COMBINATORICS OF THE FIGURE EQUATION ON DIRECTED GRAPHS

TAYLOR E. COON

Abstract. There are many ways of computing a graph's characteristic polynomial; a lesser known method is a formula called the figure equation. The figure equation provides a direct link between a graph's structure and the coefficients of its characteristic polynomial. This method does not use determinants, but calculates the characteristic polynomial of any graph by counting cycles. We give a complete combinatorial analysis of four increasingly complex graph families, which yields closed formulae for their characteristic polynomials.

1. Introduction

Every finite directed graph has a characteristic polynomial. Originally, the characteristic polynomial was believed to be a complete invarient, or unique to a graph and all its isomorphisms. Though there are cases where two structurally different graphs share the same characteristic polynomial, characteristic polynomials are highly studied because they provide much information about a graph in a concise form. They are useful in chemistry, economics, and physics, among others. For example, a graph's *spectrum*, the roots of its characteristic polynomial, has significance in atomic structure. For more applications of the characteristic polynomial, see [2] or chapter 5 of [4]. There are multiple ways to determine the characteristic polynomial of a graph with n vertices; the most common is to create the graph's *adjacency matrix*, an $n \times n$ matrix where the ij^{th} entry corresponds to the number of edges with source i and target$j$. The characteristic polynomial of the graph is the characteristic polynomial of its adjacency matrix. The *figure equation*, given in [2], is a relatively unfamiliar method of calculating characteristic polynomials. The figure equation calculates each coefficient of a graph's characteristic polynomial by considering the set of *linearly directed subgraphs* of a corresponding length. A linearly directed subgraph of length i consists of i vertices and i edges such that each vertex has indegree and outdegree 1.

In section 2, we explain the figure equation. Then, in sections 3 through 6, we use the figure equation to find closed formulae for four increasingly complex infinite families of graphs: the linear, cyclic, ladder, and dihedral graphs. These examples include *Cayley graphs*, the representation of a group G using generating set A. We label this type of graph $C(G, A)$, where the vertices represent group elements, and the edges represent the action of the generating set.

In section 7, we use these results to give examples of *isospectral* graphs and a *graph covering map*. We define two graphs to be isospectral if they share the same

Date: August 7, 2006.
Research supported by an NSF REU grant.
Research conducted at Canisius College under Dr. Terrence Bisson.

characteristic polynomial. We say that the graph X covers graph Y if there is some map $f : X \to Y$ that maps each vertex in X to a vertex in Y and gives a bijection between edges with sources $x \in X$ and $f(x) \in Y$. We then call f a graph covering map. If such a map exists, the characteristic polynomial of Y divides the characteristic polynomial of X See Theorem 9.3.3 of [3] for a proof.

2. The Figure Equation

The Figure Equation states that for any graph G with n vertices, the characteristic polynomial

$$\mathcal{X}(G) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_{n-1}x + c_n$$

such that for $1 \leq i \leq n$, the coefficient

$$c_i = \sum_{L \in \mathcal{L}_i} (-1)^{P(L)}$$

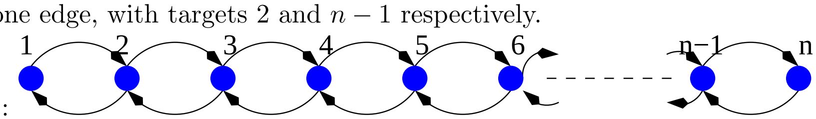
where $\mathcal{L}_i$ is the set of all linearly directed subgraphs of G and $P(L)$ is the number of *linearly directed cycles*, or the number of pieces in L. See Theorem 1.2 of [2] for a proof.

For some graphs, counting each $\mathcal{L}_i$ can be cumbersome. However, for other graphs and their families, the coefficients of the characteristic polynomials follow specific patterns, and we have closed formulas to calculate them.

3. The Linear Graph

The first example of a graph family with a closed coefficient formula is the linear graph T_n, a graph with n vertices such that each vertex $i \in \{2, 3, \ldots, n-1\}$ is the source of edges with targets $i-1, i+1$, and vertices 1 and n are each the source of only one edge, with targets 2 and $n-1$ respectively.

T_n :



By the figure equation,

$$\mathcal{X}(T_n) = x^n + t_1 x^{n-1} + t_2 x^{n-2} + \cdots + t_{n-1}x + t_n$$

and each t_i is determined by examining the set $\mathcal{L}_i$.

Lemma 1. *In the graph T_n, each $L \in \mathcal{L}_i$ consists of $\frac{i}{2}$ disjoint 2-cycles $(i \leftrightarrow i+1)$.*

Proof. Of the n vertices of T_n, numbered $1, 2, \ldots, n$, let $a \in L \in \mathcal{L}_i$ for some $1 \leq i \leq n$ be the smallest vertex of T_n in L (i.e. $a \in L$, but $\{1, \ldots, a-1\} \notin L$). Each vertex of L has indegree and outdegree 1, and as $a - 1 \notin L$, the only possible target and source for edges of vertex a is vertex $a+1$. Thus,

$$a \in L \Longrightarrow a + 1 \in L$$

Therefore, vertex $a+1$ has edges with target and source a, so $a+1$ is not connected to $a+2$ in L.

Next, consider vertex $b \in L \in \mathcal{L}_i$ such that $a + 2 \leq b \leq n-1$ and b is the closest vertex to $a+1$ in L. By the same argument,

$$b \in L \Longrightarrow b + 1 \in L$$

and $b+1$ is not connected to $b+2$ in the subgraph L.

