# Demystifying Functions: The Historical and Pedagogical Difficulties of the Concept of the Function

Melanie Jones[a]

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**Abstract**

In this study, the author discusses the concept of function from a historical and pedagogical perspective. The historical roots, ranging from ancient civilizations all the way to the twentieth century, are summarized. The author then details several different function representations that have emerged over the course of the concept's history. Special attention is paid to the idea of abstraction and how students understand functions at different levels of abstraction. Several middle school, high school, and college textbooks are then analyzed and evaluated based on their portrayal of the function concept. The author describes several common misconceptions that students have about functions and finally proposes a short educational module designed to help older high school students grow to a deeper level of understanding of this complex and often misunderstood concept.

**Keywords**: function, mathematical history, mathematical pedagogy, abstraction

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[a] Trinity University Mathematics, San Antonio, Texas, USA. Corresponding author's email is Melanie.Jones@trinity.edu.

# 1 Introduction

Irish writer Oscar Wilde once surmised that "[t]he truth is rarely pure and never simple." While his musings were most likely not directed towards the mathematics community, Wilde's statement sheds light on the nature of mathematical truth, an issue that has plagued mathematicians and mathematics educators for centuries.

Many beginning mathematics students delight in the black and white nature of math. While they struggle to decipher reality from fiction, right from wrong, and fact from opinion in their history and English courses, they perceive math to be a subject of cut-and-dry answers. In working problems or doing homework, students strive for the "right" answer and are often satisfied when reaching the solution even if they do not completely understand why their answer is correct. While teachers often stress that students "show their work," such work typically exhibits the ability of the student to reproduce a symbolic manipulation and is not always an indication of whether or not the student comprehends the mathematical subtleties.

Exposure to higher level mathematics dispels the illusion of the simplicity of math. Students realize that their prior understanding of mathematical concepts is incomplete. The study of functions, perhaps the most central concept in all mathematics, is often one of the topics where students' understanding is most incomplete. From early elementary school throughout the rest of their mathematics careers, students encounter functions in various forms and applications. While they often learn how to manipulate and "use" functions to perform tasks, students do not always grasp the complexity of the multifaceted concept. In fact, mathematicians over the past 500 years have struggled to produce an accurate definition of function. Teachers and textbooks, in their attempts to make the idea accessible, have used a myriad of pedagogical techniques to teach functions. Sometimes teachers and textbooks "water-down" the concept in order to avoid confusing students, which leaves gaps in their understanding. Other times students are overloaded with definitions and subtleties beyond the depth of their mathematical understanding. These students end up disregarding or forgetting certain aspects of what they are taught in favor of their own understanding.

In this paper, the notion of the function, its historical development, common misconceptions, and the pedagogical difficulties that arise from its complexity are discussed. How functions are taught and presented in various textbooks is analyzed, and suggestions are made about ways in which functions can be taught with increased clarity.

# 2 History

The complexity of the function concept mirrors the intricacy of its historical development. The concept has roots dating back 4000 years as ancient civilizations developed the idea of counting and the notion of correspondence between objects that is implied by a sequence of numbers [15]. From their countless tablets of corresponding numbers, it is likely that the Babylonians touched on aspects of the function idea. These tablets presented sets of ordered pairs with unknown purpose but of evident functional character [14]. While they studied specific functions, it is not likely that the Babylonians had developed a generalized concept [12]. Thus, while this early emergence demonstrates the primacy of the function within the natural world, it does not mark the beginning of a sophisticated and deliberate understanding of the concept.

In the fourteenth century, French mathematician Nicole Oresme developed the geometric theory of latitude of forms and the concept of the rectangular coordinate system [15]. In this work, Oresme introduced the idea of acceleration as the intensity of velocity and touched on several ideas about independent and dependent variables [21]. His investigation and depiction of natural laws that describe one quantity as dependent on another set the groundwork for the function concept of today [12].

During the early 17th century, as scientists discovered more about natural laws and as mathematicians began to connect the studies of algebra and geometry, the idea of function became more of a necessity. Galileo, in his study of motion, clearly grasped the concept of the relation of variables to one another, and Descartes, in his study of the algebraic expression of curves, also touched on the function concept [12].

It is essential to note that at this point in time, Newton and Leibniz were developing calculus, a subject which, in modern times, is inseparable from the function concept. Early calculus, however, was not a calculus of functions but rather was a calculus of geometric curves. In fact, most early calculus dealt with solving problems about curves and properties of curves such as tangents and areas under them. Leibniz and Newton were concerned with the geometric nature of their new understanding, and it was not until after the introduction of the function concept that calculus began to take shape in its algebraic form [8].

Neither the explicit concept nor the word 'function' were introduced until the 17th century. Within the context of the development of analytical geometry, it is not surprising that the word 'function' emerged in relation to a geometric concept. Leibniz used the word to denote a geometric object, such as the tangent associated with a curve, in 1692 [8]. Leibniz's usage of the word "function" aligns with the present-day definition, however, he described a much narrower and more restricted understanding of the concept. In fact, over the years, new examples have been a driving force in the development of the function concept. In response to these ideas, the definition has had to develop and expand to encompass the complete function concept. It was not until 1718 that Johann Bernoulli introduced the first formal definition of the function in response to the need for such a term. His definition reads,

> "One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants" [8].

This vague definition marked the beginning of the function's evolution into the multi-faceted concept that exists today.

As mathematics began to drift away from the geometric idea of analysis to the algebraic in the 1700's, the notion of the function underwent a similar transformation. Euler's definition in his 1748 work, *Introductio in Amalysin Infinitorum*, reads,

> "A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities" [8].

