PROPER COLORINGS AND p-PARTITE STRUCTURES OF THE ZERO DIVISOR GRAPH

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ABSTRACT. Let $\Gamma(\mathbb{Z}_m)$ be the zero divisor graph of the ring $\mathbb{Z}_m$. In this paper we explore the p-partite structures of $\Gamma(\mathbb{Z}_m)$, as well as determine a complete classification of the chromatic number of $\Gamma(\mathbb{Z}_m)$. In particular, we explore how these concepts are related to the prime factorization of m.

1. Introduction

The *zero divisor graph* of a commutative ring R is defined as the graph containing one vertex for each nonzero element in the set of zero divisors, $Z(R)$, with two vertices u and v connected by an edge if and only if $uv = 0$ in R. It was shown by Anderson and Livingston in [1] that the zero divisor graph, denoted $\Gamma(R)$, is always connected. By convention, $\Gamma(R)$ does not contain *loops*, i.e., edges that go from a vertex to itself, or multiple copies of the same edge, known as *multiple edges*. Thus we see that $\Gamma(R)$ is always a simple connected graph, which allows us to explore a large variety of graph theoretic concepts with regard to the zero divisor graph. In [3], Cordova et. al. have already explored diameter and girth limitations, Eulerian and Hamiltonian circuits, and the relationship between clique number and chromatic number in $\Gamma(R)$ in some detail.

Here, we will continue the explorations begun by Cordova et. al. into the chromatic number χ of the zero divisor graph. Specifically, we will focus on $\Gamma(\mathbb{Z}_m)$ and the relationship between $\chi(\Gamma(\mathbb{Z}_m))$ and m itself. We also investigate the various p-partite structures in $\Gamma(\mathbb{Z}_m)$ and their relationship to the prime factorization of m. Before presenting our results, we begin with some preliminary definitions of graph theoretic terms in the following section.

2. Preliminaries and Definitions

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. G is said to be *complete p-partite* if $V(G)$ can be partitioned into p disjoint sets $V_1, V_2, \ldots, V_p$ such that no two vertices within any V_i are adjacent, but for every $v \in V_i, u \in V_j$, u and v are adjacent. An example of a complete 2-partite (bipartite) and a complete 4-partite graph are given below.

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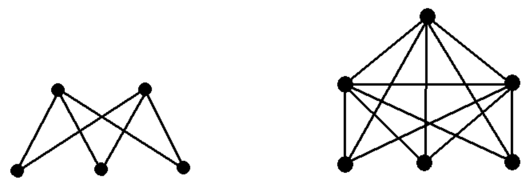
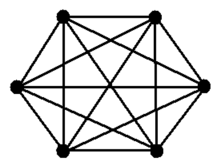


A complete bipartite graph (left) and a complete 4-partite graph (right).

Let K_n denote the *complete graph* on n vertices. Recall that a complete graph G is a simple graph with $E(G)$ containing every possible edge between the vertices in the set $V(G)$. As an example, K_6 is shown below.



K_6, the complete graph on six vertices.

A *subgraph of* G *induced* by $V'(G)$, where $V'(G) \subseteq V(G)$, is the graph H such that $V(H) = V'(G)$, and $e \in E(H)$ if and only if $e \in E(G)$. A *clique* is an induced subgraph H such that $H = K_n$ for some n. If H is the clique in G with the largest number of vertices, the *clique number* of G is said to be n.

A *proper coloring* of a graph G is a function that assigns a color to each vertex such that no two adjacent vertices have the same color. Throughout this paper, we will assign colors using the set $K = \{k_1, k_2, \ldots\}$. The *chromatic number* of G, denoted $\chi(G)$, is the smallest number of colors necessary to produce a proper coloring.

Finally, we note that throughout this paper, we will be working extensively with the prime factorization of m and its relationship to $\Gamma(\mathbb{Z}_m)$. In this vein we will write $m = {p_1}^{a_1}{p_2}^{a_2}\cdots{p_l}^{a_l}$ to mean the unique prime factorization of the integer m, where each p_i is unique and each a_i is nonzero.