This method can be continued to show that every vertex in L has an edge to and from exactly one adjacent vertex, creating a set of disjoint 2-cycles ($i \leftrightarrow i+1$) that make up L. As each $L \in \mathcal{L}_i$ has exactly i vertices, L must have exactly $\frac{i}{2}$ disjoint 2-cycles. □

Corollary 1. *For the characteristic polynomial $\mathcal{X}(T_n)$, the coefficient $t_{2j+1} = 0$ for $0 \leq j \leq \frac{n-1}{2}$.*

Proof. By Lemma 1, for any $0 \leq j \leq \frac{n-1}{2}$, $\mathcal{L}_{2j+1} = \emptyset$ as $2j+1$ vertices require $\frac{2j+1}{2}$ disjoint 2-cycles, which is not a whole number. Therefore, by the figure equation,

$$t_{2j+1} = \sum_{L \in \mathcal{L}_i} (-1)^{P(L)} = 0$$

□

To find the even coefficients, we can count $|\mathcal{L}_{2m}|$ by examining the ways to place m disjoint 2-cycles on the graph T_n. For similar combinatorial proofs and applications, see [1].

Theorem 1. $|\mathcal{L}_{2m}| = \binom{n-m}{m}$ *for* $1 \leq m \leq \frac{n}{2}$

Proof. $|\mathcal{L}_{2m}|$ is the number of ways to place m disjoint 2-cycles on T_n. First, we denote a 2-cycle by its lesser vertex (i.e. denote ($i \leftrightarrow i+1$) by i). Then, each $L \in \mathcal{L}_{2m}$ corresponds to a set of vertices $\{x_1 < x_2 < \cdots < x_m\}$ such that $x_i - x_{i-1} \geq 2$ as the 2-cycles are disjoint. Note, $1 \leq x_1$ and $x_m \leq n-1$. Now, consider

$$L' = \{x_1, x_2 - 1, x_3 - 2, \ldots, x_m - (m-1)\} = \{x'_1 < x'_2 < x'_3 < \cdots < x'_m\}$$

such that $x'_i - x'_{i-1} \geq 1$. Note, $1 \leq x'_1$ and $x'_m \leq (n-1) - (m-1)$.

Therefore, each $L' \in \mathcal{L}_{2m}$ is a choice of the m lesser vertices from $n-m$ possibilities. Thus, $|\mathcal{L}_{2m}| = \binom{n-m}{m}$. □

Corollary 2. *The even coefficients,* $t_{2m} = (-1)^m \binom{n-m}{m}$

Proof. Each $L \in \mathcal{L}_{2m}$ is made up of m disjoint 2-cycles, so $P(L) = m$. Therefore,

$$t_{2m} = (-1)^m |\mathcal{L}_{2m}| = (-1)^m \binom{n-m}{m}$$

□

We now have a closed formula for the coefficients of the characteristic polynomial of T_n.

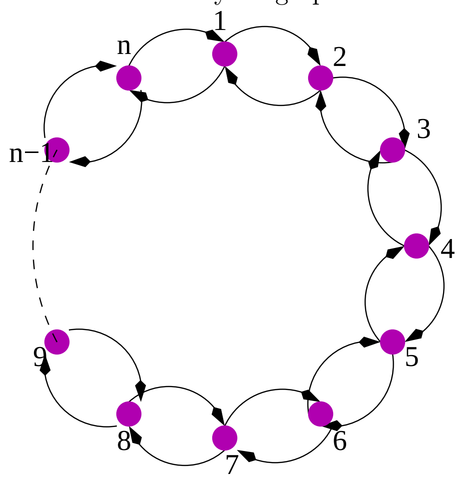
$$\mathcal{X}(T_n) = x^n + t_2 x^{n-2} + t_4 x^{n-4} + \cdots + t_{n-1} x + t_n$$

where

$$t_{2m} = (-1)^m \binom{n-m}{m}$$

4. The Cyclic Graph

This section discusses $C(\mathbb{Z}_n, \pm 1)$, the Cayley graph of the cyclic group of order n using ± 1 to generate the group. We achieve this graph by taking T_n and joining its ends. Thus, the coefficient formula for the graph T_n has a direct connection to the coefficient formula for the cyclic graph.



$C(\mathbb{Z}_n, \pm 1)$

$$\mathcal{X}(C(\mathbb{Z}_n, \pm 1)) = x^n + g_1 x^{n-1} + g_2 x^{n-2} + \cdots + g_{n-1} x + g_n$$

Using the figure equation and the results from Section 3, we can find a closed formula for the coefficients g_i.

First, we consider $\mathcal{L}_i$ for $i < n$, the set of subgraphs of $C(\mathbb{Z}_n, \pm 1)$. For each $L \in \mathcal{L}_i$, there is at least one vertex $v \in C(\mathbb{Z}_n, \pm 1)$ such that $v \notin L$. Take one such v and number the vertices of $C(\mathbb{Z}_n, \pm 1)$ such that $v = 1$. Now, by our definition of vertex 1, vertex n has no edges with source or target 1. Therefore, Lemma 1 holds, and as a corollary, we have that in the graph $C(\mathbb{Z}_n, \pm 1)$,

$$|\mathcal{L}_{2j+1}| = \emptyset \text{ for } j < \frac{n-1}{2} \Rightarrow g_{2j+1} = 0$$

Now, using a similar counting method as in Section 3, we can show

Theorem 2. $|\mathcal{L}_{2m}| = \frac{n}{n-m}\binom{n-m}{m}$ *for each* $2 \leq 2m < n$,

