Referree's Report
Isoperimetric Regions in Spaces

This is a very nice paper, looking at isoperimetric problems in various surfaces, though the paper's title indicates it considers isoperimetric problems in general spaces. It would be a much better and unified paper, in particular more accessible to undergraduates, if the abstract and title reflected that the results center on abstract surfaces rather than abstract spaces. I would recommend that the paper be accepted with some minor corrections, though some careful rewriting would make it a much better paper. Specific comments and suggestions are given below.

Abstract: The abstract should be enlarged to give the reader a better overview of the paper and attract more readers. I would also suggest a better title.

Section 1: The classic isoperimetric problem is to seek the region that encloses the largest area among all regions with the same perimeter (isoperimetric means same perimeter), i.e. Queen Dido's problem (cf. "The Parsimonious Universe" by Hildebrandt and Tromba). You are really considering the dual problem. The solutions of the problems are equivalent, but the phrasing of the two problems are different.

The introduction indicates that you are considering isoperimetric problems in various surface, but the title indicates your are considering abstract spaces. It would worthwhile to unify the title and the introduction and the paper.

I would suggest a slight reordering of the sections and the introduction. In particular, you should consider starting with strips in $\mathbb{R}^2$ and $\mathbb{S}^2$ before considering two-dimensional twisted chimney space. Strips in $\mathbb{R}^2$ and $\mathbb{S}^2$ are less complex and more easily understood than two-dimensional twisted chimney space. Always start with the more basic material, and then add complexity. When you start mentioning that two-dimensional twisted chimney space is one of the analogues of the ten flat orientable models for the universe, a reference would be useful. You give the reference three pages later, but may have lost readers by not giving it early.

The lead in to the spaces that will be considered suggests that the results on these spaces will not be discussed, but rather only the spaces described. Trying to describe the results in detail can scare readers from continuing, as they start reading the technical results, methods and references. This could stop them from reading further. I would suggest just describing the spaces and the results, leave the references till the later sections.

Section 2: This is a good section with the right amount of detail. My only suggestions are to change *Proof Sketch* to *Sketch of Proof* and in the last sentence of Remark 2.2 to expound upon what you mean by a free boundary. This is a technical term that normally encountered in the undergraduate curriculum. Not explaining this could lead to confusion in the next sections.

Section 3: This is fine, as mentioned above I would only suggest interchanging sections 3 and 4 because of the added topological complexity in twisted chimney space.

Section 4: Again a good section, however, confusion could be avoided by mentioning that the perimeter of a region A is calculated only as the portion in the interior of S, much like the comment in Ros ....

Section 5: This section is the most technical and requires in my opinion that most care in writing. For instance, it is unclear whether Proposition 5.1 is a new result as it heavily references other work. Most of the proof as written is unaccessible with the references to [Co1], a preprint from the SMALL Program. This proposition is also never used later in the paper, except as a motivation for Conjecture 5.6. If this is not a new result, it would almost be worthwhile to leave it out, and state the result and maybe sketch the proof. (This proof seems more like a sketch, as it heavily depends on results in a preprint.)

The figure on Page 9 needs to be referenced in the text, so that the reader understands exactly what the figure is to help them understand. Otherwise, it looks like a intriguing picture that the reader has to place in context.

Propositions 5.2 and 5.3 should state explicitly that horizontal lines are lines of the form $y =$ constant and vertical lines are lines with $x =$ constant. My biggest complaint in the entire paper is the proof of Lemma 5.4. It seems either the result is a direct consequence of Remark 2.2, since the strip has a free boundary the minimizer intersects the free boundary, or else you need to directly prove that a minimizer can not be a circle of radius centered at the origin, as any rotation about the origin will not have this curve intersect the boundary. I have the same complaints about the proof of Lemma 5.5. The conjecture is great, but the supporting justification needs some explanation. Specifically, what is an unduloid, and why when it is projected would it be a nonlinear curve from one boundary to the other?

One last comment, why are Lemmas 5.4 and 5.5 actually Lemmas, when they are not used to prove a larger theorem, the hallmark of a lemma.

Section 6: This seems to be the main emphasis of the paper as it is the largest section in page length. First, there appear to be a few typos in this section. Above the Conjecture on Page 13, I believe you mean Mathematica not Mathematic. On the bottom of page 13, I believe you want 0 instead of κ_ψ as the left hand side of equation (3).

There needs to be some explanation of a 2α-cone before Proposition 6.7

Remark 6.9, Proposition 6.10, and Proposition 6.11 require some explanation of log-concave density or else just state that ψ need to be strictly concave.

Proposition 6.12 seems to completely unused in the paper. If necessary you need explain in some detail. There has no mention of mass, halflines, and perimeters in halflines. The propositions leading to Theorem 6.23 are good and lead to a nice result. But the propositions and lemmas after Theorem 6.23 require some explanation. It is pure theorem proof, with no explanation as to what these results say.

Remark 6.16 does not extend monotonicity result to $\mathbb{R}^n$. It indicates that results should extend not do extend. Either the result do extend or do not extend. Subsection 6.4 is the only results for higher dimensions and none of the results is outstanding. I would suggest scrapping these results for the paper to be unified and easier to understand.

Section 7: This is a nice extension to some product spaces and continues the theme of results on strips of surfaces.