Referee's Report for *Proper Colorings and p-Partite Structures of the Zero Divisor Graph*

Overview: Zero divisor graphs of many different structures, such as commutative rings, noncommutative rings, and semigroups, have been studied by undergraduates, graduate students, and university faculty, with a significant amount of results appearing in just the past eight years. The interplay between graph theory and these other structures is especially intriguing, and the topic's accessibility to undergraduates and undiscovered knowledge appeal to a wide audience. Algebraically, the zero divisors do not exhibit much structure; indeed, the set of zero divisors is not necessarily closed under addition. So, any avenue of study that eases the study of such an unwieldy object should be explored. In this paper, the author studies the zero divisor graphs of $\mathbb{Z}_m$, or $\Gamma(\mathbb{Z}_m)$, focusing on p-partite subgraphs of $\Gamma(\mathbb{Z}_{p^n})$, as well as classifying the chromatic number of $\Gamma(Z_m)$ for all $m \geq 2$. The results presented in this paper are new, non-trivial, and easily accessible to undergraduate students who have a background in elementary graph and ring theory, insomuch as the reader should know the basic definitions of "graph," "commutative ring," and "zero divisor." After some minor editorial changes, I highly recommend this paper for publication.

Copy-editing: By and large, this paper is well-written. The material flows in a logical order, examples are presented early to motivate the topic at hand, and there is not the overuse of symbolism commonly displayed by young, budding mathematicians. This author should be commended for showing this level of mathematical writing maturity at this point in the author's education. Though there are many comments that follow, many of them deal with what appear to be LaTeX issues and not with the author's writing ability, and some of them deal with mathematical writing conventions, such as do not begin a sentence with mathematical symbols.

Figures: Is it possible to label the figures of the p-partite graphs that show the vertex sets being described at the bottom of page 1? This would help illuminate the definition.

Editorial suggestions: The numbering scheme that follows give the page and line number where there is a change needed. If a minus sign appears in front of a line number, count up from the bottom of the page.

p. 2, l. -7: change "theoretical" to "theoretic"

p. 2, l. -6: delete the comma following "namely"

p. 2, l. -5: insert the word "the" after the word "examines" and move the words "for which" after the "m"

p. 3, l. 1: replace "so" with "giving that"

p. 3, l. 3: delete ", namely, that"

p. 3, l. 9: insert "The graph" before "$\Gamma(\mathbb{Z}_{p^3})$"

p. 3, l. 23: insert "values of" after "certain"

p. 3, l. 24: insert "The graph" before "$\Gamma(\mathbb{Z}_{p^5})$"

p. 3, l. -19: insert a period after "$Z(\mathbb{Z}_{p^4})$", capitalize "thus", and insert a comma after "Thus"

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p. 3, l. -16: consider replacing "|" with ":" in the definition of V'; all of the vertical bars in that set make for a difficult read

p. 3, l. -15: replace "|/" with "|/"

p. 3, l. -14: insert a comma after the ellipsis

p. 3, l. -11: the V_i'''s and the V_j'''s have not been defined anywhere.

p. 3, l -11: delete the extraneous "!"

p. 3, l. -8: insert "The graph" before "$\Gamma\left(\mathbb{Z}_{p^q}\right)$"

p. 3, l. -8: either specify that q is an odd prime, or replace "$(q-1)/2$" with "$\lfloor (q-1)/2 \rfloor$"

p. 3, l. -4: replace "|/" with "|/"

p. 3, l. -3: insert "Theorem" before "3.3"

p. 4, l. -22: replace "$pq|a,b$" with "$pq|a$ and $pq|b$" and specify that $a \neq b$

p. 4, l. -18: delete ",not including $pq^2 = 0$ itself,"

p. 4, l. -18: insert a period before "so" and capitalize "so"

p. 4, l. -17: replace "|/" with "|/"

p. 4, l. -16: replace "|/" with "|/" (both instances)