Proof. The set $\mathcal{L}_{2m}$ consists of all possible placements of m disjoint 2-cycles onto the graph $C(\mathbb{Z}_n, \pm 1)$. Consider $\mathcal{L}_{2m}$ as the sum of the set that uses the 2-cycle $(1 \leftrightarrow 2)$ and the set that does not. We can see that the latter set has the same cardinality as the set $\mathcal{L}_{2m}$ in the graph T_n, or $\binom{n-m}{m}$ What remains to be counted is the ways to choose m disjoint 2-cycles using 2-cycle $(1 \leftrightarrow 2)$, or equivalently, the ways to choose $m-1$ disjoint 2-cycles using the $n-2$ remaining vertices. But this, in the graph T_{n-2}, is the set $\mathcal{L}_{2(m-1)}$ with cardinality $\binom{n-2-(m-1)}{m-1}$

Therefore, for $2 \leq 2m < n$, $|\mathcal{L}_{2m}| = \binom{n-m-1}{m-1} + \binom{n-m}{m} = \frac{n}{n-m}\binom{n-m}{m}$ □

Corollary 3. *The coefficients* $g_{2m} = (-1)^m \frac{n}{n-m}\binom{n-m}{m}$

Proof. Each $L \in \mathcal{L}_{2m}$ is made up of m disjoint 2-cycles, so $P(L) = m$. So,

$$g_{2m} = \sum_{L \in \mathcal{L}_{2m}} (-1)^{P(L)} = (-1)^m \frac{n}{n-m}\binom{n-m}{m}$$

□

Next, consider the set $\mathcal{L}_n$:

Proposition 1. *The coefficient* $g_n = \begin{cases} -2, & n \text{ odd} \\ -2 + 2(-1)^{\frac{n}{2}} & n \text{ even} \end{cases}$

Proof. In the graph $C(\mathbb{Z}_n, \pm 1)$, there are two linearly directed cycles of length n; let C_n^+ be the n-cycle using generator $+1$ and C_n^- be the n-cycle using generator -1. Following from Lemma 1, if n is odd, $\mathcal{L}_n = \{C_n^+, C_n^-\}$ and $P(C_n^+) = P(C_n^-) = 1$, so for n odd, $g_n = -2$.

If n is even, $\mathcal{L}_n = \{C_n^+, C_n^-\} \cup \mathcal{N}$ where $\mathcal{N}$ is the set of placements of $\frac{n}{2}$ 2-cycles on the graph $C(Z/_n, \pm 1)$ such that $|\mathcal{N}| = 2\binom{\frac{n}{2}}{\frac{n}{2}} = 2$, each with $\frac{n}{2}$ pieces.

Therefore, for n even, $g_n = \sum_{L \in \mathcal{L}_n} (-1)^{P(L)} = -2 + 2(-1)^{\frac{n}{2}}$

□

Now, we have a closed formula for the coefficients of the characteristic polynomial of $C(\mathbb{Z}_n, \pm 1)$.

$$\mathcal{X}(C(\mathbb{Z}_n, \pm 1)) = x^n + g_2 x^{n-2} + g_4 x^{n-4} + \cdots + g_{n-1} x + g_n$$

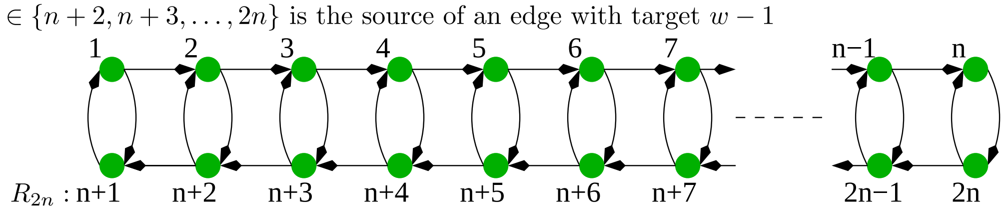
Where

$$g_{2m} = (-1)^m \frac{n}{n-m} \binom{n-m}{m}, \; 2m \neq n$$

$$g_n = \begin{cases} -2, & n \text{ odd} \\ -2 + 2(-1)^{\frac{n}{2}} & n \text{ even} \end{cases}$$

5. The Ladder Graph

In this section, we will discuss R_{2n}, the *ladder graph* with vertices $\{1, 2, \ldots, 2n\}$ with sets of 2-cycles $(c \leftrightarrow (n + c))$ that form rungs of the ladder. Each vertex $v \in \{1, 2, \ldots, n-1\}$ is the source of an edge with target $v + 1$ and each vertex $w \in \{n + 2, n + 3, \ldots, 2n\}$ is the source of an edge with target $w - 1$



By the figure equation,

$$\mathcal{X}(R_{2n}) = x^{2n} + \ell_1 x^{2n-1} + \ell_2 x^{2n-2} + \ell_3 x^{2n-3} + \cdots + \ell_{2n-1} x + \ell_{2n}$$

where the coefficients ℓ_i are as defined by the figure equation.

Lemma 2. *For any $L \in \mathcal{L}_j$ of the graph R_{2n}, let C_k be any linearly directed cycle of length k such that $C_k \subseteq L$. Then, k is even.*

Proof. For the graph R_{2n}, let vertex $a \in C_k \subseteq L \in \mathcal{L}_j$ be the smallest of the $2n$ vertices that is a vertex in C_k. Let vertex $b \in C_k \subseteq L \in \mathcal{L}_j$ be the largest vertex in C_k.

