Randomly Generated Triangles whose Vertices are Vertices of a Regular Polygon

Anna Madras
Drury University

Shova KC
Hope College

October 2, 2006

Abstract

We generate triangles randomly by uniformly choosing a subset of three vertices from the vertices of a regular polygon. We determine the expected area and perimeter in terms of the number of sides of the polygon. We use combinatorial methods combined with trigonometric summation formulas arising from complex analysis. We also determine the limit of these equations to compare with a classical result on triangles whose vertices are on a circle.

1 Introduction

This paper is an exploration into a method of randomly generating triangles. We start with a regular n-gon inscribed in a unit circle. Then out of $\binom{n}{3}$ possible subsets of vertices of size three, one subset is chosen uniformly. The three vertices form our randomly generated triangle.

Research into this method of generating triangles has lead us to four main questions:

- What is the average area of a triangle generated by this process?
- What is the limit of this area as n tends to infinity?
- What is the average perimeter of a triangle generated by this process?
- What is the limit of this perimeter as n tends to infinity?

In this paper, we present the answers to the above questions. In Section 2, we present background information, including the solution to a related problem dealing with three randomly chosen points on a unit circle. In particular, we detail the following classical result:

Theorem 1: Suppose that three points are randomly chosen uniformly and independently on the circumference of a unit circle. Then the expected area of the triangle formed is $\frac{3}{2\pi}$ and the expected perimeter of the triangle formed is $\frac{12}{\pi}$.

In Section 3, we give examples of finding expected area and perimeter in our situation for certain small values of n.

In Section 4, we present the main theorems and proofs. These are summarized in the following result.

Theorem 2: Let P be a regular n-gon inscribed in a unit circle. If a set of three vertices of P is chosen uniformly at random, then the expected area $\overline{A_n}$ of the triangle formed is

$$\overline{A_n} = \frac{3n\cot(\pi/n)}{2(n-1)(n-2)}.$$

The expected perimeter $\overline{P_n}$ of the triangle formed is

$$\overline{P_n} = \frac{6(\csc(\pi/n)+\cot(\pi/n))}{(n-1)}.$$

Moreover, we have

$$\lim_{n\to\infty} \overline{A_n} = \frac{3}{2\pi} \qquad \text{and} \qquad \lim_{n\to\infty} \overline{P_n} = \frac{12}{\pi},$$

which corresponds to the results given by Theorem 1.

In Section 5, we extend our main result to determine the expected area and perimeter of quadrilaterals, pentagons, and other polygons that are generated randomly by choosing vertices of a regular n-gon. We give two equations that can be used to find both the expected area and expected perimeter of an m-gon generated randomly from the vertices of a regular n-gon.

2 Related Problem on a Circle

Our research was motivated by the classical problem dealing with a unit circle [2]. Let P, Q, and R be points chosen randomly, uniformly, and independently on the circumference of a unit circle. Two questions naturally arise:

- What is the expected area of $\triangle PQR$?
- What is the expected perimeter of $\triangle PQR$?

The solution to the first question can be found online [2] and can be summarized as follows:

Theorem 2.1 *If three points are chosen randomly, uniformly, and independently on the circumference of a circle with radius one, then the expected area of the triangle inscribed in the circle is* $\frac{3}{2\pi}$.

Proof. (following [2]) One of the points can be taken to be $(1,0)$ without loss of generality. Let θ_1 and θ_2 be the polar angles of the other two points, as seen in Figure 1. Due to symmetry, we are able to assume without loss of generality that θ_1 is in the interval $[0,\pi]$, while θ_2 is allowed to take on any value in the interval $[0,2\pi)$.

The area of the triangle formed is given by

$$A = A(\theta_1,\theta_2) = \left| 2\sin\left(\frac{1}{2}\theta_1\right)\sin\left(\frac{1}{2}\theta_2\right)\sin\left(\frac{1}{2}(\theta_1-\theta_2)\right)\right|.$$

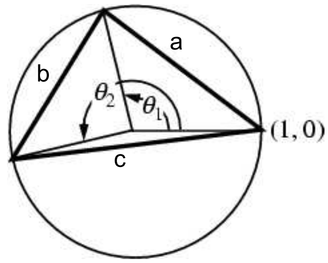

Figure 1: Circle with Inscribed Triangle (adapted from [2])

The expected area is given by

$$\overline{A} = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{\pi} A(\theta_1, \theta_2)\, d\theta_1\, d\theta_2.$$

This integral can be evaluated to give $\dfrac{3}{2\pi}$. See [2] for details of the integration. □

A solution for the average perimeter can be found in a similar way.

Theorem 2.2 *If three points are chosen randomly, uniformly, and independently on the circumference of a unit circle, then the expected perimeter of the triangle inscribed in the circle is* $\dfrac{12}{\pi}$.