Euler's definition is nearly identical to Bernoulli's, however, the addition of the term "analytical expression" is significant because it shifts the notion from the geometric to the algebraic. Euler's term "analytical expression" is not to be confused with the modern definition of analytic which refers to a type of function locally given by a convergent power series. Euler refers to a more broad concept, and although he does not explicitly define the term in his work, he does include expressions with the four algebraic operations, roots, exponentials, logarithms, trigonometric functions, derivatives and integrals in his list of acceptable "analytic expressions." From here on in this paper the term "analytic expression" will be used to refer to this concept defined by Euler. At a time when geometry and algebra were viewed as distinct mathematical subjects, Euler's definition emphasizes the idea of using algebra to represent a geometric object. He links the geometric with the algebraic. [8].

At around this same time, Euler was also engaged in a debate with Jean d'Alembert and Daniel Bernoulli concerning a famous problem concerning vibrating strings. The problem involves determining the function that describes the shape of an elastic string with fixed ends at a specific time, t, after it has been released to vibrate from some initial shape. During this era, many mathematicians ascribed to an "article of faith" which asserted that if two analytic expressions agree on an interval, they must agree everywhere. Since, at that time, analytic expressions were generally thought to be those described by Euler, this assumption was believable.

French mathematician d'Alembert, a believer in the aforementioned "article of faith," took a mathematically rigorous perspective on this problem. In 1747, he developed the wave equation, given by

$$\left(\frac{\delta^2 y}{\delta t^2}\right) = a^2 \left(\frac{\delta^2 y}{\delta x^2}\right) \tag{1}$$

where $a$ is a constant. From this partial differential equation, d'Alembert produced his "most general" solution to the vibrating string problem:

$$y(x,t) = \left(\frac{1}{2}\right)[\varphi(x+at) + \varphi(x-at)],$$

where $\varphi$ is a function determined on the interval between the fixed ends by the initial shape of the string. Most significant to this discussion of functions was d'Alembert's belief that $\varphi$ must be what Euler referred to as an analytic expression that, since it satisfied the wave equation (1), was twice differentiable. D'Alembert's contention that he had produced the most general solution emphasized his belief that such analytic expressions were the only permissable functions [8].

The next year, Euler proposed his findings on the problem. He agreed with d'Alembert's solution, but disagreed with his assertions about $\varphi$. Euler contended that the original function did not necessarily have to be representable by one analytic expression. In fact, he argued that a more general solution to the problem could be given by extending $\varphi$ to include initial shapes of the string represented by multiple analytic expressions defined on different subintervals of the string. Even more broadly, Euler believed that $\varphi$ could represent any curve drawn freehand. Since $\varphi$ is a function, Euler's claim implies that the function concept extends to include piecewise functions, or functions defined individually over different intervals, and freehand functions that cannot be expressed by any combination of analytic expressions. The physical considerations in this problem caused Euler to alter his conception of function significantly so that in 1755 he wrote

> "If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities" [8].

Euler's observations led to a broader view of the function concept.

In 1753, Daniel Bernoulli challenged both d'Alembert's and Euler's solutions to the vibrating string problem. Bernoulli was more of a physicist than a mathematician, and he found d'Alembert's and Euler's solutions to lack consistency with the physics of the problem. He even went so far as to say in reference to the solutions, "beautiful mathematics but what has it to do with vibrating strings?" [8].

Bernoulli based his solution to the vibrating string problem on his understanding of musical vibrations. He knew that vibrating strings have infinitely many fundamental vibrations, and thus concluded that given the constants $b_n$ and $a$ and the endpoints of the string $(0, \ell)$, the solution to the problem could be expressed as

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi at}{\ell}\right).$$

While Bernoulli was not interested in debating the nature of functions, his solution, when solved for initial conditions at $t = 0$, implies that any arbitrary function can be represented by the series

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right).$$

Euler and d'Alembert rejected Bernoulli's solution. While Bernoulli understood the physics behind the problem, Euler contended that he did not consider its implication concerning functions. Euler believed that all series of sines must be odd and periodic, and thus, according to Euler, the existence of even and non-periodic functions disproves Bernoulli's solution. Euler's argument rested on his belief in the "article of faith," mentioned earlier [20]. The debate surrounding the vibrating string problem eventually died down, but the controversy led to increased thought and discourse about the definition of the term function [8].

In 1807, Joseph Fourier, as a result of his study of heat conduction, developed a theorem about functions that, similar to Bernoulli's, deals with a series of sines and cosines. His theorem states: Any function f(x) defined over ($-\ell, \ell$) is representable over this interval by a series of sines and cosines,

$$f(x) = \left(\frac{a_o}{2}\right) + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right],$$

where the coefficients $a_n$ and $b_n$ are given by

$$a_n = \left(\frac{1}{\ell}\right) \int_{-\ell}^{\ell} f(t) \cos\left(\frac{n\pi t}{\ell}\right) dt$$

and

$$b_n = \left(\frac{1}{\ell}\right) \int_{-\ell}^{\ell} f(t) \sin\left(\frac{n\pi t}{\ell}\right) dt.$$

Fourier's proof was questionable, and his theorem was eventually studied by German mathematician Johann Dirichlet. In 1829 Dirichlet developed sufficient conditions for Fourier's theorem. He stated that all functions with only finitely many discontinuities and finitely many maxima and minima on an interval can be represented by a Fourier series. Fourier's work is significant because it disproved the "article of faith" assumption and because it re-emphasized analytic expressions as functions. The result proved that two analytic expressions could agree on one interval without necessarily agreeing outside that interval [8].