Note that $1 \leq a \leq n$ as if not, $a - 1 \notin C_k$ implies that the edge with source a can only have target $n - a$, but this contradicts a being the smallest vertex. Similarly, $n + 1 \leq b \leq 2n$.

Now, each vertex in C_k must have indegree and outdegree 1, and since $a-1 \notin C_k$, the edge with target a must have source $n+a \in C_k$. Thus,

$$a \in C_k \iff n+a \in C_k$$

If the edge with source a has target $n+a$, then C_k is a 2-cycle, $k=2$, and we are done.

Otherwise, the edge with source a has target $a+1 \in C_k$, and therefore, the edge with target $n+a$ must have source $n+a+1$. Thus,

$$a+1 \in C_k \Leftrightarrow n+a+1 \in C_k$$

So, $\{a, a+1, n+a, n+a+1\} \subseteq C_k$. Now, if the edge with source $a+1$ has target $n+a+1$, $k=4$ and we are done. Otherwise, the edge has target $a+2 \in C_k$ and the edge with target $n+a+1$ has source $n+a+2$. Therefore,

$$a+2 \in C_k \Leftrightarrow n+a+2 \in C_k$$

This argument continues to show that

$$c \in C_k \Leftrightarrow n+c \in C_k$$

The cycle will close at vertex b, as the edge with source b can only have target $b-n$ as b is the largest vertex in the cycle. Thus, C_k is comprised of pairs of vertices $\{c, n+c\}$ and so k is even. □

Corollary 4. *For the characteristic polynomial $\mathcal{X}(R_{2n})$, the coefficient $\ell_{2j+1} = 0$ for $j \leq \frac{2n-1}{2}$.*

Proof. We have that any C_k in R_{2n} is an even cycle, so each $L \in \mathcal{L}_{2j+1}$ must be a sum of disjoint even cycles. Therefore, $\mathcal{L}_{2j+1} = \emptyset$, so

$$\ell_{2j+1} = \sum_{L \in \emptyset} (-1)^{P(L)} = 0$$

□

To calculate a formula for the even coefficients ℓ_{2m}, we must examine the set $\mathcal{L}_{2m}$ of linearly directed subgraphs of R_{2n}. We will find that this set breaks into a sum of smaller sets, and we will find cancellations that enable us to calculate the coefficients. First, we consider $\mathcal{A} \subseteq \mathcal{L}_{2m}$ where

$$\mathcal{A} = \{L \in \mathcal{L}_{2m} | L \text{ consists of } m \text{ disjoint 2-cycles } (i \leftrightarrow n+1)\}$$

For $0 \leq c \leq m$, let $L_c \in \mathcal{A} \subseteq \mathcal{L}_{2m}$ consist of a *chain of c adjacent 2-cycles* and $m-c$ non-*adjacent* 2-cycles. (We define the 2-cycle $(i \leftrightarrow (n+1))$ to be adjacent to the 2-cycles $((i-1) \leftrightarrow (n+i-1))$ and $((i+1) \leftrightarrow (n+i+1))$ and a chain of c adjacent 2-cycles to be $\{(i \leftrightarrow (n+i)), ((i+1) \leftrightarrow (n+i+1)), \ldots, ((i+c-1) \leftrightarrow (n+i+c-1))\}$.

Next, consider the family $\mathcal{B} \subseteq \mathcal{L}_{2m}$ where

$$\mathcal{B} = \{L \in \mathcal{L}_{2m} | \; L \text{ uses the same } 2m \text{ vertices as } L_c\}$$

Lemma 3. $\sum_{L \in \mathcal{B}} (-1)^{P(L)} = 0$

Proof. To prove this claim, we must consider $P(L)$ for each $L \in \mathcal{B}$. For example, if $c=2$, then $\mathcal{B}$ has two elements, one with m pieces, using m disjoint 2-cycles, and one with $m-1$ pieces, using $m-2$ disjoint 2-cycles and one 4-cycle.

For each $L \in \mathcal{B}$, the $m-c$ non-adjacent 2-cycles always contribute $m-c$ pieces. Thus, we focus on $C \subseteq L$, the chain of adjacent vertex pairs, where $1 \leq P(C) \leq c$. I claim that there are $\binom{c-1}{k-1}$ different ways to draw C in k pieces.

For $P(C) = 1$, a cycle of length $2c$ is the only option, and $1 = \binom{c-1}{0}$

For $P(C) = 2$, we fix each option for the first cycle and make the remaining vertices into a second cycle. There are $\sum_{i+1}^{c-1} 1 = c-1 = \binom{c-1}{1}$ options.

For $P(C) = k-1$, assume that there are $\binom{c-1}{k-2}$ ways to form the $2c$ vertices into $k-1$ disjoint cycles.