Proof. The same assumptions can be made as in the previous proof without loss of generality. Let θ_1 and θ_2 be defined the same as in the previous proof. The lengths of the sides, a, b, and c, of the inscribed triangle are

$$a = 2\sin\frac{\theta_1}{2} \qquad b = 2\sin\frac{\theta_2}{2} \qquad c = 2\sin\frac{|\theta_2 - \theta_1|}{2}$$

where $\theta_1 = [0, \pi]$ and $\theta_2 = [0, 2\pi)$ (the absolute value is used for c because θ_2 may be less than θ_1). Since the perimeter of the triangle is $a + b + c$, the integral for finding expected perimeter is

$$\begin{aligned}
\overline{P} &= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{2\pi} (a + b + c)\, d\theta_2\, d\theta_1 \\
&= \frac{1}{2\pi^2} \int_0^{\pi} \int_0^{2\pi} \left(2\sin\frac{\theta_1}{2} + 2\sin\frac{\theta_2}{2} + 2\sin\frac{|\theta_2 - \theta_1|}{2} \right) d\theta_2 d\theta_1 \\
&= \frac{1}{2\pi^2} \int_0^{\pi} \left[\int_0^{2\pi} 2\sin\frac{\theta_1}{2} d\theta_2 + \int_0^{2\pi} 2\sin\frac{\theta_2}{2} d\theta_2 + \int_{\theta_1}^{2\pi} 2\sin\frac{(\theta_2 - \theta_1)}{2} d\theta_2 \right. \\
&\qquad \left. + \int_0^{\theta_1} 2\sin\frac{(\theta_1 - \theta_2)}{2} d\theta_2 \right] d\theta_1 \\
&= \frac{1}{2\pi^2} \int_0^{\pi} \left[4\pi\sin\frac{\theta_1}{2} + 8 - 4\cos\left(\pi - \frac{\theta_1}{2}\right) + 8 - 4\cos\frac{\theta_1}{2} \right] d\theta_1 \\
&= \frac{1}{2\pi^2} \left(-8\pi\cos\frac{\theta_1}{2} + 16\theta_1 + 8\sin\left(\pi - \frac{\theta_1}{2}\right) - 8\sin\frac{\theta_1}{2} \right) \Bigg|_0^{\pi} \\
&= \frac{12}{\pi}.
\end{aligned}$$

□

3 Examples and General Formulas

In this section, we give examples relating to randomly choosing a subset consisting of three vertices of a regular n-gon. We also show how the methods used in these specific cases lead to the formula for general values of n that will be needed in the next section.

3.1 Area Examples

Given a regular n-gon R, we number the vertices 0, 1, …, $n-1$. By $\triangle(a,b,c)$, we will mean the triangle inscribed in R whose vertices are numbered a, b, and c. We will denote the area of $\triangle(a,b,c)$ as $A(a,b,c)$ and the perimeter of $\triangle(a,b,c)$ as $P(a,b,c)$. We will assume for convenience that $0 \le a < b < c \le n-1$.

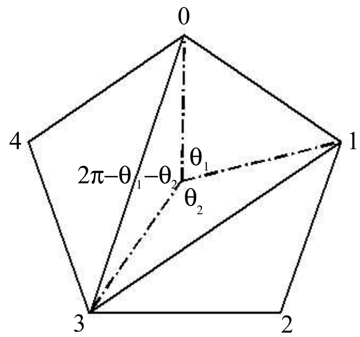

Figure 2: Pentagon

As an example, we calculate the area of $\triangle(0,1,3)$ inside a regular pentagon inscribed in a unit circle, as seen in Figure 2. To find the area of $\triangle(0,1,3)$, we add up the areas of the three triangles within the triangle, which gives us the equation

$$A(0,1,3) = \frac{1}{2}\left(2\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2} + 2\sin\frac{\theta_2}{2}\cos\frac{\theta_2}{2} + 2\sin\frac{2\pi-\theta_1-\theta_2}{2}\cos\frac{2\pi-\theta_1-\theta_2}{2}\right).$$

Then using the double angle identity we can simplify this equation to

$$A(0,1,3) = \frac{1}{2}\left(\sin\theta_1 + \sin\theta_2 + \sin(2\pi-\theta_1-\theta_2)\right)$$

where $\theta_1 = \frac{2\pi}{5}$ and $\theta_2 = \frac{4\pi}{5}$. Note that θ_1 and θ_2 are both multiples of the central angle of pentagon which is equal to $\frac{2\pi}{5}$. So, the area of $\triangle(0,1,3)$ can be written as

$$A(0,1,3) = \frac{1}{2}\left[\sin\frac{2\pi}{5} + \sin\left(2\cdot\frac{2\pi}{5}\right) + \sin\left(2\cdot\frac{2\pi}{5}\right)\right].$$

This can also be written as

$$A(0,1,3) = \frac{1}{2}\left[\sin\left((1-0)\cdot\frac{2\pi}{5}\right) + \sin\left((3-1)\cdot\frac{2\pi}{5}\right) + \sin\left((0-3+5)\cdot\frac{2\pi}{5}\right)\right].$$

This analysis easily generalizes to yield a general equation for finding the area of $\triangle(a,b,c)$ in any regular n-gon inscribed in a unit circle:

$$A(a,b,c) = \frac{1}{2}\left[\sin\left((b-a)\frac{2\pi}{n}\right) + \sin\left((c-b)\frac{2\pi}{n}\right) + \sin\left((a-c+n)\frac{2\pi}{n}\right)\right].$$

We remark that, in Figure 2, $\triangle(0,1,3)$ contains the center of the circle and therefore its area is computed as the sum of the areas of three smaller triangles. If this is not the case, the above formula for $A(a,b,c)$ still applies. For example, when computing the area of $\triangle(0,1,2)$ inside the regular pentagon, one can still use the three triangles determined by the center C of the circle and vertices 0, 1, and 2 of the pentagon. However, in obtaining the area of $\triangle(0,1,2)$, the area of the triangle determined by C and vertices 0 and 2 must be subtracted off of the sum of the other two areas. The formula given above does this automatically, since the appropriate sine function will be negative in this case.