Fourier's theorem also forced mathematicians to re-evaluate the concept of function both because of the confusion caused by his results and because of the looseness of his proof techniques. His work prompted the investigation of other mathematicians like Dirichlet, who once said, "To make sense out of what he [Fourier] did took a century of effort by men of "more critical genius," and the end is not yet in sight" [8]. In 1829, Dirichlet produced a counterexample to Fourier's original theorem. The Dirichlet function, which assigns one value to all rationals and another to all irrationals, can not be represented by a Fourier series and was the first clear example of a function that was neither an analytic expression or a curve drawn freehand. It was also the first function to be discontinuous everywhere. Rather importantly, the Dirichlet function highlights the concept of arbitrary pairing. While many mathematicians had acknowledged the arbitrary nature of functions prior to him, Dirichlet was the first to give a concrete example of an arbitrary function [8].

In 1939, the group of mathematicians that wrote under the allonym Nicolas Bourbaki defined functions in the following manner:

> "Let $E$ and $F$ be two sets, which may or may not be distinct. A relation between a variable element $x$ of $E$ and a variable element $y$ of $F$ is called a *functional relation* in $y$ if for all $x \in E$ there exists a unique $y \in F$ which is in the given relation with $x$" [8].

The Bourbaki definition became the first to define function in terms of a set of ordered pairs. This idea of functions as sets of ordered pairs has since been accepted by many mathematicians as an all-encompassing and succinct manner in which to understand functions. In fact, many algebra and higher level textbooks present this definition as the primary (or glossary) definition.

The function concept has undergone a drastic transformation over the course of more than 300 years since Leibniz introduced the term. What began as a word coined to describe a purely geometric idea has evolved into a concept of importance in nearly every field of mathematics.

## 3 Function Representations

In light of the lengthy development of the function and debates within the mathematical world about its nature, it is no wonder that students struggle to grasp the concept. Annie and John Selden, in their 1992 summary of recent research on students' conceptions of functions, explain that, "there is an unavoidable conflict (tension) between the structure of mathematics and the cognitive process of concept acquisition. Whereas it can only take one sentence to state a definition, 'unpacking' a definition is a hard cognitive task" [18]. This task is especially difficult when students face concepts, like functions, with many different definitions and representations. New ways of representing functions have emerged continually over the course of its development. Each of these representations is important in understanding a specific aspect of the idea and each is strongly tied to the others, but as a collection, they may overwhelm and confuse students [18].

True understanding of functions runs deeper than just the ability to regurgitate definitions. Three basic conceptions of functions are prominent throughout the literature concerning the teaching of functions. These three categories, named and defined in Dubinsky and Harel's "The Nature of the Process Conception of Function," indicate the level of understanding achieved by students. These are not three completely different notions, but rather, they represent a continuum of abstraction [4].

The most basic concept of function is that of an action. Dubinsky and Harel call action "a repeatable mental or physical manipulation of objects" [4]. Students at this phase of understanding need evidence of a concrete action in order to grasp that something is a function. They see functions as an explicit rule which takes an input, transforms it by means of a specific algorithm, and then produces an output. Students with an action conception easily comprehend two of the most common representations of functions.

Graphs are quite possibly the most recognizable representation of functions. This representation, however, is not always directly connected with the idea of function in students' minds since graphs are often taught several years before the term function is ever introduced. Graphs help students understand useful information about functions including maxima and minima as well as the concepts of increasing and decreasing [18]. In fact, in his article about functions, Theodore Eisenberg goes so far as to assert, "Single valued, real variable functions should be thought of as being inherently tied to a graphical representation, and . . . all elementary concepts concerning functions (should) be defined in a visual format" [6]. His argument rests on Israel Kleiner's contention that concepts should be viewed from as many points of view as possible. When students are introduced to functions, Eisenberg believes that the visual representation should be specifically emphasized. Unfortunately, students often struggle to develop visualization skills, especially when they encounter non-typical functions such as the Dirichlet function [6].

The formula representation of functions is also another important aspect of the function concept. This idea restricts the notion of function in a way similar to Euler's original definition of function [18]. While this representation is especially useful in calculus and pre-calculus, students who are only exposed to this definition have difficulty understanding that functions can have completely arbitrary pairings. Functions do not necessarily have a formulaic representation nor do they even have to concern numbers. This representation also leads to a misunderstanding about the existence of discontinuous functions. Even functions depicted by two or more formulas defined over different parts of the domain are difficult for students to perceive as functions [18].

The next level of understanding is the process conception. This conception involves a deeper understanding of a function as something that takes in an object, transforms it and produces a completely new object. Rather than needing an explicit formula or rule, students at this level of understanding are willing to accept functions that involve vague transformations.

The idea of a function machine is a common tool used by teachers to help students view functions from the process conception. This technique presents a function as a machine or a box that accepts an input and produces an output. With the process understanding, students have no need to know the contents of the box; rather, the existence of the machine alone is enough to convince them that they are dealing with a function. While some researchers suggest that "the function machine provides a powerful foundation and is a cognitive root for developing understanding of the concept of function" [10], others see the representation as ineffective for students who do not understand how machines process numbers [18]. Research also shows that students have difficulty connecting the function machine concept with graphs and other such representations of functions [10]. While it offers a simplistic approach to help younger students understand functions as a process, like each of the other representations, the function machine conception does not provide a complete picture of the notion.