Then, for $P(C) = k$, fix each option for the first cycle and sum each way to form the remaining vertices into $k-2$ disjoint cycles. The number of options is

$$\binom{c-1-1}{k-2} + \binom{c-1-2}{k-2} + \binom{c-1-3}{k-2} + \cdots + \binom{k-2}{k-2} = \binom{c-1}{k-1}$$

Now, for $L \in \mathcal{B}$, $P(L) = k + (m-c)$, and using the above claim,

$$\sum_{L\in\mathcal{B}} (-1)^{P(L)} = (-1)^{m-c} \sum_{k=1}^{c} (-1)^k \binom{c-1}{k-1} = 0$$

□

In Lemma 3, L_c had one chain of adjacent 2-cycles, but $\mathcal{A}$ contains elements with multiple chains. Consider some $L_{\{c\}} \in \mathcal{A}$ with $\{c\} = \{c_1, c_2, \ldots, c_i\}$, a set of i chains of any length, and $m - \{c\}$ leftover non-adjacent 2-cycles. Let the family $\mathcal{C} = \{L \in \mathcal{L}_{2m} | L$ uses the same $2m$ vertices as $L_{\{c\}}\}$, and the sum over $\mathcal{C}$ breaks into a sum of sums over each chain:

$$\sum_{L\in\mathcal{C}} (-1)^{P(L)} = (-1)^{m-\{c\}} \sum (-1)^{P(C_1)} \sum (-1)^{P(C_2)} \sum \cdots \sum (-1)^{P(C_i)}$$

However, each individual sum equals zero by Lemma 3, so the total sum is also zero.

Therefore, the only elements of $\mathcal{L}_{2m}$ that contribute to ℓ_{2m} are those made up of m non-adjacent 2-cycles. Let $\mathcal{R} \subseteq \mathcal{L}_{2m}$ be this set. Then, $\ell_{2m} = \sum_{L\in\mathcal{R}_{2m}} (-1)^{P(L)}$, but, as $P(L) = m$ for every $L \in \mathcal{R}_{2m}$,

$$\ell_{2m} = (-1)^m |\mathcal{R}_{2m}|$$

Theorem 3. *$|\mathcal{R}_{2m}| = \binom{n+1-m}{m}$ for $1 \leq m \leq n$*

Proof. $|\mathcal{R}_{2m}|$ is the number of ways to place m non-adjacent 2-cycles on R_{2n}. First, we denote a 2-cycle by its lesser vertex (i.e. denote $(i \leftrightarrow n+i)$ by i). Then, each $L \in \mathcal{R}_{2m}$ corresponds to a set of vertices $\{x_1 < x_2 < \cdots < x_m\}$ such that $x_i - x_{i-1} \geq 2$ as the 2-cycles are disjoint. Note, $1 \leq x_1$ and $x_m \leq n$. Now, consider

$$L' = \{x_1, x_2 - 1, x_3 - 2, \ldots, x_m - (m-1)\} = \{x'_1 < x'_2 < x'_3 < \cdots < x'_m\}$$

such that $x'_i - x'_{i-1} \geq 1$. Note, $1 \leq x'_1$ and $x'_m \leq n - (m-1)$.

Therefore, each $L' \in \mathcal{R}_{2m}$ is a choice of the m lesser vertices from $n-m$ possibilities. Thus, $|\mathcal{R}_{2m}| = \binom{n+1-m}{m}$. □

Corollary 5. *The coefficients $\ell_{2m} = (-1)^m \binom{n+1-m}{m}$*

Proof. Each $L \in \mathcal{R}_{2m}$ is made of m non-adjacent 2-cycles, so $P(L) = m$. Therefore,

$$\ell_{2m} = \sum_{L \in \mathcal{R}} (-1)^{P(L)} = (-1)^m \binom{n+1-m}{m}$$

□

Corollary 6. *For $m \geq \frac{n+2}{2}$, $|\mathcal{R}_{2m}| = 0$*

Proof. Using the above result,

$$|\mathcal{R}_{n+2}| = \binom{\frac{n}{2}}{\frac{n+2}{2}} = 0$$

So for $2m > n+2$, $\binom{n+1-m}{m} = 0$ □

Therefore, the coefficients

$$\ell_{n+2} = \ell_{n+3} = \cdots = \ell_{2n} = 0$$

Now, we have simplified $\mathcal{X}(R_{2n}) = x^{2n} + \ell_1 x^{2n-1} + \ell_2 x^{2n-2} + \cdots + \ell_{2n-1} + \ell_{2n}$ to

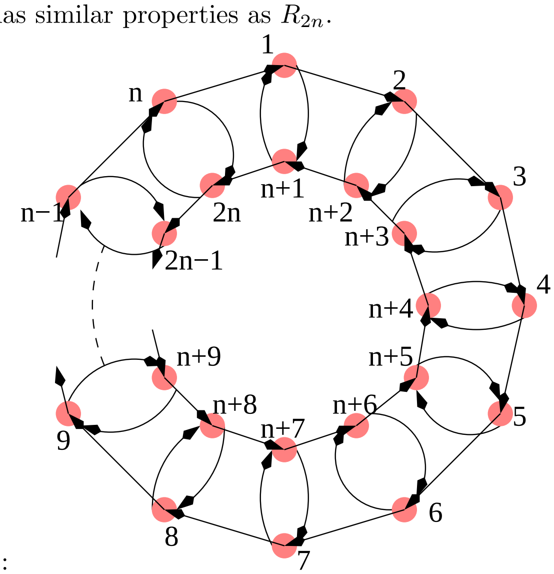
$$\mathcal{X}(R_{2n}) = x^{n-1}\left[x^{n+1} + \ell_2 x^{n-1} + \ell_4 x^{n-3} + \ell_6 x^{n-5} + \cdots + \ell_n x + \ell_{n+1}\right]$$

where

$$\ell_{2m} = (-1)^m \binom{n+1-m}{m}$$

6. The Dihedral Graph

In this section, we focus on $C(D_{2n}, \{\mu, \rho\})$ the Cayley graph of the dihedral group of order $2n$ using μ, a reflection of order 2, and ρ, a rotation of order n to generate the group. We join the ends of the ladder graph to form the dihedral graph. Thus, it has similar properties as R_{2n}.



