  if and only if  $n$ 's prime factorization is of the form  $n = 2^m p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$  where  $m \in \{0, 1\}$ ,  $e_1, e_2, \dots, e_s \in \mathbb{Z}_+$  and  $p_1, p_2, \dots, p_s$  are odd rational primes congruent to 1 modulo 4.*

*Proof.* If  $n = x^2 + y^2 \equiv 0 \pmod{4}$  then since squares mod 4 are 0 or 1, one has  $x^2 \equiv y^2 \equiv 0 \pmod{4}$  which implies  $x \equiv y \equiv 0 \pmod{2}$  yielding  $\gcd(x, y) \neq 1$ . Thus those  $n$  which are divisible by 4 cannot be written in the desired way. Note that  $n = x^2 + y^2$  for  $x, y \in \mathbb{N}$  if and only if  $n = (x + iy)(x - iy)$  for  $x, y \in \mathbb{N}$ . We want  $\gcd(x, y) = 1$  so for all rational primes  $p \mid n$  we must have  $p \nmid (x + iy)$ . By theorems (3.2) and (3.3) we observe that if a rational integer  $n \in \mathbb{N}$  can be written as the sum of two squares then its decomposition into Gaussian primes is of the form:  $n = \pi_1 \bar{\pi}_1 \pi_2 \bar{\pi}_2 \dots \pi_m \bar{\pi}_m$  for  $m \in \mathbb{N}$  and  $\pi_1, \bar{\pi}_1, \dots, \pi_m, \bar{\pi}_m$  Gaussian primes, not all necessarily distinct. Then

$$x + iy = \prod_{i \in I} \pi_i \cdot \prod_{j \in J} \bar{\pi}_j \quad (7)$$

for some sets  $I, J \subset \{1, \dots, m\}$  such that  $I \cap J = \emptyset$  and  $I \cup J = \{1, \dots, m\}$  (a partition of the set  $\{1, \dots, m\}$ ). Thus in order for  $n$  to be written as the sum of two squares which are coprime it is enough to see if there exist sets  $I, J$  such that for any prime  $p$  dividing  $n$ :  $p \nmid (x + iy) = \prod_{i \in I} \pi_i \cdot \prod_{j \in J} \bar{\pi}_j$ .

- a. Suppose  $p \mid n$ ,  $p \equiv 3 \pmod{4}$  and suppose without loss of generality that  $\pi_1 = \bar{\pi}_1 = p$ . Then for any choice of sets  $I, J$  one has  $p \in I$  or  $p \in J$ . This forces  $p \mid \prod_{i \in I} \pi_i \cdot \prod_{j \in J} \bar{\pi}_j$ .
- b. Suppose  $p \mid n$ ,  $p \equiv 1 \pmod{4}$  and suppose without loss of generality that  $\pi_1 \bar{\pi}_1 = p$ . Then let  $I = \{1, \dots, m\}$  and  $J = \emptyset$  i.e.  $x + iy = \prod_{i=1}^n \pi_i$ . Then  $p = \pi_1 \bar{\pi}_1 \mid x + iy = \prod_{i=1}^n \pi_i \Rightarrow \bar{\pi}_1 \mid \prod_{i=2}^n \pi_i$  or there exists  $k \in \{1, \dots, m\}$  such that  $\bar{\pi}_1 \mid \pi_k$ . This is impossible since we know by theorem 3.3 that if  $p = \pi_k \bar{\pi}_k$  then  $\pi_k$  and  $\bar{\pi}_k$  are *not* associate and if  $p \neq p' = \pi_k \bar{\pi}_k$  then  $N(\pi_k) = N(\bar{\pi}_k) = p'^2$ . Thus  $p \nmid x + iy = \prod_{i=1}^n \pi_i$ . Also, notice that the choice of the sets  $I, J$  does not depend on the rational prime  $p$ .
- c. Suppose  $2 \mid n$ , then we can use the same reasoning as above by picking sets  $I, J$ :  $I = \{1, \dots, m\}$  and  $J = \emptyset$  such that  $\pi_m = i(1 - i)$ . This partition can clearly be made compatible with the one from b.