A second process-oriented view deals with the idea of functions as a correspondence between two sets. This understanding is similar to Dirichlet's 1837 definition, and basically says that a function is a correspondence such that for every element of the first set there corresponds one element of the second set. This correspondence idea forces students to abandon their need for an algorithm and instead focus more on the idea of mapping one set to another [18].

The most sophisticated understanding is that of functions as objects. With this conception, the "machine" is no longer necessary, and students see functions as entities in and of themselves that can be transformed and operated upon. The concept is best encapsulated by the ordered-pair representation.

The ordered pair definition of function, as introduced by Bourbaki in 1939, is arguably the most mathematically accurate in the sense that it completely captures the essence of a function. This representation describes a function as a possibly infinite set of ordered pairs (x,y) in which each x-coordinate is paired with only one y-coordinate. It is important because it can accurately describe discontinuous functions, arbitrary pairings, and can even be extended to account for functions whose domain and range are not numbers. Also, the set concept gives rise more readily to the notion of function as an object. Unfortunately, many researchers feel that this definition is too abstract for students in high school or below. In order to fully grasp this definition, students must have a fairly firm understanding of set theory. Most junior high and high school students, however, have had little exposure to set theory, and as a result, use of words such as set and subset may add to their confusion. In addition, students must also deal with the idea of infinity and, more perplexing, what it means to have an infinite set. Researchers contend that while students are able to reproduce the ordered pair definition formally, they seem to ignore the definition in application and practice and instead default to their own intuitive understanding [18]. In this manner, there is a disconnect between this definition and the adolescent's concept of function.

In her article about the formation of the view of functions as objects, Anna Sfard asserts that this view, called a structural conception, is important for mathematicians in the sense that it makes cognitive processes efficient [19]. Sfard uses the term operational conception to refer to the view of a function as a process and argues that students should first be taught to see functions as operations so that they will more naturally develop a sense of functions as objects. She contends that students should not be expected to obtain a structural understanding of functions until they reach higher-level theory in which such an understanding is necessary [19]. While this issue is debateable, the importance of the object conception of function is undeniable. Leading students to an object understanding of function is the ultimate goal of function understanding, but Sfard argues that students will not be able to completely grasp the concept of functions as object until they use functions as such in their course work.

## 4 Textbook Analyses

Textbooks often serve as an authority to the students they serve. Many teachers use their students' textbooks as their primary resource. They develop their lesson plans directly from the exercises and activities within the texts, and often, some of the teachers' own understanding of the concepts they teach are derived from these books. Even when teachers do not teach directly from the book, their students still have a copy of the text for reference. When they encounter a concept that they do not know or cannot remember, it is likely that they turn to the glossaries of their books for a definition. Under such conditions, the way information is presented within each book is vitally important to the students' understanding of the concepts.

Textbook authors face a dilemma when writing about concepts as complex as the function. When developing their presentation of the concept, they must consider the mathematical maturity of their audience without losing sight of the core mathematical principles involved. They must also evaluate the pedagogical issues that underlie all of education. Should they introduce a specific example first and then develop outward to a general definition? Or should they clearly define the concept first and then list specific examples? In all of these matters, which subtleties will help the most students?

Nine texts ranging in level from pre-algebra through college mathematics were analyzed for this study to determine how authors present the concept of function to students of varying mathematical maturity. Four of these texts are pre-algebra and algebra books, all published between 1990 and 1999. Calculus books published between 1994 and 2002 account for three of the texts, and a discrete mathematics text makes eight. Also, the handbook for the TAKS (Texas Assessment of Knowledge and Skills) exit level mathematics test is analyzed in order to understand what the state expects of students.

While traces of the concept are taught as early as kindergarten, students in the United States are not usually introduced to the mathematical use of the word function until they enter pre-algebra or algebra when they are somewhere between 12 and 15 years old. At this point in their mathematical careers, students will begin to encounter variables and equations for the first time. They will first begin to classify and describe relationships between variables. While the algebra curriculum is fairly uniform across textbooks, the manner in which the material is presented varies greatly from book to book. Each of the four algebra textbooks studied here emphasizes different concepts, and with respect to functions, gives students different perceptions of the subject.

As stated previously, graphs are often introduced to students before the concept of function. In the 1998 Glencoe *Algebra 1: Integration, Applications, Connections* [2], functions are introduced directly as graphs. The textbook authors take the concept of graphs, which are familiar to students, and build the function idea on it. The word function is defined as

> ". . . a relationship between input and output. In a function the output depends on the input. There is exactly one output for each input" [2].

The authors immediately illustrate the concept with a graph and then foreshadow the idea of sets of ordered pairs and relations. Over two hundred pages after its initial definition in the 748 page, two volume set of books, a function is redefined as "a relation in which each element of the domain is paired with exactly one element of the range" [2]. Later, functions are addressed again within the context of describing linear, quadratic, and exponential functions. For each of these sections, the authors place a heavy emphasis on graphical representation. At this entry level text into understanding the function concept, the authors chose to focus on the concrete, visual representation rather than the more abstract notion of functions as sets of ordered pairs.

Some educators believe that the discovery method of teaching is the most effective teaching approach. They believe that helping students to explore a concept so that they will discover it on their own rather than explicitly teaching the topic allows students to more successfully internalize what they are learning. Some textbooks were written with this philosophy in mind. HRW's *Algebra: Integrating* [17] is one such text with respect to its presentation of functions. The first mention of the word function occurs during its introduction of linear functions. The book describes a linear equation and explains that it is a linear function because one of the variables depends on another. It goes on to "preview" several different types of functions, including step, exponential, and quadratic, without ever specifically defining the term function. While this text digs deeper into non-traditional functions like absolute value, integer, and piecewise functions in a way that gives students a broader understanding of the concept, the book does not actually define function until towards the end of the book when it gives the ordered pair definition. While it does elaborate on the function machine concept of function, it does not explicitly make connections between different representations of function.