$C(D_{2n}, \{\mu, \rho\})$:

By the figure equation, we write

$$\mathcal{X}(C(D_{2n}, \{\mu, \rho\})) = x^{2n} + d_1 x^{2n-1} + d_2 x^{2n-2} + d_3 x^{2n-3} + \cdots + d_{2n-1} x + d_{2n}$$

From the figure equation and the results from Section 5, we can find a closed formula for the coefficients d_i by analyzing the sets $\mathcal{L}_i$ using a similar strategy as that from Section 3.

Lemma 4. *For $C(D_{2n}, \{\mu, \rho\})$, $\mathcal{L}_{2j+1} = \emptyset$ for $j \neq \frac{n-1}{2}$*

Proof. In the dihedral graph, we first consider $C_k \subseteq L \in \mathcal{L}_{2j+1}$, a linearly directed cycle of length k, in three cases.

Case 1: $k < 2n$**,** $k \neq n$

As $k < 2n$, there is at least one vertex $v \in C(D_{2n}, \{\mu, \rho\})$ such that $v \notin C_k$. Let $v = 1$ and renumber the graph accordingly. Then, we follow Lemma 2 and note that for b, the largest vertex in C_k, $b \neq n+1$ as $1 \notin C_k$. Lemma 2 implies k is even.

Case 2: $k = n$

If n is even, C_n can be a cycle using $\frac{n}{2}$ adjacent vertex pairs $\{c, n+c\}$, or C_n can be the n-cycle C_n^o that uses vertices $\{1, 2, \ldots, n\}$ with edges representing ρ or the n-cycle C_n^i that uses vertices $\{n+1, n+2, \ldots, 2n\}$ with edges representing ρ.

If n is odd, the only possible n-cycles are C_n^o and C_n^i.

Case 3: $k = 2n$

Clearly, any 2n-cycle is of even length.

Each $L \in \mathcal{L}_{2j+1}$ is made up of products of disjoint linearly directed cycles with lengths that sum to $2j+1$. By the above case-wise analysis, we know that the only possible cycles of odd length are C_n^o and C_n^i when n is odd. However, from Lemma 2, we know that any C_k where $k \neq n$ uses a set of adjacent vertex pairs of the form $\{c, n+c\}$. Therefore, for all $C_k \neq C_n^i \neq C_n^o$, we have $C_k \cap C_n^o \neq \emptyset$ and $C_k \cap C_n^i \neq \emptyset$ as for each vertex pair $\{c, n+c\} \in C_k$, $c \in C_n^o$ and $n+c \in C_n^i$. Because $2j+1 \neq n$, any $L \in \mathcal{L}_{2j+1}$ must be a sum of disjoint even cycles. As $2j+1$ is odd, such an L does not exist. So, $\mathcal{L}_{2j+1} = \emptyset$. □

Corollary 7. *The coefficients $d_{2j+1} = 0$ for $j \neq \frac{n-1}{2}$*

Proof. By Lemma 4, the set $\mathcal{L}_{2j+1} = \emptyset$. Therefore,

$$d_{2j+1} = \sum_{L \in \mathcal{L}_{2j+1}} (-1)^{P(L)} = 0$$

□

Next, we examine the sets $\mathcal{L}_{2m}$, $\mathcal{L}_n$, and $\mathcal{L}_{2n}$ in cases.

Case 1: $\mathcal{L}_{2m}$ **for** $m < n$**,** $2m \neq n$

The set $\mathcal{L}_{2m}$ of subgraphs of $C(D_{2n}, \{\mu, \rho\})$ behaves similarly to the set $\mathcal{L}_{2m}$ of the graph R_{2n}. From Lemma 2, we have that vertices

$$c \in C_k \subseteq L \in \mathcal{L}_{2m} \Leftrightarrow n + c \in C_k \subseteq L \in \mathcal{L}_{2m}$$

Because $m < n$, for each $L \in \mathcal{L}_{2m}$, we have some vertex $v \notin L$, which implies $n + v \notin L$. As there is at least one vertex pair of $C(D_{2n}, \{\mu, \rho\})$ that is not in L, each $L \in \mathcal{L}_{2m}$ for the dihedral graph maps surjectively to some $L' \in \mathcal{L}_{2m}$ for the ladder graph. Therefore, the properties from Section 5 hold in this case, and we need to count $\mathcal{R}_{2m} \subset \mathcal{L}_{2m}$, the set of placements of m non-adjacent 2-cycles on $C(D_{2n}, \{\mu, \rho\})$.

Using a method like that of Theorem 2, we have

Theorem 4. $|\mathcal{R}_{2m}| = \frac{n}{n-m}\binom{n-m}{m}$ *for each $m < n$ and $2m \neq n$.*

Proof. The set $\mathcal{R}_{2m}$ consists of all possible placements of m non-adjacent 2-cycles of the form $(c \leftrightarrow n+c)$ onto the graph $C(D_{2n}, \{\mu, \rho\})$ We consider R_{2m} the sum of the set that uses the 2-cycle $(1 \leftrightarrow n+1)$ and the set that does not. We can see that the latter has the same cardinality as $\mathcal{R}_{2m}$ in the graph $R_{2(n-1)}$, which is $\binom{(n-1)+1-m}{m}$. What remains to be counted is the ways to choose m non-adjacent 2-cycles using $(1 \leftrightarrow n+1)$ as one of the choices, or equivalently, the ways of choosing $m-1$ nonadjacent 2-cycles from the $n-3$ remaining options. This, is the set $\mathcal{R}_{2(m-1)}$ in the graph $R_{2(n-3)}$, with cardinality $\binom{(n-3)+1-(m-1)}{m-1}$.