3. p-Partite Structures

In previous studies of $Z(R)$, complete bipartite structures have proven to be useful in realizing many interesting algebraic aspects of R. Axtell et. al. began the exploration of complete bipartite structures in [2], their primary finding being that if a complete bipartite subgraph can be obtained by removing only edges, then the zero divisors form an ideal. In [4], Martinez and Skalak continued the work on this topic. These findings suggest that extending this idea to the more general graph theoretic concept of complete p-partite graphs may be worthwhile. Thus we begin by looking at the p-partite algebraically simple rings, namely $\mathbb{Z}_m$, focusing on the prime factorization of m. Our first result examines the values of m for which $\Gamma(\mathbb{Z}_m)$ is a complete graph.

Theorem 3.1. *The zero divisor graph of* $\mathbb{Z}_{p^2}$ *is* K_{p-1}.

Proof. If $a \in Z(\mathbb{Z}_{p^2})$, then $p|a$. Thus for any two zero divisors a, b, $p^2|ab$, so $ab = 0$. Giving that, in $\Gamma(\mathbb{Z}_{p^2})$, each vertex is adjacent to every other vertex. Furthermore, there are $p-1$ nonzero elements that are divisible by p, so $\Gamma(\mathbb{Z}_{p^2})$ has $p-1$ vertices. Therefore, $\Gamma(\mathbb{Z}_{p^2})$ is K_{p-1}. □

Complete graphs are a very specific type of p-partite graph in which each of the p sets of vertices contains a single vertex. Although these graphs are undoubtedly interesting in their own right, we want a more general classification. In the following result, we see that for all primes p, there is at least one complete p-partite graph which is realizable as a zero divisor graph. the relationship between p and m in the ring $\mathbb{Z}_m$.

Theorem 3.2. *The graph* $\Gamma(\mathbb{Z}_{p^3})$ *is complete* p*-partite, where* p *is prime.*

Proof. If $a \in Z(\mathbb{Z}_{p^3})$, then $p|a$. In order for $ab = 0$, $a, b \neq 0$, $p^2|a$ or b or both. Furthermore, if $p^2|a$, then $ab = 0$ for all $b \in Z(\mathbb{Z}_{p^3})$. Thus in $\Gamma(\mathbb{Z}_{p^3})$, a is adjacent to all other vertices. In $\mathbb{Z}_{p^3}$, there are $p-1$ nonzero elements divisible by p^2. Of the remaining zero divisors, no two annihilate each other and thus in $\Gamma(\mathbb{Z}_{p^3})$, there are no edges other than those between elements divisible by p^2 and all others. Thus we can split $V(\Gamma(\mathbb{Z}_{p^3}))$ into p independent sets $V_1, V_2, \ldots, V_p$. Each of $V_1, \ldots, V_{p-1}$ will contain exactly one vertex, divisible by p^2, and V_p will contain those zero divisors not divisible by p^2. Thus $\Gamma(\mathbb{Z}_{p^3})$ is p-partite. Since each $a \in V_1, \ldots, V_{p-1}$ annihilates all other elements, all possible edges between these sets exist, so $\Gamma(\mathbb{Z}_{p^3})$ is complete p-partite. □

As the reader can see in the previously mentioned research, complete bipartite structures are not only interesting when they comprise the entire graph, but also when they are merely a piece of the whole. In the following two results, we look at the underlying p-partite structures in $\mathbb{Z}_m$ for certain values of m.

Theorem 3.3. *The graph* $\Gamma(\mathbb{Z}_{p^5})$*, where* p *is prime, has an induced complete* p*-partite subgraph.*

