S1 × G1

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January 9, 2006

Remark 0.1. $S^1 \times G^1$ has finite total area and so minimizers exist ([**?**], 5.5, 9.1) and are smooth curves of constant curvature ([**?**], Section 3.10).

Existence: Given an area, consider regions of that area. The set of their perimeters has an infimum because every set of positive real numbers has an infimum. Take a sequence of regions whose perimeters converge to the infimum. Since the perimeters converge, the perimeters are bounded. Since area is given it is bounded. Therefore by the local Compactness Theorem (see [**?**], 5.5, 9.1) the space of these regions is compact. So there is a subsequence of regions that converge. The limit has the right area because the area of $S^1 \times G^1$ is finite. A limit can have no more perimeter than the regions in the sequence so it must be equal to the infimum of the perimeters.

Lemma 0.2. *In $S^1 \times G^1$ a minimizer cannot be a homotopically trivial closed curve.*

Proof. Take a homotopically trivial closed curve C in $S^1 \times G^1$. Then slice the region enclosed by C with vertical slices and replace each slice with a halfline of equal weighted length. Each slice is G^1 and in G^1 halflines are minimizing. So the new curve has equal area but less perimeter than C. □

Lemma 0.3. *In $S^1 \times G^1$ there exist minimizers that have reflectional symmetry about the y-axis.*

Proof. Take a minimizer C in $S^1 \times G^1$. Then there is a pair of antipodal vertical line that slices the area in half. Take one of the halves and it and its reflection will still be minimizing. Translate this new curve left or right until the line of symmetry is the y-axis. □

Lemma 0.4. *In $S^1 \times G^1$ minimizers cannot intersect a horizontal line more than twice.*

Proof. Take a curve C that encloses an area in $S^1 \times G^1$ that intersects a horizontal line more than twice. Then slice the region enclosed horizontally. Since the density is constant in a horizontal line, single segments or the whole line are minimizing. So replace each slice with a single segment or whole line with equal weighted length. This process decreases perimeter and preserves area. □

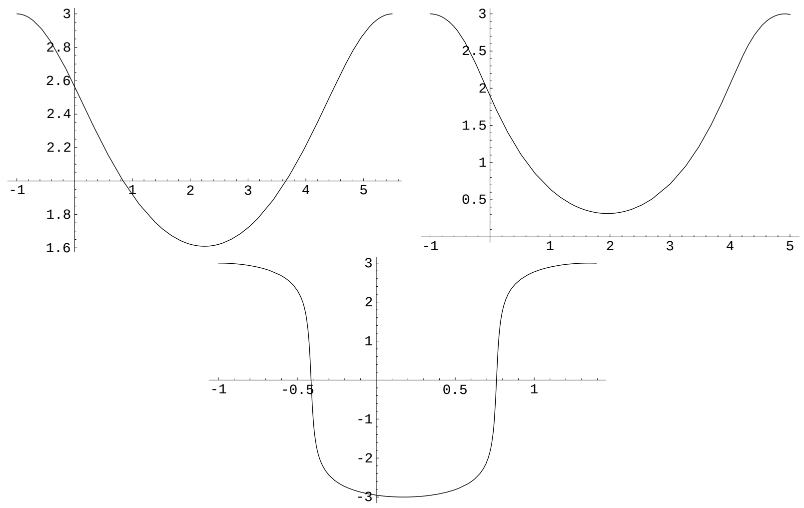

Figure 1: Examples of constant curvature curves in $S^1 \times G^1$

Proposition 0.5. *In $S^1 \times G^1$, with radius 1, all smooth constant-curvature curves with downward unit normal satisfy the following differential equation:*

$$x'(s)y''(s) - y'(s)x''(s) + x'(s)y(s) = \kappa_\phi \tag{1}$$

where s is the arc length, with the constraints

$$\begin{aligned} x'(s)^2 + y'(s)^2 &= 1 \\ x(0) &= -1 \\ x'(0) &= 1 \\ y'(0) &= 0. \end{aligned}$$

Proof. Equation **??** follows from the definition of $\phi - curvature$ and the fact that for arc length parametrization Euclidean curvature equals $x'(s)y''(s) - y'(s)x''(s)$. $\square$

Conjecture 0.6. *In $S^1 \times G^1$ minimizers are always horizontal lines if when the length of S^1 is small and horizontal lines or a pair of vertical lines.*

Support for the conjecture: I performed estimates on perimeter and area of some of these constant curvature curves and compared them to a horizontal line and found that the horizontal line seems to do better.

NOTE: When then length of S^1 is less than $2\sqrt{2\pi}$ the perimeter of the half vertical strip is always less than the perimeter of the vertical strip enclosing the same area.

References

[M1] Frank Morgan, *Geometric Measure Theory: a Beginner's Guide*, Academic Press, 2000.

[M2] Frank Morgan, *Regularity of isoperimetric hypersurfaces in Riemannian manifolds*, Trans. Amer. Math. Soc. **355** (2003), 5041-5052.