By a., b., c. above and by the first result, namely  $4 \nmid n$ , we can conclude the proof. □

## 4 Three Dimensional Estimates

One could extend the Orchard Problem by considering its 3-d generalization. Let  $O := (0, 0, 0) \in \mathbb{R}^3$ ,  $\Lambda := \mathbb{Z}^3 \setminus O$ ,  $D' := \Lambda \cap D$  and let  $D := \overline{B(0, R)} \subset \mathbb{R}^3$  be a closed sphere of radius  $R$ , centered at the origin of  $\mathbb{R}^3$ . Trees will be represented by closed spheres of radius  $r$ , centered at all the points of  $D'$ . What is the minimal radius  $\rho$  of the spheres such that every ray from the origin intersects at least one of the spheres before it hits  $\partial D$ ? We adapt some of the techniques

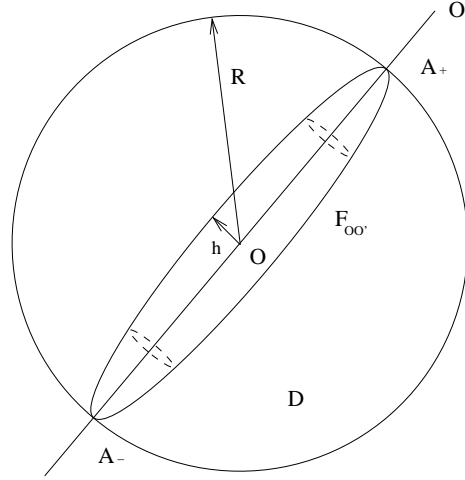


Figure 3: Picture depicting the use of Minkowski's theorem.

used by R. Honsberger([5]) and T.T. Allen([1]) in order to give some bounds for  $\rho$ . Minkowski's Theorem will be a very useful tool for giving  $\rho$  an upper bound.

**Theorem 4.1.** (*Minkowski's Theorem*) Suppose  $m \in \mathbb{Z}_+$  and  $F \subset \mathbb{R}^n$  satisfy the following:

- i.  $F$  is symmetric with respect to the origin  $O$  of  $\mathbb{R}^n$ .
- ii.  $F$  is convex.
- iii.  $\text{vol}(F) \geq m2^n$ .

Then  $F$  contains at least  $m$  pairs of points  $\pm A_i \in \mathbb{Z}^n \setminus O$  ( $1 \leq i \leq m$ ) which are distinct from each other.

We will also make use of the formula giving the distance between a line and a point in  $\mathbb{R}^3$ .

**Proposition 4.2.** Let  $\vec{x}_0, \vec{x}_1$  and  $\vec{x}_2$  be points in  $\mathbb{R}^3$ . The distance between  $\vec{x}_0$  and the line passing through  $\vec{x}_1$  and  $\vec{x}_2$  is given by:  $d(\vec{x}_0, \overline{\vec{x}_1\vec{x}_2}) = \frac{|(\vec{x}_2 - \vec{x}_1) \times (\vec{x}_1 - \vec{x}_0)|}{|\vec{x}_2 - \vec{x}_1|}$

**Proposition 4.3.** If the radius  $R$  of the Orchard satisfies  $R^3 \geq \frac{6}{\pi}$  then  $\rho \leq \sqrt{\frac{6}{\pi}} \cdot \frac{1}{\sqrt{R}}$ .