Proof. In $Z(\mathbb{Z}_{p^5})$, there are exactly $p-1$ elements divisible by p^4. As in the proof above, those elements annihilate all other $a \in Z(\mathbb{Z}_{p^5})$. Thus, they are adjacent to all other vertices in $\Gamma(\mathbb{Z}_{p^5})$. Specifically, the elements divisible by p^4 annihilate all elements divisible by p but not p^2. None of the latter set can annihilate each other, so in $\Gamma(\mathbb{Z}_{p^5})$, no edges exist between their vertices. Let $V' = \{v \in V(\Gamma(\mathbb{Z}_{p^5})) : p^4|v$, or $p|v$ and $p^2 \nmid v\}$. We can divide V' into p independent sets, $V_1', V_2', \ldots, V_p'$, where $V_i' = \{a_i p^4\}$ for some unique vertex $a_i p^4 \in V'$ where $1 \leq i < p$, and $V_p' = \{v : p|v$ and $p^2 \nmid v\}$. Clearly $V_1' \cup V_2' \cup \ldots \cup V_p' = V'$, and $V_1' \cap V_2' \cap \ldots \cap V_p' = \emptyset$. Without removing any edges from within the vertices of V', we can see that for any given V_i', no edges exist w! ithin its vertices, and for any V_j', all possible edges exist between the vertices of V_i' and V_j'. Therefore, V' forms a complete p-partite induced subgraph of $\Gamma(\mathbb{Z}_{p^5})$. □

Theorem 3.4. *The graph* $\Gamma(\mathbb{Z}_{p^q})$*, where* p *and* q *are prime and* q *is odd, has* $(q-1)/2$ *induced complete* p*-partite subgraphs.*

Proof. In $Z(\mathbb{Z}_{p^q})$, there are exactly $p-1$ elements divisible by p^{q-1}. These elements annihilate all others. For any $i \leq (q-1)/2$, there is a distinct set $A \subset Z(\mathbb{Z}_{p^q})$ such that for all $a \in A$, $p^i | a$ but $p^{i+1} \nmid a$. Since $i \leq (q-1)/2$, there exist no $a, b \in A$ such that $ab = 0$. Thus we can use an argument identical to the one of Theorem 3.3 to show that the vertices corresponding to the elements of A form an induced complete p-partite subgraph with the elements divisible by p^{q-1}. There are exactly $(q-1)/2$ such subsets, so $\Gamma(\mathbb{Z}_{p^q})$ has $(q-1)/2$ induced complete p-partite subgraphs of this type. □

One limitation of Theorems 3.3 and 3.4 is that they deal with induced subgraphs. In order to form any nontrivial induced subgraph, vertices must be removed from the original graph. In the context of the zero divisor graph, the removal of a vertex is tantamount to removing a zero divisor within the ring. Therefore, the algebraic repercussions of these results in and of themselves might be minimal. One direction for further research could be to extend the above results to deal with subgraphs in which only edges are removed, as these graphical structures have proved much more useful in finding algebraic connections than the induced subgraphs used above.

4. Proper Colorings of $\mathbb{Z}_m$

One of the most interesting concepts in graph theory is that of a proper coloring. Thus far, little research has been done on proper colorings of the zero divisor graph, and so we start as we did before with algebraically simple structures in order to obtain more complex graph theoretical results. Throughout this section, we will discover that the prime factorization of m is intricately tied to the chromatic number of $\mathbb{Z}_m$. Here we give a complete classification of the chromatic number for each such m, but our first three results give preliminary proofs for the least complex $\mathbb{Z}_m$.

Theorem 4.1. *$\Gamma(\mathbb{Z}_{pq^2})$ is q-colorable, where p and q are distinct primes.*

Proof. If a and b are zero divisors in $\mathbb{Z}_{pq^2}$ such that $pq|a$ and $pq|b$ where $a \neq b$, then a and b are adjacent in the zero divisor graph of $\mathbb{Z}_{pq^2}$. Thus all such zero divisors form a clique in $\Gamma(\mathbb{Z}_{pq^2})$. There are

$$\frac{pq^2}{pq} - 1 = q - 1$$

such zero divisors. So the clique contains $q-1$ vertices and can be colored on $k_1, k_2, \ldots, k_{q-1}$. Now, if c is any other zero divisor, either $p \nmid c$ or $q \nmid c$, because otherwise c would be in the clique. If $q \nmid c$, then c is not adjacent to any vertex in the clique, so we can color it with k_1. If $q|c$, then c is adjacent to every member of the clique, so we must color it with k_q. We can color all such zero divisors with k_q, because if we have c and d not in the clique and $q|c$ and $q|d$ where $c \neq d$, then $p \nmid c, d$, by the fact that c and d are outside the clique. Thus c and d cannot be adjacent, and so can be colored with the same color. Thus we have a proper coloring of $\Gamma(\mathbb{Z}_{pq^2})$ on exactly q colors. □