The 1999 Glencoe *Pre-Algebra* book [9] takes a more straightforward approach to its introduction of function. The text covers slightly different material than the other algebra books since it is intended for younger students, but towards the end it begins to discuss the function concept. First it introduces relations as a set of ordered pairs, and calls the set of all the first coordinates the domain and the set of all the second coordinates the range. Functions are then presented as a special type of relation "in which each element of the domain is paired with exactly one element in the range" [9]. The authors then move into a lengthy discussion of graphical representations of different functions and specifically focus on scatter plots and linear relations. One of the most notable aspects of this book is the way that it attempts to continue the ordered pairs concept beyond one isolated part of the text. Even in its discussion of equations, it states, "Since the solutions of an equation in two variables are ordered pairs, such an equation describes a relation" [9]. Rather than abandoning the relation concept after a quick cursory introduction, the authors attempt to make connections for their young students that will help them have a deeper understanding of functions.

A similar approach to introducing functions was taken by the 1990 Heath *Algebra I* [3] textbook. Relations appear first and are established as a set of ordered pairs. The authors then demonstrate several different representations of relations including lists of pairs, tables, and graphs. Unlike the other algebra texts, this book details these differing representations of the relation and, by extension function concepts, so that students are encouraged to connect and reconcile them as different facets of the same idea. This text also defines functions directly from the relation idea. It explains,

> "A relation is a function if and only if each first component in the relation is paired with exactly one second component" [3].

With its use of the expression "if and only if," this definition may be too complex for young students who have not yet developed an understanding of the nuances of definitional language. While it does not elaborate on the concept as a process or leave room for the extension of the function idea to entities other than numbers, this text does provide a straightforward function definition.

On the whole, the algebra books studied for this paper presented the function in a basic, simple manner. Each book tended to describe the idea from only one perspective, often tailoring their emphasis to the aspect that would be most useful later in the text. Some may see this technique as inadequate, however, it can be argued that providing too much information about the complexity of functions may serve only to confuse and frustrate students, especially at an early level of understanding. In fact, some, including Anna Sfard, whose research will be discussed later, contend that gradual instruction beginning with a lower level of expectation for understanding will actually lead to a better understanding of the subtleties in the future. [19]

The TAKS test determines whether or not students in Texas graduate, and within the public education system, teachers in Texas feel pressure to focus their instruction solely on state-mandated curriculum. For many students, the perception of the function presented by the TAKS test determines their understanding of the concept. While algebra marks the beginning of the study of functions, the TAKS standards mark what students are expected to know about functions by the end of their high school careers. Function-related concepts are addressed by four of the ten TAKS objectives for the exit level mathematics test. This percentage of objectives demonstrates the high level of attention that the Texas Education Agency (TEA) feels the topic deserves [1].

The TAKS study guide was developed by the TEA as a resource primarily for students and their parents [1]. The guide is designed specifically for struggling students and is free to students who have previously failed the test. The book collectively addresses the test objectives by giving explanations, practice questions, and answer keys.

Objective one states,"The student will describe functional relationships in a variety of ways" [1]. This section of the study guide introduces functions and defines a function as a set of ordered pairs in which "each x-coordinate is paired with only one y-coordinate" [1]. More significant than the actual definition is the stress that the objective places on the student's understanding that functions can be represented in multiple ways. The study guide immediately gives multiple examples of different representations including a table, an input/output function machine, a list of ordered pairs, a graph, and a word problem describing a functional relationship.

Objective two concerns the students' understanding of the properties and attributes of functions. It deals with parent functions, domains and ranges, correlation of scatterplots, and representing patterns with algebraic expressions. Objective three focuses on linear functions and objective five deals with quadratic and other nonlinear functions.

It is evident that the TEA wants students to have as broad an understanding of functions as possible. The guide, however, never encourages students to think of functions outside of the numerical realm nor does it expose them to non-traditional functions like piecewise functions or the absolute value function. The addition of such material might serve to confuse students more than it would actually enhance their understanding of function. In a text designed to make concepts as simple as possible, the authors may have found such material inappropriate.

While the TAKS test measures the success of students in mastering high school mathematics in Texas, the study of calculus marks the beginning of advanced mathematical study. Many students never take a course in calculus, and those who do enter a course do not usually do so until their last year of high school or during college. By the time they reach calculus, most students have been exposed to functions for several years. The study of calculus, however, sheds new light on functions as the ideas of differentiation and integration clarify the notion of functions as objects. The three texts studied for this paper each use different techniques to present the concept, but while the algebra books focused on functions as either graphs or ordered pairs, each of the calculus texts take more of an action approach and refer to functions as rules.

In Edwards and Penney's *Calculus* [5], the authors explain that understanding the relationships between variables is often crucial in order to mathematically analyze geometric and scientific situations. They state that these relationships can often be expressed as formulas in which one variable is a function of another and write,

> "A real-valued function f defined on a set D of real numbers is a rule that assigns to each number x in D exactly one real number, denoted by f(x)" [5].

While this definition leaves room for a broad, process-oriented "rule" view of function, the book focuses mainly on the formula idea. This trend is natural considering that much of the time involved in a first year calculus course revolves around learning how to differentiate and integrate functions in their formula form.