We now turn our attention to the $n=5$ case of the first general question of interest. In particular, if R is a regular pentagon inscribed in a unit circle, and a set three vertices of R are chosen uniformly at random, then we compute the expected area of the triangle formed. In order to do this, we need to find the areas of all possible triangles $\triangle(a,b,c)$ where $0 \le a < b < c \le 4$. We then calculate $(b-a),(c-b)$ and $(a-c+n)$ for all those triangles and apply the formula derived above. By adding the areas of all of these triangles and dividing the result by the number of all such triangles, we will find the expected area.

The table below gives all possible triangles in pentagon along with $(b-a),(c-b)$ and $(a-c+n)$.

(a,b,c)	$(b-a)$	$(c-b)$	$(a-c+n)$
(0,1,2)	1	1	3
(0,1,3)	1	2	2
(0,1,4)	1	3	1
(0,2,3)	2	1	2
(0,2,4)	2	2	1
(0,3,4)	3	1	1
(1,2,3)	1	1	3
(1,2,4)	1	2	2
(1,3,4)	2	1	2
(2,3,4)	1	1	3

Table 1: Triangles Generated from a Regular Pentagon

The expected area of a randomly generated triangle based on the vertices of a regular pentagon inscribed in a unit circle is

$$\overline{A_5} = \frac{1}{\binom{5}{3}} \sum_{0\le a<b<c\le 4} A(a,b,c).$$

By examining Table 1, we count fifteen 1's, ten 2's, and five 3's that occur as either $(b-a)$, $(c-b)$, or $(a-c+n)$. By collecting all like terms and factoring out 1/2, the equation for the expected area reduces to

$$\overline{A_5} = \frac{1}{2(10)}\left(15\sin\frac{2\pi}{5} + 10\sin\frac{2\times 2\pi}{5} + 5\sin\frac{3\times 2\pi}{5}\right).$$

We now turn our attention to carrying out the analogous computation when $n = 6$. In other words, we are generating random triangles by choosing a set of three vertices of a regular hexagon inscribed in a unit circle. Again, we list all such triangles in order to collect like terms in the sum which gives the expected area of the triangle formed. In this case, there are $\begin{pmatrix} 6 \\ 3 \end{pmatrix} = 20$ possible triangles since $n = 6$.

(a,b,c)	$(b-a)$	$(c-b)$	$(a-c+n)$	(a,b,c)	$(b-a)$	$(c-b)$	$(a-c+n)$
(0,1,2)	1	1	4	(1,2,3)	1	1	4
(0,1,3)	1	2	3	(1,2,4)	1	2	3
(0,1,4)	1	3	2	(1,2,5)	1	3	2
(0,1,5)	1	4	1	(1,3,4)	2	1	3
(0,2,3)	2	1	3	(1,3,5)	2	2	2
(0,2,4)	2	2	2	(1,4,5)	3	1	2
(0,2,5)	2	3	1	(2,3,4)	1	1	4
(0,3,4)	3	1	2	(2,3,5)	1	2	3
(0,3,5)	3	2	1	(2,4,5)	2	1	3
(0,4,5)	4	1	1	(3,4,5)	1	1	4

Table 2: Triangles Generated from a Regular Hexagon

In the above table, there are twenty-four 1's, eighteen 2's, twelve 3's, and six 4's that occur as $(b-a)$, $(c-b)$, or $(a-c+n)$. Similar to the case of the pentagon, we can collect like terms in the sum

$$\overline{A_6} = \frac{1}{\begin{pmatrix} 6 \\ 3 \end{pmatrix}} \sum_{0\le a<b<c\le 5} A(a,b,c)$$

to obtain the result

$$\overline{A_6} = \frac{1}{2(20)}\left(24\sin\frac{2\pi}{6} + 18\sin\frac{2\times 2\pi}{6} + 12\sin\frac{3\times 2\pi}{6} + 6\sin\frac{4\times 2\pi}{6}\right).$$

The general equation for finding the expected area of a triangle formed by choosing three vertices of a regular n-gon inscribed in a unit circle is given by

$$\overline{A_n} = \frac{1}{2\begin{pmatrix} n \\ 3 \end{pmatrix}} \sum \left[\sin\left((b-a)\frac{2\pi}{n}\right) + \sin\left((c-b)\frac{2\pi}{n}\right) + \sin\left((a-c+n)\frac{2\pi}{n}\right)\right],$$

where the sum is taken over all integers a, b, c such that $0 \le a < b < c \le (n-1)$. In the next section, we will explore methods of simplifying this sum by collecting like terms as above, and then applying trigonometric summation identities.