Corollary 4.2. *For $\Gamma(\mathbb{Z}_{pq^2})$, we have $\chi(\Gamma(\mathbb{Z}_{pq^2})) = q$.*

Proof. From the previous proof, we know that $\Gamma(\mathbb{Z}_{pq^2})$ has a clique of size $q-1$, so clearly we must have $\chi(\Gamma(\mathbb{Z}_{pq^2})) \geq q-1$. Also from Theorem 4.1, we see that since $\Gamma(\mathbb{Z}_{pq^2})$ is q-colorable, $\chi(\Gamma(\mathbb{Z}_{pq^2})) \leq q$. Suppose $\chi(\Gamma(\mathbb{Z}_{pq^2})) = q-1$. Then we can color every vertex with a color used in the clique consisting of elements divisible by pq. Suppose a is a member of the clique. Then $pq^2|qa$, so q is adjacent to all such a. Thus there is no color in the clique with which we can color q, so we have a contradiction. Thus $\chi(\Gamma(\mathbb{Z}_{pq^2})) = q$. □

Theorem 4.3. *The graph $\Gamma(\mathbb{Z}_{p^2q^2})$ is $(pq-1)$-colorable, where p and q are distinct primes.*

Proof. As in Theorem 4.1, we observe that for all zero divisors a, b such that $pq|a$ and $pq|b$ where $a \neq b$, a and b are adjacent in the zero divisor graph of $\mathbb{Z}_{p^2q^2}$. These vertices form a clique on

$$\frac{p^2q^2}{pq} - 1 = pq - 1$$

vertices. Thus we can color these zero divisors with $k_1, k_2, \ldots, k_{pq-1}$. For every other zero divisor c, we see that either $p \nmid c$ or $q \nmid c$. Regardless which case we have, we see that c is not adjacent to the zero divisor pq. In the former case, we would see that $p^2 \nmid pqc$, and in the latter, $q^2 \nmid pqc$. Either way, $p^2q^2 \nmid pqc$, so c and pq are not adjacent in $\Gamma(\mathbb{Z}_{p^2q^2})$. Thus we can use the color already assigned to the vertex pq to color all other vertices in the zero divisor graph, and we have a proper coloring $pq - 1$ colors. □

Corollary 4.4. *For $\Gamma(\mathbb{Z}_{p^2q^2})$, we have $\chi(\Gamma(\mathbb{Z}_{p^2q^2})) = pq - 1$.*

Proof. In Theorem 4.3, we saw that $\Gamma(\mathbb{Z}_{p^2q^2})$ has a clique of size $pq-1$, so $\chi(\Gamma(\mathbb{Z}_{p^2q^2})) \geq pq-1$. Since $\Gamma(\mathbb{Z}_{p^2q^2})$ is $pq-1$-colorable, $\chi(\Gamma(\mathbb{Z}_{p^2q^2})) \leq pq-1$. Thus $\chi(\Gamma(\mathbb{Z}_{p^2q^2})) = pq - 1$. □

Theorem 4.5. *$\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})$ is 4-colorable, where p_1, p_2, p_3, p_4 are distinct primes.*

Proof. Define K as in Section 2. For every zero divisor a, we must have that at least one of p_1, p_2, p_3 or p_4 does not divide a, since if all of these primes divide a, $a = p_1p_2p_3p_4 = 0$. Thus if $p_i \nmid a$ where i is minimal, we color a with k_i. Using this scheme, if a and b have the same color, say k_j, then $j \nmid a$ or b, so $j \nmid ab$, and thus $ab \neq 0$. Thus a and b cannot be adjacent. Therefore, we have a proper coloring on four colors. □

Corollary 4.6. *For $\mathbb{Z}_{p_1p_2p_3p_4}$, we have $\chi(\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})) = 4$.*

Proof. From Theorem 4.5, we know $\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})$ is 4-colorable, so $\chi(\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})) \leq 4$. Furthermore, we have that the elements $p_1p_2p_3$, $p_2p_3p_4$, $p_1p_3p_4$, and $p_1p_2p_4$ form a clique on four vertices. Thus $\chi(\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})) \geq 4$, so it must be that $\chi(\Gamma(\mathbb{Z}_{p_1p_2p_3p_4})) = 4$. □

The following three theorems give a complete classification of the chromatic number of $\mathbb{Z}_m$. The previous three cases are contained within the following three, but we have retained the preliminary results for the purposes of illustrating the proof techniques that will be used in the general case.