Therefore, in the dihedral graph, for $m < n$ and $2m \neq n$,

$$|\mathcal{R}_{2m}| = \binom{n-m}{m} + \binom{n-m-1}{m-1} = \frac{n}{n-m}\binom{n-m}{m}$$

□

Corollary 8. *The coefficients* $d_{2m} = (-1)^m \frac{n}{n-m}\binom{n-m}{m}$

Proof. Each $L \in \mathcal{R}_{2m}$ is made up of m non-adjacent 2-cycles, so $P(L) = m$. Thus,

$$d_{2m} = \sum_{L \in \mathcal{R}_{2m}} (-1)^{P(L)} = (-1)^m \frac{n}{n-m}\binom{n-m}{m}$$

□

Corollary 9. *For* $n > m \geq \frac{n+1}{2}$, $|\mathcal{R}_{2m}| = 0$

Proof. Using the above result,

$$|\mathcal{R}_{n+1}| = \frac{2n}{n-1}\binom{\frac{n-1}{2}}{\frac{n+1}{2}} = 0$$

So, for $n > m \geq \frac{n+1}{2}$, $\binom{n-m}{m} = 0 \Rightarrow |\mathcal{R}_{2m}| = 0$. □

Therefore, the coefficients

$$d_{n+1} = d_{n+2} = \cdots = d_{2n-2} = 0$$

Case 2: $\mathcal{L}_n$

Proposition 2. *The coefficient* $d_n = \begin{cases} -2, & n \text{ odd} \\ -2 + 2(-1)^{\frac{n}{2}} & n \text{ even} \end{cases}$

Proof. In the dihedral graph, for n odd, $\mathcal{L}_n = \{C_n^o, C_n^i\}$, where $P(C_n^o) = P(C_n^i) = 1$. Thus, $d_n = (-1)^1 = (-1)^1 = -2$

For n even, $\mathcal{L}_n = \{C_n^o, C_n^i\} \cup \mathcal{R}_n$. By Theorem 4, $|\mathcal{R}_n| = 2$ where each $L \in \mathcal{R}_n$ has $P(L) = \frac{n}{2}$. Thus, $d_n = -2 - 2(-1)^{\frac{n}{2}}$ □

Case 3: $\mathcal{L}_{2n}$

In this case, $\mathcal{L}_{2n} = \mathcal{K} \cup \{C_n^o + C_n^i\}$ Where $\mathcal{K}$ is the set of all possible combinations of even cycles with lengths that sum to $2n$. Note, $P(\{C_n^o + C_n^i\}) = 2$.

Then, each $L \in \mathcal{K}$ comes from taking all vertices of $C(D_{2n}, \mu, \rho\})$ and creating k pieces, $1 \leq k \leq n$. For each k, the number of ways to create k pieces is $\binom{n}{k}$ where each k represents a space where adjacent vertex pairs have no edges between them, and there are n such places where adjacent vertex pairs could be disconnected. Thus, for each $L \in \mathcal{K}$, $P(L) = k$. Therefore,

$$d_{2n} = \sum_{L \in \mathcal{L}_{2n}} (-1)^{P(L)} = 1 + \sum_{L \in \mathcal{K}} (-1)^{P(L)} = 1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k}$$

Now, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 = \binom{n}{0} + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \Rightarrow \sum_{k=1}^{n} (-1)^k \binom{n}{k} = -1$. Thus,

$$d_{2n} = 1 - 1 = 0$$

Therefore, we have simplified

$$\mathcal{X}(C(D_{2n}, \{\mu, \rho\})) = x^{2n} + d_1 x^{2n-1} + d_2 x^{2n-2} + \cdots + d_{2n-1} x + d_{2n}$$

to

$$\mathcal{X}(C(D_{2n}, \{\mu, \rho\})) = x^n \left[x^n + d_2 x^{n-2} + d_4 x^{n-4} + \cdots + d_{n-1} x + d_n \right]$$

Where

$$d_{2m} = (-1)^m \frac{n}{n-m} \binom{n-m}{m}, \; 2m \neq n$$

$$d_n = \begin{cases} -2, & n \text{ odd} \\ -2 + 2(-1)^{\frac{n}{2}} & n \text{ even} \end{cases}$$

7. Graph Coverings and Families of Isospectral Graphs

In the above sections, we have used the figure equation to classify the coefficients of the characteristic polynomials of four graph families. Now, we examine our results and notice some interesting connections.

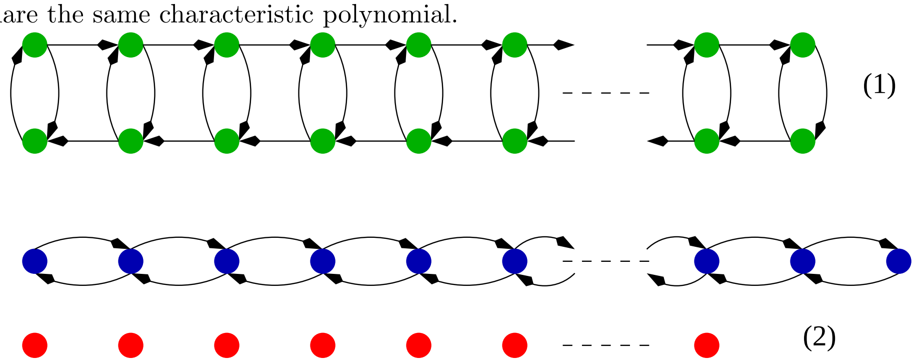
We see from the above sections that:

