

TWO QUESTIONS ON CONTINUOUS MAPPINGS

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ABSTRACT. In this paper, it is shown that a mapping from a sequential space is continuous iff it is sequentially continuous, which improves a result by relaxing first-countability of domains to sequentiality. An example is also given to show that open mappings do not imply *Darboux*-mappings, which answers a question posed by Wang and Yang.

A mapping $f : X \rightarrow Y$ is continuous if $f^{-1}(U)$ is open in X for every open subset U of Y . In [5], Wang and Yang give some interesting generalizations of continuous mappings.

Definition 1. Let $f : X \rightarrow Y$ be a mapping.

(1) f is called a *sequentially continuous mapping* if for every sequence $\{x_n\}$ converging to x in X , $\{f(x_n)\}$ is a sequence converging to $f(x)$ in Y .

(2) f is called a *Darboux-mapping* if $f(F)$ is connected in Y for every connected subset F of X .

It is a standard result that every continuous mapping is both a sequentially continuous mapping and a *Darboux*-mapping, but neither sequentially continuous mappings nor *Darboux*-mappings need to be continuous[5, 1]. However, the following result is well known(see [1], for example).

Theorem 2. Let $f : X \rightarrow Y$ be a mapping, where X is first countable. If f is sequentially continuous, then f is continuous.

Take the above theorem into account, the following question naturally arises.

Question 3. Can first-countability of X in Theorem 2 be relaxed?

On the other hand, Wang and Yang posed the following question in [5].

Question 4. Does there exist an open mapping $f : X \rightarrow Y$ such that f is not a *Darboux*-mapping?

In this paper, we investigate the above Questions. We show that we can relax first-countability of X in Theorem 2 to sequentiality, which gives an affirmative answer for Question 3. We also give an example to answer Question 4 affirmatively.

Throughout this paper, all spaces are assumed to be T_1 . The set of all natural numbers is denoted by \mathbb{N} . A sequence is denoted by $\{x_n\}$, where the n -th term is x_n . Let X be a space and $P \subset X$.

Definition 5. Let X be a space.

(1) A sequence $\{x_n\}$ converging to x in X is *eventually in P* if $\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$.

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(2) Let $x \in X$. A subset P of X is called a sequential neighborhood of x if every sequence $\{x_n\}$ converging to x is eventually in P , and a subset U of X is called sequentially open if U is a sequential neighborhood of each of its points.

(3) X is called a Fréchet-space[2] if for every $P \subset X$ and for every $x \in \overline{P}$, there exists a sequence $\{x_n\}$ in P converging to the point x .

(4) X is called a sequential space[4] if for every $A \subset X$, A is closed in X iff $A \cap S$ is closed in S for every convergent sequence S (containing its limit point) in X .

(5) X is called a k -space[3] if for every $A \subset X$, A is closed in X iff $A \cap K$ is closed in K for every compact subset K of X .

Remark 6. It is well known that first countable spaces \implies Fréchet-spaces \implies sequential spaces \implies k -spaces (see [4], for example).

Lemma 7. Let X be a space. The following are equivalent.

(1) X is a sequential space.

(2) For every non-closed subset F of X , there exists a sequence $\{x_n\}$ in F converging to x for some $x \in X - F$.

(3) Every sequentially open subset of X is open in X .

Proof. (1) \implies (2): Let F be a non-closed subset of X . Since X is a sequential space, there exists a sequence S converging to a point $x \in X$ such that $F \cap S$ is not closed in S . It is clear that $F \cap S$ is infinite. So there exists a subsequence $\{x_n\}$ of S such that $x_n \in F$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x . Put $L = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. If $x \in F$, then $x \in F \cap S$, thus $F \cap S$ is closed in S , a contradiction. So $x \in X - F$.

(2) \implies (3): Let U be a sequentially open subset of X . If U is not open in X , that is, $X - U$ is not closed in X , then there exists a sequence $\{x_n\}$ in $X - U$ converging to x for some $x \in U$. Thus U is not a sequentially open subset of X , a contradiction.

(3) \implies (1): If X is not a sequential space, then there exists a non-closed subset F of X such that $F \cap S$ is closed in S for every convergent sequence S in X , where S containing its limit point. Since $X - F$ is not open in X , $X - F$ is not a sequentially open subset of X , so there exist a point $x \in X - F$ and a sequence $\{x_n\}$ converging to x such that $\{x_n\}$ is not eventually in $X - F$. Thus there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \notin X - F$ for all $n \in \mathbb{N}$, that is, $y_n \in F$ for all $n \in \mathbb{N}$. Put $S = \{y_n : n \in \mathbb{N}\} \cup \{x\}$, then $x \notin F \cap S$. Note that x is a cluster point of $F \cap S$, $F \cap S$ is not closed in S . This is a contradiction. \square

Theorem 8. Let $f : X \longrightarrow Y$ be a mapping, where X is sequential. If f is sequentially continuous, then f is continuous.

Proof. Let $f : X \longrightarrow Y$ be sequentially continuous and let U be an open subset of Y . Since X is a sequential space, it suffices to prove that $f^{-1}(U)$ is a sequentially open subset of X from Lemma 7.

Let $x \in f^{-1}(U)$ and $\{x_n\}$ be a sequence converging to x . Since $f : X \longrightarrow Y$ is sequentially continuous, $\{f(x_n)\}$ is a sequence converging to $f(x) \in U$. Note that U is an open neighborhood of $f(x)$, there exists $k \in \mathbb{N}$ such that $f(x_n) \in U$ for all $n > k$. So $x_n \in f^{-1}(U)$ for all $n > k$, thus $\{x_n\}$ is eventually in $f^{-1}(U)$. This proves that $f^{-1}(U)$ is a sequentially open subset of X . \square

The above theorem improves Theorem 2 and gives an affirmative answer for Question 3. However, the following question is still open.

Question 9. Let $f : X \longrightarrow Y$ be a mapping, where X is a k -space. If f is sequentially continuous, is f continuous?

The following example answers Question 4 affirmatively.

Example 10. There exists an open mapping $f : X \longrightarrow Y$ such that f is not a Darboux-mapping.

Proof. Let $X = \mathbb{R}$ with the Euclidean topology and $Y = \mathbb{R}$ with the discrete topology, where \mathbb{R} is the set of all real numbers. Let $f : X \longrightarrow Y$ be the identity mapping. Then f is an open mapping because every subset of discrete space Y is open in Y . Notice that X is a connected space and $Y = f(X)$ is a discrete space, thus Y is not connected. So f is not a Darboux-mapping. \square

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