Population Dynamics with Nonlinear Diffusion

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Abstract

We consider reaction diffusion models in population dynamics where the per capita growth rate is a logistic type or a weak Allee type. In particular, we study the effects of nonlinear diffusion (arising due to aggregative population movements) on the steady states. We obtain our results via the quadrature method.

1 Introduction

In this paper we discuss steady state reaction diffusion equations with Dirichlet boundary conditions arising in population dynamics. These equations are of the form:

$$\begin{cases} -(d\phi(u)u')' = u\tilde{f}(u), \quad x \in (0,1) \\ \quad u(0) = 0 = u(1). \end{cases} \tag{1.1}$$

Here u is the population density, $\tilde{f}(u)$ is the per capita growth rate, $d\phi(u)$ represents the diffusion coefficient where $d > 0$ is a constant and $\phi(u) > 0$ for $u \geq 0$.

When $\phi(u) = u$ and $\tilde{f}$ is of logistic type ($\tilde{f}(0) > 0$, $\tilde{f}$ decreasing and becomes negative for large u), there is a rich history of results on the steady states and on the dynamics of the associated time dependent problem. In particular, for any positive initial data, the solution to the time dependent problem eventually dies out if $\lambda < \frac{\pi^2}{\tilde{f}(0)}$, and approaches a unique positive steady state if $\lambda > \frac{\pi^2}{\tilde{f}(0)}$ (unconditional persistence). Here $d = \frac{1}{\lambda}$. However when $\phi(u) = u$ and $\tilde{f}$ is of the Allee effect type ($\tilde{f}$ increasing for small values of u but eventually decreases and becomes negative for large u), unconditional persistence no longer persists for all λ. In particular, when $\tilde{f}(0) > 0$ (weak Allee effect), the solution for the time dependent problem eventually dies out if $\lambda < \lambda_*$ (some) with $\lambda^* < \frac{\pi^2}{\tilde{f}(0)}$, and approaches unique positive steady state if $\lambda_* > \frac{\pi^2}{\tilde{f}(0)}$, while if $\lambda \in (\lambda_*, \frac{\pi^2}{\tilde{f}(0)})$ the steady state of the solution depends on the size of the initial data (conditional persistence). In the case when $\tilde{f}(0) < 0$ (strong Allee effect), the solution for the time dependent problem eventually dies out if $\lambda < \lambda_{**}$ (some) while if $\lambda > \lambda_{**}$, the steady state of the solution depends on the size of the initial data (conditional persistence). Typical bifurcation diagrams for the logistic and Allee type problems are given below (see Figure 1).