*Proof.* Suppose  $F$  is an ellipsoid with semi-axes of lengths  $R, h$ , and  $h$ . Also, say  $F$  is centered at  $O$ . By Minkowski's Theorem 4.1 we see that if  $\text{vol}(F) \geq 2^3 = 8$  then  $F \cap \Lambda' \neq \emptyset$ . The volume of an ellipsoid is given by  $\text{vol}(F) = \frac{4\pi}{3} abc$ , where  $a, b$  and  $c$  are the lengths of the three semi-axes. So if  $\text{vol}(F) = \frac{4\pi}{3} Rh^2 = 8$  then there exists some integer pair of points  $\pm P \in F \cap \Lambda'$ . This gives us  $h = \sqrt{\frac{6}{\pi}} \frac{1}{\sqrt{R}}$ . Now take any ray through  $O$ , say it is  $OO'$ , passing through the point  $O' \in \mathbb{R}^3 \setminus D$ . The line defined by  $OO'$  will intersect the boundary of  $D$ ,  $\partial D$ , in two symmetric points  $A_+, A_- \in \partial D$ . Now let  $F_{OO'}$  (Figure 3) be the ellipsoid, centered at  $O$ , with semi-axes of lengths  $R, h$  and  $h$  and whose

semi-axis of length  $R$  is along the line  $OO'$ . If  $h = \sqrt{\frac{6}{\pi}} \frac{1}{\sqrt{R}}$  we know by the above reasoning that  $F_{OO'}$  contains at least two lattice points other than  $O$ . Also, the distance  $d(x, OO')$  from any point  $x \in F_{OO'}$  to the segment  $OO'$  will satisfy  $d(x, OO') \leq h$ . Thus, if  $r = h$  the ray  $OO'$  will intersect one of the spheres from  $\pm P$ . Note that we want  $\pm P \in D$  so for sufficiency we need to have:  $F_{OO'} \subset D \Leftrightarrow h \leq R \Rightarrow R^3 \geq \frac{6}{\pi}$ , which gives the reason why we suppose this condition in the statement of the proposition.  $\square$

We can now use Proposition 4.2 to give a lower bound for  $\rho$ .

**Proposition 4.4.** *The minimal radius of the spheres satisfies the inequality  $\rho \geq \frac{1}{d}$ , where  $d$  is the distance from  $O$  to the closest integer point, having coprime coordinates, that lies outside the orchard.*

*Proof.* Take a ray  $OX$  through  $O$  and  $X := (x, y, z) \in \Lambda \setminus D'$  and suppose this ray does not hit any integer points from  $D$ . The distance from any point  $X' := (x', y', z') \in D'$  to the ray  $OX$  can be computed explicitly using the formula given in Proposition 4.2:

$$d^2(X', OX) = \frac{(z'y - y'z)^2 + (x'z - z'x)^2 + (y'x - x'y)^2}{x^2 + y^2 + z^2} \quad (8)$$

Now, by assumption  $X \neq X'$  so not all  $(z'y - y'z), (x'z - z'x), (y'x - x'y)$  are zero. This yields that for any  $X' \in D'$ :  $d^2(X', OX) \geq \frac{1}{x^2 + y^2 + z^2}$ . Thus, if the radii  $r$  of the spheres are smaller than  $\frac{1}{\sqrt{x^2 + y^2 + z^2}}$ , the ray  $OX$  does not intersect any sphere. So in order to be able to be hit by at least one sphere, the minimal radius  $\rho$  is bound to satisfy  $\rho \geq \frac{1}{\sqrt{x^2 + y^2 + z^2}}$  for all  $(x, y, z) \in \Lambda \setminus D'$  with  $\gcd(x, y, z) = 1$ . The point  $X_* := (x_*, y_*, z_*) \in \Lambda \setminus D'$  that has coprime coordinates and is closest to the origin gives  $d = \sqrt{x_*^2 + y_*^2 + z_*^2}$  and thus yields the desired result  $\rho \geq \frac{1}{d}$ .  $\square$

## 5 Examples

In the following we will look at different planar shapes and find  $\rho$  in each case.