Theorem 4.7. *$\Gamma(\mathbb{Z}_{p^n})$ is $p^{\lfloor n/2 \rfloor}$-colorable, where p is prime and $n \in \mathbb{Z}$.*

Proof. Define our set of colors K as always. If we have two zero divisors, u and v, such that $p^{\lceil n/2 \rceil}|u$ and $p^{\lceil n/2 \rceil}|v$ where $u \neq v$, then u and v must be adjacent. There are $(p^n)/(p^{\lceil n/2 \rceil}) - 1 = p^{\lfloor n/2 \rfloor} - 1$ such zero divisors, thus we have a clique on those vertices on which we can use the colors $k_1, k_2, \ldots, k_{p^{\lfloor n/2 \rfloor}-1}$. If n is odd, all multiples of $p^{\lfloor n/2 \rfloor}$ less than $p^{\lfloor n/2 \rfloor + 1}$ will not be included in the clique since they will not be adjacent to each other, but they will be adjacent to every vertex in the clique. Because none of these vertices can be adjacent to one another, so we can color all of them with $k_{p^{\lfloor n/2 \rfloor}}$. If a is any vertex not colored thus far, we know that $ap^{\lceil n/2 \rceil} \neq 0$. So we can use the color for $p^{\lceil n/2 \rceil}$ to color all such a. Thus we have a proper coloring on $p!^{\lfloor n/2 \rfloor}$ colors. □

Corollary 4.8. *If n is odd, $\chi(\Gamma(\mathbb{Z}_{p^n})) = p^{(n-1)/2}$. Otherwise, $\chi(\Gamma(\mathbb{Z}_{p^n})) = p^{n/2} - 1$.*

Proof. If n is odd, the result follows clearly from the previous theorem. If n is even, no vertex outside of the clique defined above can be attached to $p^{n/2}$, so we can use the color for this vertex to color everything else. Thus in the latter case we only require $p^{n/2} - 1$ colors for a proper coloring. Therefore $\chi(\Gamma(\mathbb{Z}_{p^n})) = p^{n/2} - 1$. □

Theorem 4.9. *Suppose $m = p_1 p_2 \cdots p_n$, where each p_i is a distinct prime. Then $\Gamma(\mathbb{Z}_m)$ is n-colorable.*

Proof. Define our set of colors K as usual. For every $a \in Z(\mathbb{Z}_m)$, there must exist some i such that $p_i \nmid a$. For each vertex v, take the smallest such i and color v with k_i. To show that this coloring is proper, suppose two vertices, v_1 and v_2, are both colored k_i. Then $p_i \nmid v_1$ or v_2, so $p_i \nmid v_1 v_2$. Therefore, $v_1 v_2 \neq 0$ so these two vertices are not adjacent. Thus we have a proper coloring on n colors. □

Theorem 4.10. *Suppose $m = {p_1}^{a_1} {p_2}^{a_2} \cdots {p_n}^{a_n}$, where each p_i is a distinct prime, $n \geq 2$, $a_i > 0$ for all i, and $a_i > 1$ for some i. Let $s = {p_1}^{\lfloor a_1/2 \rfloor} {p_2}^{\lfloor a_2/2 \rfloor} \cdots {p_n}^{\lfloor a_n/2 \rfloor}$. Then $\Gamma(\mathbb{Z}_m)$ is s-colorable.*