3.2 Perimeter Examples

We now proceed to employ similar methods to compute the expected perimeter of a triangle randomly generated from the vertices of a regular pentagon or hexagon inscribed in a unit circle. As before, we use these results to indicate the proper form for the corresponding expression involving a regular n-gon.

Let $\triangle(a, b, c)$ be defined using vertices of a regular n-gon inscribed in the unit circle as above. Using basic trigonometry, it follows that the perimeter of $\triangle(a, b, c)$ is given by

$$P(a, b, c) = 2\left[\sin\left((b-a)\frac{\pi}{n}\right) + \left(\sin(c-b)\frac{\pi}{n}\right) + \left(\sin(n+a-c)\frac{\pi}{n}\right)\right].$$

Let R be a regular pentagon inscribed in a unit circle. The expected perimeter of a triangle chosen by randomly choosing a subset of three vertices of R is now given by

$$\overline{P_5} = \frac{1}{\binom{5}{3}} \sum_{0 \le a < b < c \le 4} P(a, b, c).$$

Using Table 1 this can be simplified to give

$$\overline{P_5} = \frac{2}{10}\left(15\sin\frac{\pi}{5} + 10\sin\frac{2\pi}{5} + 5\sin\frac{3\pi}{5}\right).$$

Now let R be a regular hexagon inscribed in a unit circle. The expected perimeter of a triangle chosen by randomly choosing a subset of three vertices of R is now given by

$$\overline{P_6} = \frac{1}{\binom{6}{3}} \sum_{0 \le a < b < c \le 5} P(a, b, c).$$

Using Table 2 this can be simplified to give

$$\overline{P_6} = \frac{2}{20}\left(24\sin\frac{\pi}{6} + 18\sin\frac{2\pi}{6} + 12\sin\frac{3\pi}{6} + 6\sin\frac{4\pi}{6}\right).$$

The general equation for finding the expected perimeter of a triangle formed by choosing three vertices of a regular n-gon inscribed in a unit circle is given by

$$\overline{P_n} = \frac{2}{\binom{n}{3}} \sum \left[\sin\left((b-a)\frac{\pi}{n}\right) + \sin\left((c-b)\frac{\pi}{n}\right) + \sin\left((a-c+n)\frac{\pi}{n}\right)\right],$$

where the sum is taken over all integers a, b, c such that $0 \le a < b < c \le (n-1)$. We will simplify this expression in the next section.

4 The Main Theorems

Theorem 4.1 *Let R be a regular n-gon inscribed in a unit circle. If a subset consisting of three distinct vertices of R is chosen uniformly at random, then the expected area of the triangle formed is given by*

$$\overline{A_n} = \frac{3n}{2(n-1)(n-2)} \cot\frac{\pi}{n}.$$

Proof. From the preceding section, it follows that

$$\overline{A_n} = \frac{1}{2\binom{n}{3}} \sum \left[\sin\left((b-a)\frac{2\pi}{n}\right) + \sin\left((c-b)\frac{2\pi}{n}\right) + \sin\left((a-c+n)\frac{2\pi}{n}\right)\right],$$

where the sum is taken over all integers a, b, c such that $0 \leq a < b < c \leq (n-1)$.

For each k such that $0 \leq k \leq n-2$, let $p(n,k)$ be defined to be the number of times k occurs as $(b-a)$,$(c-b)$, or $(a-c+n)$ in the above sum, so that

$$\overline{A_n} = \frac{1}{2\binom{n}{3}} \sum_{k=1}^{n-2} p(n,k) \sin\frac{2k\pi}{n} = \frac{1}{2\binom{n}{3}} \sum_{k=0}^{n-2} p(n,k) \sin\frac{2k\pi}{n}.$$

In the last expression, we have added the $k=0$ term to the sum for ease of computation later. It makes no difference in the value of the sum, since $\sin(0) = 0$.

Claim: $p(n,k) = n(n-1-k)$ for all $n \geq 3$ and $1 \leq k \leq (n-2)$.

Proof of Claim:[1] Let $n \geq 3$, and let k be an integer satisfying $1 \leq k \leq (n-2)$. We count the number of times that a difference of size k occurs due to a triple of the form $(0, b, c)$ (with $0 < b < c \leq n-1$), where 0 is used in the subtraction (that is, $k = b - 0$ or $k = 0 - c + n$). There are $n-1-k$ triples of the form $(0, k, c)$ and $n-1-k$ triples of the form $(0, b, n-k)$, and therefore a difference of size k occurs in this way $2(n-1-k)$ times. By symmetry, for any j such that $1 \leq j \leq n-1$, a difference of size k occurs from a triple in a subtraction involving j exactly $2(n-1-k)$ times. There are n choices for j (counting $j = 0$), and $2(n-1-k)$ differences of size k for each j. This process counts each difference of size k exactly twice (once for each number involved in the subtraction), and therefore, $p(n,k) = n(n-1-k)$, as desired.