### 5.1 Circle

The first mathematicians concerned with the Orchard Problem considered circular domains  $D$  centered at the origin with radius  $R \in \mathbb{Z}_+$ . By Theorem 2.1 we see that  $\rho = \frac{1}{\sqrt{1+R^2}}$  since  $(R, 1)$  is the closest point to the origin that lies outside the orchard which has coprime coordinates. Now consider a circular orchard  $D$  of any radius  $R \in \mathbb{R}_+$  with  $R \geq 1$ . We can let  $S$  be the set of integers that can be written as the sum of two squares,  $x^2 + y^2$ , where  $x$  and  $y$  are coprime integers. Theorem 3.4 describes these numbers. Clearly  $S$  is unbounded, so if we order  $S$  in the usual way we can find unique consecutive integers  $a_1, a_2 \in S$  such that  $a_1 \leq R^2 < a_2$ . Since  $a_2 \in S$ , there exist  $x, y \in \mathbb{Z}_{\geq 0}$ , coprime, with  $x^2 + y^2 = a_2$ . Then  $(x, y)$  lies outside the circle. Suppose  $(x', y')$  is another lattice point with coprime coordinates closer to the origin than  $(x, y)$ . Then  $x'^2 + y'^2 \leq a_1$  since

$x'^2 + y'^2 \in S$  and  $S$  is ordered with  $a_2$  following  $a_1$ . Then  $(x', y')$  is inside the orchard, so the closest distance to the origin of a point outside  $D$  is  $d = \sqrt{a_2}$ . Therefore  $\rho = \frac{1}{\sqrt{a_2}}$ . The above cases have already been studied by T.T. Allen in [1].

## 5.2 Square

Another interesting shape to consider is the square  $D := \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq m\}$  for some fixed  $m \in \mathbb{Z}_+$ . Note that by solving this problem for  $m \in \mathbb{Z}_+$  we have solved this problem for all  $m \in \mathbb{R}_+$  because the lattice points inside the countour  $|x| + |y| = m$  for  $m \in \mathbb{R}_+$  are the same lattice points as the ones inside  $|x| + |y| = \lfloor m \rfloor$ .

**Proposition 5.1.** *If the orchard  $D$  is the square whose boundary is given by  $|x| + |y| = m$  where  $m$  is a positive integer, then*

$$\rho = \begin{cases} 1/\sqrt{2} & \text{if } m = 1, \\ 1/\sqrt{2k^2 + 2k + 1} & \text{if } m = 2k, \\ 1/\sqrt{2k^2 + 4k + 4} & \text{if } m = 2k + 1 \text{ for } k \text{ odd,} \\ 1/\sqrt{2k^2 + 4k + 10} & \text{if } m = 2k + 1 \text{ for } k \geq 2, k \text{ even.} \end{cases}$$

*Proof.* By symmetry we only need to consider the part of the square that lies in the first quadrant, namely  $O_m := \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0 \text{ and } y \leq -x + m\}$ . The line through the origin  $O$  that is perpendicular to  $y = -x + m$  is given by  $y = x$ . By taking the next square  $O_{m+1}$  and intersecting the line  $y = x$  with its boundary we see that the point that is on the line  $y = -x + m + 1$  and is closest to  $O$  is  $(\frac{m+1}{2}, \frac{m+1}{2})$ . If  $m$  is of the form  $m = 2k$  for some  $k \in \mathbb{Z}_+$  then by plugging in we find that the two closest lattice points outside the square  $O_m$  are  $(k + 1, k)$  and  $(k, k + 1)$ . These points always have coprime coordinates so for  $m$  even we get  $\rho = \frac{1}{\sqrt{2k^2 + 2k + 1}}$ . If  $m$  is odd, let  $m = 2k + 1$  for some  $k \in \mathbb{Z}_+$ , then the closest point is  $(k + 1, k + 1)$ . This point never has coprime coordinates unless  $k = 0$  for which we get the special case  $m = 1$  and  $\rho = \frac{1}{\sqrt{2}}$ . The next closest points are  $(k, k + 2)$  and  $(k + 2, k)$ . These have relatively prime coordinates if  $k$  is odd. Thus if  $m = 2k + 1$  for some odd positive integer  $k$  then  $\rho = \frac{1}{\sqrt{2k^2 + 4k + 4}}$ . Now, by taking the next points out we have  $(k - 1, k + 3)$  and  $(k + 3, k - 1)$ . These points have relatively prime coordinates for  $k$  even, so if  $m = 2k + 1$  for some even positive integer  $k$  then  $\rho = \frac{1}{\sqrt{2k^2 + 4k + 10}}$ . This completes the proof.  $\square$