$$\mathcal{X}(R_{2n}) = x^{n-1} \mathcal{X}(T_{n+1})$$

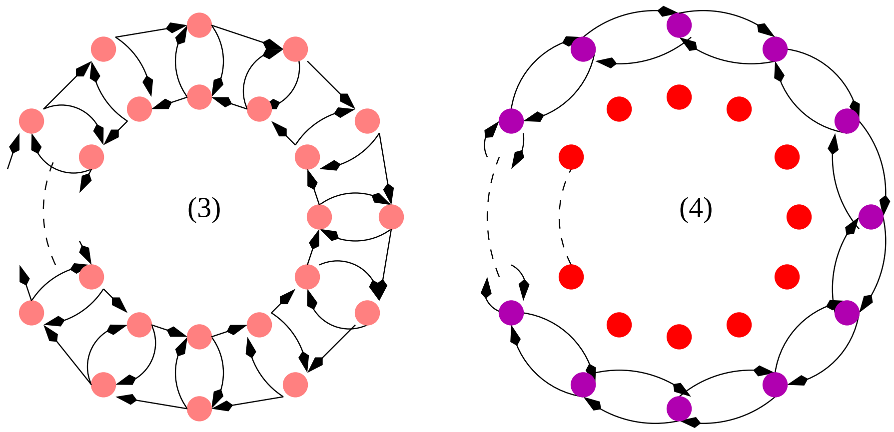
and

$$\mathcal{X}(C(D_{2n}, \{\mu, \rho\})) = x^n \mathcal{X}(C(\mathbb{Z}_n, \pm 1))$$

Therefore, we have established the following families of isospectral graphs, which share the same characteristic polynomial.



The graph (1) $= R_{2n}$ is isospectral to the graph (2) $= T_{n+1} + (N-1)$ where $(N-1)$ is the graph of $n-1$ vertices and no edges.



The graph (3) $= C(D_{2n}, \{\mu, \rho\})$ is isospectral to the graph (4) $= C(\mathbb{Z}_n, \pm 1) + (N)$ where (N) is the graph of n vertices and no edges.

Next, we show that the equality $\mathcal{X}(C(D_{2n}, \{\mu, \rho\})) = x^n \mathcal{X}(C(\mathbb{Z}_n, \pm 1))$ can be explained by a graph covering map, a function $f : X \to Y$ that maps every vertex in X to a vertex in Y and preserves the out degree of each vertex.

Theorem 5. *For $n > 2$, there exists a covering map $f : C(D_{2n}, \{\mu, \rho\}) \longrightarrow C(\mathbb{Z}_n, \pm 1)$ defined by $f : \{\rho^i, \mu\rho^{i+1}\} \longrightarrow i$. This covering map does not define a group action.*

Proof. Previously, we have numbered the vertices of these graphs for convenience. Now, we label each vertex by the group element it represents. For each vertex $1 \leq v \leq 2n$ of $C(D_{2n}, \{\mu, \rho\})$, we let $v' = \begin{cases} \rho^{v-1}, & 1 \leq v \leq n \\ \mu\rho^{v-n-1}, & n+1 \leq v \leq 2n \end{cases}$ be the new vertex labeling.

For each vertex $1 \leq w \leq n$ of $C(\mathbb{Z}_n, \pm 1)$, we let $w' = w - 1$ be the new vertex labeling. Then, we define f such that

$$f : \{\rho^i, \mu\rho^{i+1}\} \longrightarrow i$$

and we map the edges accordingly.

If f was the result of a group action, we would have

$$f : C(D_{2n}, \{\mu, \rho\}) \longrightarrow Act(\frac{D_{2n}}{K}, \{\mu, \rho\}) = C(\mathbb{Z}_n, \pm 1)$$

for some subgroup $K \subset D_{2n}$ where $|K| = 2$. Then, each vertex $i \in Act(\frac{D_{2n}}{K}, \{\mu, \rho\}) = C(\mathbb{Z}_n, \pm 1)$ would be the coset $\rho^i K$.

We have defined $f : \{\rho^i, \mu\rho^{i+1}\} \longrightarrow i$, so our subgroup must be

$$K = \{\rho^0, \mu\rho^1\} \cong \mathbb{Z}_2$$

Therefore, the coset

$$\rho K = \{\rho^1, \rho\mu\rho^1\} = \{\rho, \mu\rho^0\} = 1$$

However, for $i = 1$, we mapped vertices $f : \{\rho, \mu\rho^2\} \longrightarrow 1$. Therefore, for f to define a group action, we must have

$$\{\rho, \mu\} = \{\rho, \mu\rho^2\}$$

However, $\rho^2 \neq \rho^0$ unless $\rho = \rho^{-1}$ and this is only the case when $n = 2$. Therefore, for $n > 2$, f does not result from any group action. □

References

[1] A.T. Benjamin, J.J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof,* Mathematical Association of America, 2003.
[2] D.M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs,* Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 3rd revised and enlarged ed., 1995.
[3] C. Godsil, G. Royle, *Algebraic Graph Theory,* Springer, New York-Berlin-Heidelberg, 2001.
[4] N. Trinajstić, *Chemical Graph Theory*, CRC Press, Boca Raton-Ann Arbor-London-Tokyo, 2nd edition, 1992.

Department of Mathematics, University of Rochester, Rochester, NY 14627
E-mail address: tcoon@mail.rochester.edu