In the quintessential reform calculus text, *Calculus* [7], the authors hint at two different function definitions, both at the very beginning of the book. The first definition is basically the ordered pair concept; however, the authors avoid using set notation or vocabulary. They write, "One quantity, H, is a function of another, t, if each value of t has a unique value of H associated with it" [7]. This definition sufficiently depicts the arbitrary nature of function pairings, but it fails to capture the idea of functions as entities. In the second part of the definition, the authors tell readers to "think of t as the input and H as the output" [7]. The input/output concept reflects the process concept of function as it implies that somehow the input is manipulated to produce the output. The book then gives three examples of function representations: tables, graphs, and formulas.

Arnold Ostebee and Paul Zorn's *Calculus: From Graphical, Numerical, and Symbolic Points of View* [13], takes a similar discovery learning approach as the HRW algebra book. The text begins by presenting and discussing five different examples of functions: an equation, a relationship expressed in words, a piecewise function, a graph, and a table. The definition is finally presented only after examples are discussed at which point the authors write, "A function is a rule for assigning to each member of one set, called the domain, one member of another set, called the range" [13]. While this definition and the examples help to dispel many common function misconceptions, the text does not mention the set of ordered pairs definition nor does it attempt to reconcile the entity nature of a function with the idea that functions are rules.

Generally, calculus texts seem to focus on the idea of functions as rules or formulas. Interestingly, while calculus texts are designed for students of greater mathematical maturity, they avoid using ordered pairs and set notation even more notably than the algebra texts. While they probably avoid this notation for the sake of simplicity and to prevent confusion over vocabulary, calculus books are capable of doing this due to the way functions are used in the course. In the author's opinion, however, this discrepancy in definition between levels of mathematics study may cause an even greater disconnect in students' understandings of the function concept. Students may not see that the idea of function that they studied in algebra is actually the same entity as the concept of function that they learn about in calculus.

University students studying mathematics should achieve a broader view of functions. As they enter more challenging courses, they will be required to think and learn about functions in new ways, and ideally they will begin to connect the bits of isolated definitions that they have absorbed throughout their mathematical careers thus far. In the abstract math text *Mathematics: A Discrete Introduction* [16], Edward Scheinerman explains that, "Intuitively, a function is a 'rule' or 'mechanism' that transforms one quantity into another." He goes on to state that this text will "develop a more abstract and rigorous view of functions" [16] at which point he defines functions in terms of relations:

> "A relation f is called a function provided (a,b) in f and (a,c) in f imply b=c" [16].

This definition serves as an umbrella definition for all the other function representations before it. This definition packs a lot of meaning into a concise statement, but is difficult to understand without knowledge of mathematical terminology. In the author's opinion, students need exposure to various function representations in order to gain the intuitive understanding of functions necessary to make sense of such technical definitions.

## 5 Common Misconceptions

As the human brain attempts to understand new information, it also works to categorize and synthesize it with its existing knowledge base [21]. Throughout these acts of understanding, however, it is the author's opinion that students have the tendency to develop incorrect assumptions and conceptions about new ideas, especially with respect to complex concepts like functions. Several misconceptions obscure students' perceptions of functions.

Students often believe that functions must be continuous and differentiable in order to truly be considered functions. Anna Sierpinska explains this phenomenon: "As, normally the first examples of functions encountered by a student are everywhere continuous, non-differentiable in at most finite number of points, built up of one piece of a curve in the graphical representation, given by a single formula; such rare functions constitute, in the student's mind, the prototype of a function" [21]. Students, when faced with a function unlike the examples that they have been taught, will hesitate before accepting it as a function. Many students base their understanding of what a function is on their reserve of examples rather than the definitions they have been taught. For her study on students' function conceptions, Professor Anna Sfard evaluated 22-25 year old university students who had completed a foundational mathematics course covering introductory set-theory, algebra, and calculus. She found that many students were unfamiliar with piecewise functions and tended to view expressions defined by cases over different sections of domain as multiple functions rather than one [19]. In a questionnaire, she asked students to state whether the following example describes a function (x and y are natural numbers):

If x is an even number then $y = 2x + 5$
Otherwise (x is an odd number) $y = 1 - 3x$.

Only 50% of the students surveyed believed that this proposition describes a function even though many willingly suggested that it describes two separate functions. Some students also have difficulty accepting that a graphical representation of a discontinuous curve represents one function rather than several. Such perplexity is natural considering that even great mathematicians like d'Alembert did not accept split-domain functions [19]. In fact, the idea of a nowhere-differentiable function was so new and disputed that in 1893 mathematician Hermite declared, "I turn away with fright and horror from this lamentable evil of functions which do not have derivatives" [8].

Similarly, some mathematicians felt that functions must have a rule or algorithm behind them, leading to a second common misconception. Sierpinska relates, "algebraic skill accompanied by the belief in the power of algebra to solve almost automatically many kinds of problems, may be an impediment to understanding the general concept of function" [21]. Students who are accustomed to working with functions in algebraic expression notation may have difficulty believing that functions may be constructed arbitrarily. Just as mathematicians prior to Dirichlet did not accept the concept of a function as an arbitrary correspondence, students fail to understand the notion because of its abstract nature [8]. Such a function was not conjured out of necessity or great usefulness to the world outside of mathematics, but its development was of vital importance to those trying to understand and develop a deeper understanding of functions. In 1899 Poincaré expressed his frustration with the new developments in thought about function: "In former times when one invented a new function it was for a practical purpose; today one invents them purposely to show up defects in the reasoning of our fathers and one will deduce from them only that" [8]. While arbitrary functions may seem impractical, their existence highlights the deeper concept that algebraic expressions are only means of describing functions. Students, however, tend to see formulas as things in themselves rather than as representations of other entities [19]. In her study, Sfard presented students with the following true or false statements:

1.) Every function expresses a certain regularity (the values of x and y cannot be matched in a completely arbitrary manner).
2.) Every function can be expressed by a certain computational formula.