Proof. Define K as usual. Every zero divisor v such that $v = {p_1}^{l_1} {p_2}^{l_2} \cdots {p_n}^{l_n}$ where $l_i \geq \lceil a_i/2 \rceil$ for all i is adjacent to every other zero divisor of this form. Each such v is a multiple of ${p_1}^{\lceil a_1/2 \rceil} {p_2}^{\lceil a_2/2 \rceil} \cdots {p_n}^{\lceil a_n/2 \rceil}$. Thus there are

$$\frac{{p_1}^{a_1} {p_2}^{a_2} \cdots {p_n}^{a_n}}{{p_1}^{\lceil a_1/2 \rceil} {p_2}^{\lceil a_2/2 \rceil} \cdots {p_n}^{\lceil a_n/2 \rceil}} - 1 = {p_1}^{\lfloor a_1/2 \rfloor} {p_2}^{\lfloor a_2/2 \rfloor} \cdots {p_n}^{\lfloor a_n/2 \rfloor} - 1 = s$$

such nonzero zero divisors in $\mathbb{Z}_m$. These vertices form a clique and can be colored on $k_1, k_2 \ldots, k_{s-1}$. If any a_i is odd, then all multiples of s less than ${p_1}^{\lceil a_1/2 \rceil} {p_2}^{\lceil a_2/2 \rceil} \ldots {p_n}^{\lceil a_n/2 \rceil}$ will be outside the previously defined clique, but all such elements will be adjacent to all the elements in the clique. We can color these vertices with k_s. For every other zero divisor u, there must exist an i such that ${p_i}^{\lfloor a_i/2 \rfloor} \nmid u$. We see that u cannot be adjacent to ${p_1}^{a_1} {p_2}^{a_2} \cdots {p_i}^{\lfloor a_i/2 \rfloor} \cdots {p_n}^{a_n}$ or to any other vertex w such that ${p_i}^{\lfloor a_i/2 \rfloor} \nmid w$, so for every such u, we can take the smallest such i, and color u with k_t, where the vertex $w = {p_1}^{a_1} {p_2}^{a_2} \cdots {p_i}^{\lfloor a_i/2 \rfloor} \cdots {p_n}^{a_n}$ is colored on k_t. Thus we have a proper coloring on s colors. □

Corollary 4.11. *If m and s are defined as in the previous theorem, then $\chi(\Gamma(\mathbb{Z}_m)) = s$ if at least one a_i is odd. Otherwise, $\chi(\Gamma(\mathbb{Z}_m)) = s - 1$.*

Proof. We saw in the previous theorem that for all such m, $\Gamma(\mathbb{Z}_m)$ has a clique of size $s-1$. Thus $\chi(\Gamma(\mathbb{Z}_m)) \geq s-1$. We also saw from this result that $\chi(\Gamma(\mathbb{Z}_m)) \leq s$ for all m. Define the vertex $v = s$. In the case that some a_i is odd, we see that $s \neq {p_1}^{\lceil a_1/2 \rceil} {p_2}^{\lceil a_2/2 \rceil} \cdots {p_n}^{\lceil a_n/2 \rceil}$, but $s{p_1}^{\lceil a_1/2 \rceil} {p_2}^{\lceil a_2/2 \rceil} \cdots {p_n}^{\lceil a_n/2 \rceil} = 0$, so v is attached to all the vertices in the clique. Therefore, we are forced to use another color and we have $\chi(\Gamma(\mathbb{Z}_m)) = s$. If no a_i is odd, $\lfloor a_i/2 \rfloor = \lceil a_i/2 \rceil$ for all i. So for each vertex u, ${p_j}^{\lfloor a_j/2 \rfloor} \nmid u$ for some $j \leq n$, and we can color all vertices outside the clique using the same process we did in the previous theorem. Thus in this case, $\chi(\Gamma(\mathbb{Z}_m)) = s - 1$. □

5. Conclusion and Acknowledgments

To finish, we would like to pose some suggestions for further research in this area. As previously mentioned, a more useful p-partite structure in the zero divisor graph would be one obtained by only removing edges. One open question is how these structures can be realized in $\Gamma(\mathbb{Z}_m)$. With regard to colorability, little research has yet been done on the chromatic numbers of zero divisor graphs of more complex rings. Using the methods and findings of this paper, it may be possible to extend the results to, for example, polynomial rings, power series rings, and other interesting algebraic structures.

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6. References

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