**Remark:** *A simple computation yields that the integer points from the second closest square  $O_{m+2}$  are farther from the origin  $O$  than the closest integer points from  $O_{m+1}$  with coprime coordinates.*

## 5.3 Rhombus

A generalization of the square is the rhombus. Consider the domain  $D := \{(x, y) \in \mathbb{R}^2 \mid n|x| + m|y| \leq nmk\}$  for some positive integers  $n, m$  and  $k$ . An easier type of rhombus that we have solved for some specific cases is  $D := \{(x, y) \in \mathbb{R}^2 \mid n|x| + |y| \leq m\}$  for fixed positive integers  $n, m$  satisfying  $n \mid m$ .



Again, by taking the line  $y = -nx + m$  and its reciprocal through  $O$ , namely  $y = \frac{1}{n}x$ , we see that their point of intersection is  $(\frac{nm}{n^2+1}, \frac{m}{n^2+1})$ . For a specific value of  $n$  we must consider cases  $m \pmod{n^2 + 1}$ .

The following two propositions give the results for  $n=2, 3$ .

**Proposition 5.2.** *If the orchard  $D$  is the rhombus whose boundary is given by  $2|x| + |y| = m$  where  $m$  is a positive integer, then*

$$\rho = \begin{cases} 1/\sqrt{5k^2 + 2k + 1} & \text{if } m = 5k, \\ 1/\sqrt{5k^2 + 4k + 1} & \text{if } m = 5k + 1, \\ 1/\sqrt{5k^2 + 6k + 2} & \text{if } m = 5k + 2, \\ 1/\sqrt{5k^2 + 8k + 4} & \text{if } m = 5k + 3, \\ 1/\sqrt{5k^2 + 10k + 10} & \text{if } m = 5k + 4. \end{cases}$$

**Proposition 5.3.** *If the orchard  $D$  is the rhombus whose boundary is given by  $3|x| + |y| = m$  where  $m$  is a positive integer, then*

$$\rho = \begin{cases} 1/\sqrt{10k^2 + 2k + 1} & \text{if } m = 10k, \\ 1/\sqrt{10k^2 + 4k + 4} & \text{if } m = 10k + 1 \text{ and } k \text{ even,} \\ 1/\sqrt{10k^2 + 4k + 2} & \text{if } m = 10k + 1 \text{ and } k \text{ odd,} \\ 1/\sqrt{10k^2 + 6k + 2} & \text{if } m = 10k + 2, \\ 1/\sqrt{10k^2 + 8k + 8} & \text{if } m = 10k + 3 \text{ and } k \text{ odd,} \\ 1/\sqrt{10k^2 + 8k + 2} & \text{if } m = 10k + 3 \text{ and } k \text{ even,} \\ 1/\sqrt{10k^2 + 10k + 5} & \text{if } m = 10k + 4, \\ 1/\sqrt{10k^2 + 12k + 4} & \text{if } m = 10k + 5 \text{ and } k \text{ odd,} \\ 1/\sqrt{10k^2 + 12k + 10} & \text{if } m = 10k + 5 \text{ and } k \text{ even,} \\ 1/\sqrt{10k^2 + 14k + 5} & \text{if } m = 10k + 6, \\ 1/\sqrt{10k^2 + 16k + 8} & \text{if } m = 10k + 7 \text{ and } k \text{ odd,} \\ 1/\sqrt{10k^2 + 16k + 10} & \text{if } m = 10k + 7 \text{ and } k \text{ even,} \\ 1/\sqrt{10k^2 + 18k + 9} & \text{if } m = 10k + 8, \\ 1/\sqrt{10k^2 + 20k + 20} & \text{if } m = 10k + 9. \end{cases}$$

The proofs of Propositions 5.2 and 5.3 were omitted as they are mainly computations.

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