From the claim, it follows that

$$\overline{A_n} = \frac{1}{2\binom{n}{3}} \sum_{k=0}^{n-2} n(n-1-k) \sin\frac{2k\pi}{n}. \tag{4-1}$$

Equation (4-1) implies that

$$\overline{A_n} = \frac{1}{2\binom{n}{3}} \left((n^2 - n) \sum_{k=0}^{n-2} \sin\frac{2k\pi}{n} - n \sum_{k=0}^{n-2} k \sin\frac{2k\pi}{n} \right). \tag{4-2}$$

Let $f(x) = \sum_{k=0}^{n-2} \cos(kx)$ and $g(x) = \sum_{k=0}^{n-2} \sin(kx)$. Euler's Formula gives that

$$e^{ikx} = \cos kx + i \sin kx,$$

[1] We are thankful to the referee for providing this alternative to our original proof of the claim. This proof is simpler and leads more naturally into the extensions given in Section 5.

where $i^2 = -1$. It follows from the partial sum formula for a geometric series that

$$\sum_{k=0}^{n-2}(f(x) + i\,g(x)) = \sum_{k=0}^{n-2} e^{ikx} = \frac{1 - e^{i(n-1)x}}{1 - e^{ix}}$$

Simplifying this right hand side, and comparing real and imaginary parts yields the trigonometric summation identities

$$f(x) = \sum_{k=0}^{n-2}\cos(kx) = \frac{1 - \cos(((n-2)+1)x) - \cos(x) + \cos((n-2)x)}{2 - 2\cos(x)} \tag{4-3}$$

and

$$g(x) = \sum_{k=0}^{n-2}\sin(kx) = \frac{\sin(x) + \sin((n-2)x) - \sin(((n-2)+1)x)}{2 - 2\cos(x)}. \tag{4-4}$$

Also, by differentiating, we obtain

$$f'(x) = -\sum_{0}^{n-2} k\sin(kx) = \frac{d}{dx}\left[\frac{1 - \cos(((n-2)+1)x) - \cos(x) + \cos((n-2)x)}{2 - 2\cos(x)}\right]. \tag{4-5}$$

Equation (4-2) implies that

$$\overline{A_n} = \frac{n}{2\begin{pmatrix} n \\ 3 \end{pmatrix}}\left((n-1)g(2\pi/n) + f'(2\pi/n)\right).$$

A tedious simplification using trigonometric identities now yields

$$\overline{A_n} = \frac{3n}{2(n-1)(n-2)}\cot\frac{\pi}{n}.$$

□

The question concerning perimeter can be answered by a similar process.

Theorem 4.2 *Let R be a regular n-gon inscribed in a unit circle. If a subset consisting of three distinct vertices of R is chosen uniformly at random, then the expected perimeter of the triangle formed is given by*

$$\overline{P_n} == \frac{6(\csc\frac{\pi}{n} + \cot\frac{\pi}{n})}{(n-1)}.$$

Proof. As this proof is nearly identical to the proof of Theorem 4.1, we provide only a sketch. From the preceding section, we have

$$\overline{P_n} = \frac{2}{\begin{pmatrix} n \\ 3 \end{pmatrix}}\sum\left[\sin\left((b-a)\frac{\pi}{n}\right) + \sin\left((c-b)\frac{\pi}{n}\right) + \sin\left((a-c+n)\frac{\pi}{n}\right)\right],$$

where the sum is taken over all integers a, b, c such that $0 \le a < b < c \le (n-1)$. Using the Claim 1 from the proof of Theorem 4.1, we find that

$$\overline{P_n} = \frac{2}{\begin{pmatrix} n \\ 3 \end{pmatrix}}\sum_{k=0}^{n-2} n(n-1-k)\sin\frac{k\pi}{n}$$

Then by using the Euler's formula and the functions $f(x)$ and $g(x)$ as in the proof of Theorem 4.1, we can simplify this expression to the closed form as desired. $\square$

Now that we have derived closed form expressions for the expected area and perimeter, we can determine the limit as n tends to infinity and compare with the answers for the problem on the unit circle from Section 2.

Theorem 4.3 *Let $\overline{A_n}$ and $\overline{P_n}$ be as in Theorems 4.1 and 4.2. Then*

$$\lim_{n\to\infty} \overline{A_n} = \frac{3}{2\pi} \qquad \textit{and} \qquad \lim_{n\to\infty} \overline{P_n} = \frac{12}{\pi}.$$

Proof. We rearrange terms and apply L´Hôpital's rule.