Only 6% of the students responded that both statements were false, which suggests that the majority of the students believe that there must be an algorithm corresponding to the function in order for it to be valid. Sfard suggests that "Not only do the students seem to think about functions in terms of process rather than of permanent objects, but they also believe that the processes must be algorithmic and reasonably simple" [19]. This desire for algorithmic simplicity also accounts for students' hesitancy to accept discontinuous and non-differentiable functions.

Sierpinska suggests that introducing students to one function represented by two different formulae may help students to discriminate between the function itself and the "analytic tools" used to describe it [21]. Unfortunately, students have a difficult time accepting that algorithms that look different, but produce the same values, are actually the same function. When presented with the algorithmically different functions:

$$\mathbb{N} \text{ to } \mathbb{N}\text{: } f(x) = x^2$$
and the recursively defined
$$g(0) = 0,\ g(x+1) = g(x) + 2x + 1$$

students had difficulty believing that they were equivalent even though they produce the same values [19]. This misconception is tied to students' unfamiliarity with the ordered pair definition of function. They cannot comprehend that two different algorithms that produce the same set of ordered pairs are in fact the same function because they cannot separate their understanding of functions as rules.

When students are first introduced to the function concept, they are often taught the vertical line test to check if a graph is a function. The test instructs students to slide a vertical line across the graph they are testing. If the line ever crosses two points of the graph at once, the graph is not a function. This technique is a helpful aid for young students who have trouble comprehending what it means for a function to be unique, however, it does not prevent students, even when they are older, from confusing domain and range values. In fact, many students overcompensate and hold the misconception that functions must have a one-to-one correspondence. Ed Dubinsky and Guershon Harel write, "It is extremely common for subjects at all levels to have difficulty with this uniqueness condition and confuse it with the notion of one-to-one" [4].

This phenomenon may be connected to students' lack of ability to visualize functions. While they may remember and understand the vertical line test, the test is of little use without a graph on which to employ it. Eisenberg asserts that ". . . students have a strong tendency to think of functions algebraically rather than visually" even though visualization can be extremely helpful [6]. He argues that students resist visual representations because visual processing requires higher level skills than analytical processing [6]. While analytical processing often involves only one degree of abstraction from an expression to concrete numbers, visualization requires the ability to evaluate an expression, develop trends, and transfer all the knowledge into a visual format. Interestingly, Alexander Norman, in his study of teachers' knowledge of functions, found that in contrast to the students, teachers tend to rely on and prefer graphical representations of functions, especially when determining whether or not a particular expression is a function [11]. He explains that teachers have typically had a high degree of exposure to different types of functions. They are comfortable with standard graphs and know how to answer a variety of questions from these graphs [11]. For students, who have had considerably less exposure to standard graphs, visualization can be intimidating and can feel quite foreign. Increased exposure to graphs and visual representations may help students overcome this reluctancy.

These misconceptions are simply the manifestation of students' incomplete understanding of the function concept. While misunderstanding is to be expected as students learn new concepts, the goal of educators is to help students attain the highest possible level of understanding in the shortest amount of time. Careful consideration of students' cognitive processes and capabilities, as well as attention to outside factors that contribute to misunderstanding, are essential to the achievement of this goal.

# 6 Function Module Proposal

From the textbook analyses, it is evident that students receive a varied view of the nature of functions as they progress through middle and high school. As discussed previously, functions can be understood on three levels of abstraction: as actions, as processes, and as objects. Studies performed on college age students reveal that few have progressed beyond the action understanding and even fewer beyond the process understanding [4], [19]. Such a lack of comprehension of the abstract indicates the value of developing new ways to help students understand functions.

A possible way to approach this problem is to develop a short lesson module specifically designed to teach students about the three conceptions of functions. The following module was designed to consist of five hour-long sessions that fill the gaps in understanding. It is intended to be taught to students near the end of their algebra II course. At this point, they will be quite familiar with functions. In fact, many of the students will probably be able to produce accurate definitions and examples of functions, but as with most topics, they probably will not have thought much about the subtleties of functions or about their own thought processes. The author's hope is that a new look at a familiar subject will help students make connections and piece together ideas that have previously been separated in their minds.

Day 1: Overview of all three abstractions
The first day of the function module begins with an interactive discussion in which students are asked the question: What is a function? Students are encouraged to think back to all their previous courses to produce as many different definitions as they can. The teacher makes a running list of these definitions on the blackboard, leaving space next to each item for further notes. The teacher serves as the facilitator for this discussion, but does not make qualitative judgements about any of

the definitions produced. Next, the students are challenged to produce examples of different types of functions. The teacher encourages the students to think creatively to produce unique examples. In total this discussion lasts approximately half an hour, at which point the teacher introduces the three different levels of abstraction of the function concept. The teacher then explains that in mathematics, an abstraction is an idea that has been generalized from another concept so that it is less dependent on real world objects and closer to a mathematical construct. He or she then goes on to talk through each abstraction using the function machine idea as the base to which he or she ties each abstraction (See Figure 1). The action process is associated with a box on which a specific formula or algorithm is explicitly given. The instructor emphasizes that students often start out understanding functions only as specific rules or formulas and that this is typical of an action conception. The process conception is represented by the same function box, however, for those who have reached the process level of understanding, the box no longer has a specific algorithm written on it, but instead, the process that takes place within the function machine is unknown. Lastly, the teacher presents the students with the final conception, the object conception. The visual for this conception actually excludes the machine altogether. The instructor explains that at the most abstract level, functions are simply arbitrary pairings. At this point the instructor turns the class' focus back to the original definitions on the board and asks the students to identify which definitions fit under each abstraction. After further discussion, the teacher chooses several of the examples brainstormed earlier and, for homework, asks students to classify each of these examples based on the abstraction to which they are most related: action, process, or object.