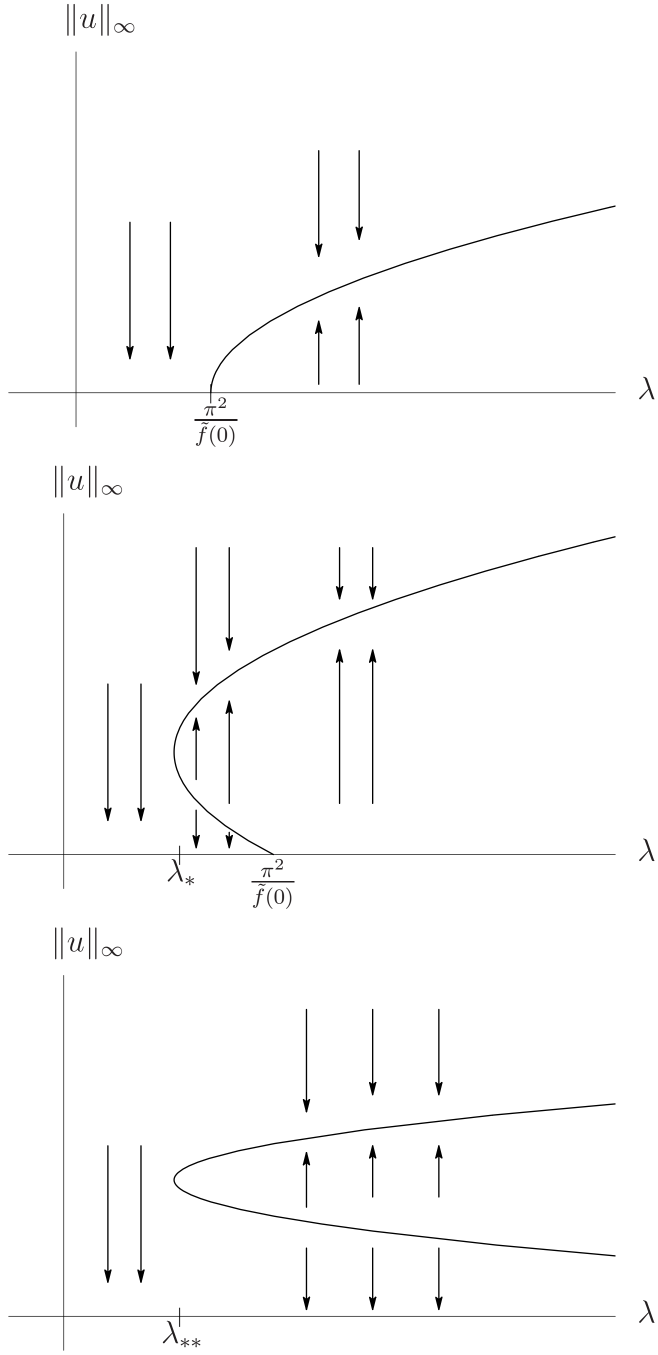

Figure 1: Bifurcation diagrams: logistic (upper); weak Allee effect (middle); strong Allee effect (lower).

In this paper, we examine effects of nonlinear diffusion. We study the combined effects of $\tilde{f}(u)$ and $\phi(u)$ on the positive steady states. In particular, we establish that the combined effects of $\tilde{f}(u)$ and ϕ can produce both Allee and logistic type reactions (conditional/unconditional persistence) from both logistic and weak Allee types $\tilde{f}$.

We note that one can rewrite (1.1) (see Section 5.1) as

$$\begin{cases} -v'' = \lambda\psi^{-1}(v)\tilde{f}(\psi^{-1}(v)), & x \in (0,1) \\ v(0) = 0 = v(1) \end{cases} \tag{1.2}$$

where $\psi(u) := \int_0^u \phi(s)ds$, and $v = \psi(u)$. We include the following examples in our study (see [3], [4], [8], [9] and [10]) for the biological relevance of many of these examples):

$$\begin{array}{ll}
\tilde{f}(u) = a - bu, \quad a, b > 0, & \text{(Logistic)} \\
\tilde{f}(u) = (u+a)(b-u), \quad b > a > 0, & \text{(Weak Allee Effect)} \\
\tilde{f}(u) = (m-u) - (\frac{K}{1+u}); \quad m > K > 0, K \leq 1 & \text{(Logistic with Predation)} \\
\tilde{f}(u) = (m-u) - (\frac{K}{1+u}); \quad m > K > 0, K > 1 & \text{(Weak Allee with Predation)} \\
\psi(u) = u & \text{(Linear Diffusion)} \\
\psi(u) = e^{\alpha u} - 1 & \text{(Nonlinear Diffusion)} \\
\psi(u) = u^3 - Bu^2 + Cu, & \text{(Nonlinear Diffusion)} \\
\quad \text{where} \quad B, C > 0, B^2 - 3C < 0 & \\
\quad (\text{when} \quad B^2 - 3C < 0, \phi(u) = \psi'(u) > 0). &
\end{array}$$

The study of positive solutions to (1.1) when $\psi(u) = u$ has significant history (see [3] and the references within), however, much less is known about the nonlinear case (arising due to aggregative population movements). In this paper we pursue this study by analyzing the combined effects of $\psi(u)$ and $\tilde{f}(u)$ on the positive steady states. We do this by studying (1.2) via the quadrature method which is described briefly in Section 2. For further information on the quadrature method see [1], [2] and [6]. In Section 3, we study (1.2) when $\psi(u) = u$ and $\tilde{f}(u)$ is logistic. In Section 4, we discuss the case when $\psi(u) = u$ and $\tilde{f}(u)$ displays Allee effect with $\tilde{f}(0) > 0$ (weak Allee effect). In Section 5, we examine the case when $\psi(u)$ is nonlinear and its effect on the two per capita growth rate cases; logistic and weak Allee effect. Finally in Section 6, we discuss population models with constant yield harvesting, comparing examples with linear and nonlinear diffusion.