$$\begin{aligned}
\lim_{n\to\infty} \overline{A_n} &= \frac{3}{2} \lim_{n->\infty} \frac{n \cot \frac{\pi}{n}}{(n-1)(n-2)} \\
&= \frac{3}{2} \lim_{n->\infty} \frac{n}{(n-1)(n-2)} \cdot \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cdot \frac{\pi}{n} \cdot \frac{n}{\pi} \\
&= \frac{3}{2} \lim_{n->\infty} \left(\frac{n^2}{(n-1)(n-2)}\right) \left(\frac{\cos \frac{\pi}{n}}{\pi}\right) \left(\frac{\pi}{n \sin \frac{\pi}{n}}\right) \\
&= \frac{3}{2} \cdot 1 \cdot \frac{1}{\pi} \cdot 1 = \frac{3}{2\pi}.
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \overline{P_n} &= \lim_{n->\infty} \frac{6(\csc \frac{\pi}{n} + \cot \frac{\pi}{n})}{n-1} \\
&= \lim_{n->\infty} \frac{6}{n-1} \cdot \frac{1+\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cdot \frac{\pi}{n} \cdot \frac{n}{\pi} \\
&= \lim_{n->\infty} 6 \left(\frac{1+\cos \frac{\pi}{n}}{\pi}\right) \left(\frac{n}{n-1}\right) \left(\frac{\pi}{n \sin \frac{\pi}{n}}\right) \\
&= 6 \cdot \frac{2}{\pi} \cdot 1 \cdot 1 = \frac{12}{\pi}
\end{aligned}$$

$\square$

We find that the limits of $\overline{A_n}$ and $\overline{P_n}$ correspond precisely with the answers for the problem on the circle found in Section 2.

5 Extensions

In this section, we study the effects of increasing the number of vertices chosen, thus constructing randomly generated polygons based on the vertices of a regular polygon inscribed in a unit circle. The analysis is very similar to the case of random triangles detailed in the last section, and therefore, we will only sketch the main details here. Further details and extensions are left to the interested reader.

We use $\overline{A_{n,m}}$ and $\overline{P_{n,m}}$ to represent the expected area and perimeter, respectively, of an m-gon generated randomly by (uniformly) choosing a subset of size m of the vertices of a regular n-gon inscribed in a unit circle.

Theorem 5.1 *Let R be a regular n-gon inscribed in a unit circle. Suppose that a subset consisting of m vertices of R is chosen uniformly at random. Then the expected area of the convex m-gon formed is given by*

$$\overline{A_{n,m}} = \frac{1}{2\begin{pmatrix} n \\ m \end{pmatrix}} \sum \sum_{j=1}^{m} \sin\left((a_{j+1} - a_j)\frac{2\pi}{n}\right),$$

where the first sum is over all $a_1, \ldots, a_m$ with $0 \leq a_1 < a_2 < \cdots < a_m \leq (n-1)$, and $a_{m+1} = a_1$. The expected perimeter of this m-gon is

$$\overline{P_{n,m}} = \frac{2}{\begin{pmatrix} n \\ m \end{pmatrix}} \sum \sum_{j=1}^{m} \sin\left((a_{j+1} - a_j)\frac{\pi}{n}\right).$$

.

The preceding result shows, for example, that if a set of 4 vertices of a regular n-gon inscribed in a unit circle is chosen uniformly at random, then the expected area and perimeter of the quadrilateral formed are given by

$$\overline{A_{n,4}} = \frac{1}{2\begin{pmatrix} n \\ 4 \end{pmatrix}} \sum \left(\sin(b-a)\frac{2\pi}{n} + \sin(c-b)\frac{2\pi}{n} + \sin(d-c)\frac{2\pi}{n} + \sin(a-d+n)\frac{2\pi}{n}\right),$$

and

$$\overline{P_{n,4}} = \frac{2}{\begin{pmatrix} n \\ 4 \end{pmatrix}} \sum \left(\sin(b-a)\frac{\pi}{n} + \sin(c-b)\frac{\pi}{n} + \sin(d-c)\frac{\pi}{n} + \sin(a-d+n)\frac{\pi}{n}\right),$$

where the sum is taken over all integers a, b, c, d such that $0 \leq a < b < c < d \leq (n-1)$.

We can collect like terms in the summations in Theorem 5.1 to prove the following result. As in the case of the triangle, the $k = 0$ term adds nothing to the sum since $\sin(0) = 0$; this term is included for ease of computation later.

Theorem 5.2 *Let R be a regular n-gon inscribed in a unit circle. Suppose that a subset consisting of m vertices of R is chosen uniformly at random. Then the expected area of the convex m-gon formed is given by*

$$\overline{A_{n,m}} = \frac{n}{2\begin{pmatrix} n \\ m \end{pmatrix}} \sum_{k=0}^{n-m+1} \begin{pmatrix} n-k-1 \\ m-2 \end{pmatrix} \sin\frac{2k\pi}{n}.$$

The expected perimeter of this m-gon is given by

$$\overline{P_{n,m}} = \frac{2n}{\begin{pmatrix} n \\ m \end{pmatrix}} \sum_{k=0}^{n-m+1} \begin{pmatrix} n-k-1 \\ m-2 \end{pmatrix} \sin\frac{k\pi}{n}.$$

Sketch of Proof: We do not include a full proof of this theorem, since it is essentially the same as the proof of the Claim in the proof of Theorem 4.1. The main difference is the appearance of the coefficient $\binom{n-k-1}{m-2}$. This occurs because, for each relevant k, there are now $\binom{n-k-1}{m-2}$ lists of integers of the form

$$0 < k < c_3 < \cdots < c_m \le n-1,$$

and $\binom{n-k-1}{m-2}$ lists of integers of the form

$$0 < c_2 < c_3 < \cdots < c_{m-1} < n-k \le n-1. \ \square$$

We now sketch the details of a method of simplifying the above summations to closed form. Since the methods for $\overline{A_{n,m}}$ and $\overline{P_{n,m}}$ are nearly identical, we will focus simply on $\overline{A_{n,m}}$ and leave the analogous results concerning $\overline{P_{n,m}}$ for the interested reader.