Day 2: Action conception in depth

The second day of the function module is devoted to the action conception. The instructor asks the students to identify the action examples from the previous night's assignment. Using these examples as a springboard, the rest of the class is spent reviewing functions in their graphical and algebraic forms and talking about action-related concepts like finding intercepts, maxima, and minima. While much of the material covered in this class is review, students are encouraged to use this time to cement and refine their base knowledge. For homework, students are assigned review problems from various sections of their textbook dealing with functions and different applications of functions. These may include problems dealing with finding function values for given inputs on graphs or with equations.

Day 3: Process conception in depth

During the third day of instruction, students examine what it means to view functions from a process point of view. The teacher begins the class by referring to the examples from day one that have been deemed process examples. The overlap and subtle distinction between action and process examples may result in difficulty for the student in identifying these. The instructor discusses these difficulties and tries to help students understand that the significance of the process conception lies mainly in the students' ability to broaden their understanding of what functions can be. The teacher then spends some time discussing processes outside the type of algebraic manipulations typically studied by mathematics students. These functions may include recursive formulas, computer algorithms, and real-life conditional statements. For example, the students The last part of the period is devoted to reviewing transformations on generalized linear, quadratic, and exponential functions. Students are taught that the ability to perform these transformations is a significant step in the abstraction of functions. Students realize that they are doing more than plugging numbers into equations, but rather, they are able to perform processes on the functions themselves. For homework, the teacher

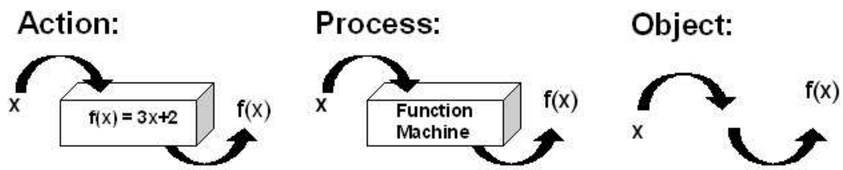


Figure 1: The function machine idea as it extends to the action, process, and object conceptions of function

selects and assigns several problems dealing with this conception. These may include problems in which the students are given a table of inputs and outputs and must determine the process that the input underwent to become the output. Students may also be asked to produce inputs and outputs from a graphical representation of a function. Some of the problems should also be devoted to transformations of linear, quadratic, and exponential functions.

Day 4: Object conception in depth

The fourth day of this module is concerned with building a function definition from the relation concept. While the topic of relations should be review for the students, many will probably have only a vague recollection of it. The teacher defines relations and then explains that functions are a specific kind of relation in which each x coordinate is paired with only one y. At this point, the instructor also introduces the concepts of onto and one-to-one and discusses in depth about domain and co-domain. The instructor introduces students to the graphical representations of these pairings consisting of circles to represent the two sets between which the mapping occurs and arrows connecting each pair of elements. The instructor is careful to emphasize that these pairs do not necessarily have to have a rule that determines them, but that those that do can still be represented in this fashion. For students the equation $f(x) = 3x + 2$, where the domain is the real numbers, may seem impossible to represent as a set of ordered pairs. Astute students will realize that simply listing a few ordered pairs that satisfy this equation will not actually accurately depict the function. The instructor, however, describes that $(x, 3x + 2)$ actually represents the same function in ordered-pair notation. The teacher also spends some time discussing inverse functions. Students are encouraged to make observations about what types (i.e. one-to-one, onto) of functions have inverses. For homework, the students are assigned several problems dealing with sets of ordered-pairs and questions of whether functions are onto, one-to-one, and if they have inverses.

Day 5: Special functions and wrap-up

The final day in this module is spent discussing special examples of functions that highlight important aspects of the function concept. The instructor begins by showing students examples of functions whose inputs and outputs are not numbers. An example of such a function is a table with a list of the people in the class and their favorite colors. Next, the teacher introduces the Dirichlet function as a means of showing students that functions can be discontinuous. The students will likely have difficulty understanding this function particularly due to the fact that it cannot be visualized or graphically displayed. Lastly, the teacher presents an example of two different processes that represent the same function. This example illustrates how the difference between the process and object conceptions can completely alter one's view of a specific function. While someone with a process conception would view these functions as different, people with object conceptions would readily agree that they are the same. The instructor ends the module by allowing students to discuss things that confuse them about functions. For homework, the students must produce and express a function from the action, process, and object conception.

# 7 Conclusion

Functions play an important role in all of mathematics. They can be found in every mathematics course from pre-algebra through graduate studies. The development of the idea now known as function has a long history only matched in complexity by the many different ways that functions can be represented. Surface familiarity with concepts of such complexity, however, often leads to misunderstanding and misconceptions. Researchers surmise that students must learn such concepts slowly and with careful attention to each level of understanding before new abstractions can be grasped. Teachers and textbooks tend to send students different signals regarding functions and many students, upon reaching college, have incomplete understandings of functions. These have often been constructed by students from the examples with which they are most familiar rather than the definitions that they have been taught. While developmental factors hinder young students from fully grasping the concept, the author believes that with careful and deliberate instruction high school juniors can understand the idea of function on an abstract level.

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