TWO QUESTIONS ON CONTINUOUS MAPPINGS

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Abstract. In this paper, it is shown that a mapping from a sequential space is continuous iff it is sequentially continuous, which improves a result by relaxing first-countability of domains to sequentiality. An example is also given to show that open mappings do not imply *Darboux*-mappings, which answers a question posed by Wang and Yang.

A mapping $f : X \longrightarrow Y$ is continuous if $f^{-1}(U)$ is open in X for every open subset U of Y. In [6], Wang and Yang give some interesting generalizations of continuous mappings.

Definition 1. *Let $f : X \longrightarrow Y$ be a mapping.*

(1) f is called a sequentially continuous mapping if for every sequence $\{x_n\}$ converging to x in X, $\{f(x_n)\}$ is a sequence converging to $f(x)$ in Y.

(2) f is called a Darboux-mapping if $f(F)$ is connected in Y for every connected subset F of X.

It is clear that every continuous mapping is both a sequentially continuous mapping and a *Darboux*-mapping, but either sequentially continuous mappings or *Darboux*-mappings need not to be continuous[6, 4]. However, the following result is well known(see [1], for example).

Theorem 2. *Let $f : X \longrightarrow Y$ be a mapping, where X is first countable. If f is sequentially continuous, then f is continuous.*

Take the above theorem into account, the following question naturally arises.

Question 3. *Can first-countability of X in Theorem 2 be relaxed?*

On the other hand, Wang and Yang posed the following question in [1].

Question 4. *Does there exist an open mapping $f : X \longrightarrow Y$ such that f is not a Darboux-mapping?*

In this paper, we investigate the above Questions. We can relax first-countability of X in Theorem 2 to sequentiality, which gives an affirmative answer for Question 3. We also give an example to answer Question 4 affirmatively. Throughout this paper, all spaces are assumed to be T_1. $\mathbb{N}$ denotes the set of all natural numbers. $\{x_n\}$ denotes a sequence, where the n-th term is x_n. Let X be a space and $P \subset X$. A sequence $\{x_n\}$ converging to x in X is eventually in P if $\{x_n : n > k\} \bigcup \{x\} \subset P$ for some $k \in \mathbb{N}$.

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Definition 5. *Let X be a space.*

(1) X is called a Fréchet-space[2] *if for every $P \subset X$ and for every $x \in \overline{P}$, there exists a sequence $\{x_n\}$ in P converging to the point x.*

(2) X is called a sequential space[5] *if for every $A \subset X$, A is closed in X iff $A \bigcap S$ is closed in S for every convergent sequence S (containing its limit point) in X.*

(3) X is called a k-space[3] *if for every $A \subset X$, A is closed in X iff $A \bigcap K$ is closed in K for every compact subset K of X.*

Remark 6. *It is well known that first countable spaces $\Longrightarrow$ Fréchet-spaces $\Longrightarrow$ sequential spaces $\Longrightarrow$ k-spaces.*

Let X be a space and $x \in X$. Recall $P \subset X$ is a sequential neighborhood of x if every sequence $\{x_n\}$ converging to x is eventually in P, and a subset U of X is called sequentially open if U is a sequential neighborhood of every of its points.

Lemma 7. *Let X be a space. The following are equivalent.*

(1) X is a sequential space.

(2) For every non-closed subset F of X, there exists a sequence $\{x_n\}$ in F converging to x for some $x \in X - F$.

(3) Every sequentially open subset of X is open in X.

Proof. (1) $\Longrightarrow$ (2): Let F be a non-closed subset of X. Then there exists a sequence $\{y_n\}$ converging to a point $x \in X$ such that $F \bigcap S$ is not closed in S, where $S = \{y_n : n \in \mathbb{N}\} \bigcup \{x\}$. It is clear that $F \bigcap S$ is infinite. So there exists a subsequence $\{x_n\}$ of $\{y_n\}$ such that $x_n \in F$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x. Put $L = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$. If $x \in F$, then $x \in F \bigcap S$, thus $F \bigcap S$ is closed in S, a contradiction. So $x \in X - F$.

(2) $\Longrightarrow$ (3): Let U be a sequentially open subset of X. If U is not open in X, that is, $X - U$ is not closed in X, then there exists a sequence $\{x_n\}$ in $X - U$ converging to x for some $x \in U$. Thus U is not a sequentially open subset of X, a contradiction.

(3) $\Longrightarrow$ (1): If X is not a sequential space, then there exists a non-closed subset F of X such that $F \bigcap S$ is closed in S for every convergent sequence S in X, where S containing its limit point. Since $X - F$ is not open in X, $X - F$ is not a sequentially open subset of X, so there exist a point $x \in X - F$ and a sequence $\{x_n\}$ converging to x such that $\{x_n\}$ is not eventually in $X - F$. Thus there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \notin X - F$ for all $n \in \mathbb{N}$, that is, $y_n \in F$ for all $n \in \mathbb{N}$. Put $S = \{y_n : n \in \mathbb{N}\} \bigcup \{x\}$, then $x \notin F \bigcap S$. Note that x is a cluster point of $F \bigcap S$, $F \bigcap S$ is not closed in S. This is a contradiction. $\square$

Theorem 8. *Let $f : X \longrightarrow Y$ be a mapping, where X is sequential. If f is sequentially continuous, then f is continuous.*

Proof. Let $f : X \longrightarrow Y$ be sequentially continuous and let U be an open subset of Y. Since X is a sequential space, it suffices to prove that $f^{-1}(U)$ is a sequentially open subset of X from Lemma 7.

Let $x \in f^{-1}(U)$. Whenever $\{x_n\}$ is a sequence converging to x. $f : X \longrightarrow Y$ is sequentially continuous, so $\{f(x_n)\}$ is a sequence converging to $f(x) \in f(f^{-1}(U)) \subset U$. Note that U is an open neighborhood of $f(x)$, there exists $k \in \mathbb{N}$ such that $f(x_n) \in U$ for all $n > k$. So $x_n \in f^{-1}f(x_n) \subset f^{-1}(U)$ for all $n > k$, thus $\{x_n\}$

is eventually in $f^{-1}(U)$. This proves that $f^{-1}(U)$ is a sequentially open subset of X. □

The above theorem improves Theorem 2 and gives an affirmative answer for Question 3. However, the following question is still open.

Question 9. *Let $f : X \longrightarrow Y$ be a mapping, where X is a k-space. If f is sequentially continuous, is f continuous?*

The following example answers Question 4 affirmatively.

Example 10. *There exists an open mapping $f : X \longrightarrow Y$ such that f is not a Darboux-mapping.*

Proof. Let $X = \mathbb{R}$ with the Euclidean topology and $Y = \mathbb{R}$ with the discrete topology, where $\mathbb{R}$ is the set of all real numbers. Let $f : X \longrightarrow Y$ be the identity mapping. Then f is an open mapping because every subset of discrete space Y is open in Y. Notice that X is a connected space and $Y = f(X)$ is a discrete space, thus Y is not connected. So f is not a *Darboux*-mapping. □

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