From Euler's formula and Theorem 5.2, we see that $\overline{A_{n,m}} = \operatorname{Im} g(n,m)$, where

$$g(n,m) = \frac{n}{2\binom{n}{m}} \sum_{k=0}^{n-m+1} \binom{n-k-1}{m-2} \alpha^k, \qquad \text{with} \qquad \alpha = e^{2\pi i/n}.$$

We note that

$$\begin{aligned}
(\alpha-1)g(n,m) &= \frac{n}{2\binom{n}{m}} \left[\sum_{k=0}^{n-m+1} \binom{n-k-1}{m-2} \alpha^{k+1} - \sum_{k=0}^{n-m+1} \binom{n-k-1}{m-2} \alpha^k \right] \\
&= \frac{n}{2\binom{n}{m}} \left[\sum_{k=1}^{n-m+2} \binom{n-k}{m-2} \alpha^k - \sum_{k=0}^{n-m+1} \binom{n-k-1}{m-2} \alpha^k \right] \\
&= \frac{n}{2\binom{n}{m}} \left[\left(\sum_{k=0}^{n-m+2} \left(\binom{n-k}{m-2} - \binom{n-k-1}{m-2} \right) \alpha^k \right) - \binom{n}{m-2} \right] \\
&= \frac{n}{2\binom{n}{m}} \left[\left(\sum_{k=0}^{n-m+2} \binom{n-k-1}{m-3} \alpha^k \right) - \binom{n}{m-2} \right] \\
&= \frac{n}{2\binom{n}{m}} \left[\frac{2}{n} \binom{n}{m-1} g(n,m-1) - \binom{n}{m-2} \right] \\
&= \frac{m}{n-m+1} g(n,m-1) - \frac{nm(m-1)}{2(n-m+2)(n-m+1)}.
\end{aligned}$$

This implies that

$$g(n,m) = \frac{m}{(\alpha-1)(n-m+1)} \left(g(n,m-1) - \frac{n(m-1)}{2(n-m+2)} \right).$$

Recall that $\alpha = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n)$. It follows that $\alpha^{-1} = \bar{\alpha} = \cos(2\pi/n) - i\sin(2\pi/n)$. We can make the denominator of the expression for $g(n,m)$ real by multiplying numerator and denominator by $(\bar{\alpha}-1)$, resulting in

$$g(n,m) = \frac{m(\bar{\alpha}-1)}{(n-m+1)(2-\alpha-\bar{\alpha})}\left(g(n,m-1) - \frac{n(m-1)}{2(n-m+2)}\right).$$

Substituting in for α and computing real and imaginary parts yields

$$\begin{aligned}
g(n,m) &= \frac{-m\left(1-\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}{4(n-m+1)\sin^2\frac{\pi}{n}}\left(g(n,m-1) - \frac{n(m-1)}{2(n-m+2)}\right) \\
&= \frac{1}{2}(1+i\cot\frac{\pi}{n})\left[\frac{-m}{n-m+1}\left(g(n,m-1) - \frac{n(m-1)}{2(n-m+2)}\right)\right] \\
&= \frac{-m}{2(n-m+1)}\left(1+i\cot\frac{\pi}{n}\right)\left(\operatorname{Re} g(n,m-1) + i\operatorname{Im} g(n,m-1) - \frac{n(m-1)}{2(n-m+2)}\right) \\
&= \frac{-m}{2(n-m+1)}\left[\operatorname{Re} g(n,m-1) - \frac{n(m-1)}{2(n-m+2)} - \operatorname{Im} g(n,m-1)\cot\frac{\pi}{n}\right. \\
&\quad \left. +i\left(\operatorname{Im} g(n,m-1) + \operatorname{Re} g(n,m-1)\cot\frac{\pi}{n} - \frac{n(m-1)}{2(n-m+2)}\cot\frac{\pi}{n}\right)\right]
\end{aligned}$$

Recall that the $\overline{A_{n,m}}$ is equal to the imaginary part of $g(n,m)$. Therefore,

$$\overline{A_{n,m}} = \frac{-m}{2(n-m+1)}\left(\operatorname{Im} g(n,m-1) + \operatorname{Re} g(n,m-1)\cot\frac{\pi}{n} - \frac{n(m-1)}{2(n-m+2)}\cot\frac{\pi}{n}\right).$$

Write $g(n,m) = a_m + ib_m$, and note that a_m and b_m are both functions of n and that $\overline{A_{n,m}} = b_m$. Let $a_2 = b_2 = 0$. The above computations imply the recurrence

$$\begin{aligned}
a_m &= \frac{m}{2(n-m+1)}\left[-a_{m-1} + b_{m-1}\cot\frac{\pi}{n} + \frac{n(m-1)}{2(n-m+2)}\right] \\
b_m &= \frac{m}{2(n-m+1)}\left[-a_{m-1}\cot\frac{\pi}{n} - b_{m-1} + \frac{n(m-1)}{2(n-m+2)}\cot\frac{\pi}{n}\right]
\end{aligned}$$