p. 4, l. -13: replace "c,d" with "c and d", replace "$pq|c,d$" with "$pq|c$ and $pq|d$", and specify that $c \neq d$

p. 4, l. -1: insert "The graph" before "$\Gamma\left(\mathbb{Z}_{p^q}\right)$" and put parentheses around "$pq-1$"

p. 5, l. 1: replace "$pq|a,b$" with "$pq|a$ and $pq|b$" and specify that $a \neq b$

p. 5, l. 6: replace "|/" with "|/" (both instances)

p. 5, l. 8: replace "|/" with "|/" (all three instances)

p. 5, l. 12: insert words before "$\chi\left(\Gamma\left(\mathbb{Z}_{p^2q^2}\right)\right)$", such as "For $\Gamma\left(\mathbb{Z}_{p^2q^2}\right)$, we have"

p. 5, l. 16: insert "The graph" before "$\Gamma\left(\mathbb{Z}_{p_1p_2p_3p_4}\right)$"

p. 5, l. 17: Since K has never been defined, you cannot "define K as usual."

p. 5, l. 19: replace "|/" with "|/"

p. 5, l. 20: replace "|/" with "|/" (both instances)

p. 5, l. -19: insert words before "$\chi\left(\Gamma\left(\mathbb{Z}_{p_1p_2p_3p_4}\right)\right)$", such as "For $\Gamma\left(\mathbb{Z}_{p_1p_2p_3p_4}\right)$, we have"

p. 5, l. -18: there are some strange characters where it should read "$\chi\left(\Gamma\left(\mathbb{Z}_{p_1p_2p_3p_4}\right)\right)$"

p. 5, l. -16: there are some strange characters where it should read "$\chi\left(\Gamma\left(\mathbb{Z}_{p_1p_2p_3p_4}\right)\right)$"

p. 5, l. -15: there are some strange characters where it should read "$\chi\left(\Gamma\left(\mathbb{Z}_{p_1p_2p_3p_4}\right)\right)$"

p. 5, l. -9: Since K has never been defined, you cannot "define our set of colors K as always."

p. 5, l. -8: replace "$p^{\lceil n/2 \rceil}|u,v$" with "$p^{\lceil n/2 \rceil}|u$ and $p^{\lceil n/2 \rceil}|v$" and specify that $u \neq v$

p. 5, l. -3: replace "on" with "with"

p. 5, l. -2: insert a period after "$ap^{\lceil n/2 \rceil} \neq 0$", capitalize "so", and insert a comma after "so"

p. 6, l. 9: Since K has never been defined, you cannot "define our set of colors K as usual."

p. 6, l. 10: replace "$|/$" with "$|\!\!\!/$"

p. 6, l. 12: replace "$|/$" with "$|\!\!\!/$" (both instances)

p. 6, l. 17: Since K has never been defined, you cannot "define K as usual."

p. 6, l. 19: insert a period before "thus", capitalize "thus", and insert a comma after "thus"

p. 6, l. 21: delete the comma after "clique"

p. 6, l. 22: Consider defining a variable to equal $p_1^{\lfloor a_1/2 \rfloor} p_2^{\lfloor a_2/2 \rfloor} \cdots p_n^{\lfloor a_n/2 \rfloor}$. It is very difficult to read this expression in the subscript and decipher what it means.

p. 6, l. -17: Same as previous comment.

p. 6, l. -16: replace "$|/$" with "$|\!\!\!/$"

p. 6, l. -14: replace "$|/$" with "$|\!\!\!/$", insert a period before "so", capitalize "so", and insert a comma before "so"

p. 6, l. -13: delete the comma after "i", insert a comma after "Thus", and consider making the subscript for k more readable (similar to the comment for p. 6, l. 22)

p. 6, l. -3: delete the extraneous "!"

p. 6, l. -1: replace "$|/$" with "$|\!\!\!/$"