In vector form, this recurrence becomes

$$\begin{pmatrix} a_m \\ b_m \end{pmatrix} = \frac{m}{2(n-m+1)}\left[\begin{pmatrix} -1 & \cot\frac{\pi}{n} \\ -\cot\frac{\pi}{n} & -1 \end{pmatrix}\begin{pmatrix} a_{m-1} \\ b_{m-1} \end{pmatrix} + \frac{n(m-1)}{2(n-m+2)}\begin{pmatrix} 1 \\ \cot\frac{\pi}{n} \end{pmatrix}\right],$$

with initial condition

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Using this recurrence, one can compute $b_m = \overline{A_{n,m}}$.

The vector recurrence can also be solved using diagonalization of the matrix $M = \begin{pmatrix} -1 & \cot\frac{\pi}{n} \\ -\cot\frac{\pi}{n} & -1 \end{pmatrix}$. The eigenvalues of M are $i\cot\frac{\pi}{n} - 1$ and $-i\cot\frac{\pi}{n} - 1$ and the respective eigenvectors are $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$. This can be used to show that

$$b_m = \frac{n}{2\begin{pmatrix} n \\ m \end{pmatrix}}\operatorname{Im}\sum_{j=1}^{m-2}\begin{pmatrix} n \\ m-1-j \end{pmatrix}\left(\frac{i\cot\frac{\pi}{n} - 1}{2}\right)^j.$$

m	b_m
3	$\frac{3n\cot\frac{\pi}{n}}{2(n-1)(n-2)}$
4	$\frac{3n\cot\frac{\pi}{n}}{(n-1)(n-2)}$
5	$\frac{5n\cot\frac{\pi}{n}(2n^2-12n+19-3\cot^2\frac{\pi}{n})}{2(n-1)(n-2)(n-3)(n-4)}$
6	$\frac{15n\cot\frac{\pi}{n}(n^2-5n+7-3\cot^2\frac{\pi}{n})}{2(n-1)(n-2)(n-3)(n-4)}$
7	$\frac{21n\cot\frac{\pi}{n}(2n^4-30n^3+160n^2-10n^2\cot^2\frac{\pi}{n}+90n\cot^2\frac{\pi}{n}-360n-230\cot^2\frac{\pi}{n}+303+15\cot^4\frac{\pi}{n})}{4(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}$
8	$\frac{7n\cot\frac{\pi}{n}(2n^4-28n^3-15n^2\cot^2\frac{\pi}{n}+139n^2+105n\cot^2\frac{\pi}{n}-287n-240\cot^2\frac{\pi}{n}+219+45\cot^4\frac{\pi}{n})}{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}$

Table 3: Expected Area $\overline{A_{n,m}}$

We may simplify this further to yield

$$\overline{A_{n,m}} = \frac{n}{2\binom{n}{m}} \sum_{l=0}^{\lfloor\frac{m-3}{2}\rfloor} \left[\sum_{j=2l+1}^{m-2} \frac{(-1)^{j+l-1}\binom{n}{m-1-j}\binom{j}{2l+1}}{2^j} \right] (\cot\frac{\pi}{n})^{2l+1}.$$

The above methods may be applied directly to compute $\overline{P_{n,m}}$. This is left to the reader. The interested reader may also like to pursue the generalization to the circle problem from Section 2 for $m > 3$, and its relationship to the results given here.

6 Acknowledgements

We would like to acknowledge our wonderful advisor, Darin Stephenson of Hope College, for guiding us through the research process. We would also like to thank Hope College and the National Science Foundation for allowing us to participate in this research program and for funding. We thank Chris Smith of the Grand Valley State University REU for help in understanding the hypergeometric summations that arise in Section 5. We are thankful to our colleague Daniela Banu for helping us with the proof of Theorem 4.3. We would also like to thank all of our brilliant research colleagues, who not only helped us out with our research but also made coming to work a joyful and memorable experience. Finally, we thank the referee, both for providing an alternate proof for the Claim in Theorem 4.1, which leads more naturally into the rationale for Theorem 5.2, and for numerous other comments which made this a better paper.

The computer software that made this research possible includes: MAPLE, Adobe Photoshop/Reader, Geometer's Sketchpad, and Winedt. We appreciate all those who

helped us to effectively use these programs, especially Mark Pearson, Darin Stephenson, and Airat Bekmetjev.

References

[1] Hogg R. and Tanis E. *Probability and Statistical Inference*, 6th Edition, Prentice Hall, 2001.

[2] Weisstein, Eric W. Circle Triangle Picking, from *MathWorld* A Wolfram Web Resource. `http://www.mathworld.wolfram.com/CircleTrianglePicking.html`.

[3] Zwillinger, D. *CRC: Standard Mathematical Tables and Formulae*, 31st. Edition, Chapman and Hall, 2000.