2 Quadrature Method

In this section we will analyze the positive solutions to the following equation:

$$\begin{cases} -u''(x) = \lambda f(u(x)), & x \in (0,1) \\ \quad u(0) = 0 = u(1), \end{cases} \tag{2.1}$$

where $f : [0, \infty) \to (0, \infty)$ is a C^1 function and λ is a nonnegative parameter. By the Picard's theorem, it follows that a positive solution must be symmetric about $x = \frac{1}{2}$ which increases on $(0, \frac{1}{2})$ and decreases on $(\frac{1}{2}, 1)$. We establish the existence of solutions to (2.1) by the following theorem:

Theorem 2.1. *Let u be a positive solution to* (2.1) *with $u(\frac{1}{2}) = \rho > 0$. Such a solution exists iff*

$$G(\rho) := \sqrt{2}\int_0^\rho \frac{dt}{\sqrt{F(\rho)-F(t)}} = \quad \sqrt{\lambda} \tag{2.2}$$

where $F(u) = \int_0^u f(s)ds$*.*
The solution $u(x)$ *is described by*

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho)-F(t)}} = \sqrt{2\lambda}x; \quad x \in [0, \tfrac{1}{2}]. \tag{2.3}$$

Proof. Let u be a positive solution to (2.1) with the property that $u'(x_0) = 0$ with $x_0 \in (0,1)$. Observe that both $v(x) := u(x_0+x)$ and $w(x) := u(x_0-x)$ satisfy the following initial value problem:

$$\begin{aligned} -z''(x) &= \lambda f(z(x)) \\ z(0) &= u(x_0) \\ z'(0) &= 0 \end{aligned}$$

for $x \in [0,c)$ where $c = \min\{x_0, 1-x_0\}$. This implies that $u(x_0+x) = u(x_0-x) \quad \forall x \in [0,c)$. Since u is a positive solution to (2.1), it must be symmetric around $x = \frac{1}{2}$, at which point it has a maximum $u(\frac{1}{2}) = \rho$.
Multiplying (2.1) by $u'(x)$,

$$-\left(\frac{[u'(x)]^2}{2}\right)' = \lambda[F(u(x))]' \tag{2.4}$$

where $F(s) = \int_0^s f(z)dz$.
Integrating both sides,

$$\frac{u'(x)}{\sqrt{F(\rho)-F(u(x))}} = \sqrt{2\lambda}; \quad x \in [0, \tfrac{1}{2}). \tag{2.5}$$

Integrating again,

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho)-F(t)}} = \sqrt{2\lambda}x; \quad x \in [0, \tfrac{1}{2}]. \tag{2.6}$$

Since $u(\frac{1}{2}) = \rho$, we have,

$$G(\rho) := \sqrt{2}\int_0^\rho \frac{dt}{\sqrt{F(\rho)-F(t)}} = \sqrt{\lambda}. \tag{2.7}$$

Therefore, if there exists a solution u such that $u(\frac{1}{2}) = \rho$, then ρ must be such that it satisfies the equation $G(\rho) = \sqrt{\lambda}$. If we have such a ρ, we can define u by

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho)-F(t)}} = \sqrt{2\lambda}x; \quad x \in [0, \tfrac{1}{2}].$$

By the Implicit Function Theorem, u is differentiable and hence

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u(x))]}.$$

Differentiating again we obtain

$$-u''(x) = \lambda f(u(x)).$$

Thus u is a positive solution to (2.1) with

$$u(\tfrac{1}{2}) = \rho \quad iff \quad \sqrt{\lambda} = G(\rho).$$

□

Remark 2.2. *Note that in the above discussion,* $f(\rho) > 0$ *and* $F(\rho) > F(s)$ $\forall s \in [0, \rho)$. *Hence* $G(\rho)$ *is well defined. In order to use the quadrature method in more general cases we must choose a* ρ *within the set* $\boldsymbol{S}$ *in which* $f(\rho)$ *is positive and* $F(\rho) > F(z), \forall z \in [0, \rho)$.

Lemma 2.3. $G(\rho)$ *is continuous and differentiable on the set* $\boldsymbol{S}$ *with*

$$G'(\rho) := \sqrt{2} \int_0^1 \frac{H(\rho) - H(\rho s)}{[F(\rho) - F(\rho s)]^{\frac{3}{2}}} ds \tag{2.8}$$

where

$$H(u) = F(u) - \frac{u}{2} f(u) \tag{2.9}$$

(See [1]).

3 Logistic Type Growth

In this section we consider the boundary value problem

$$\begin{cases} -u'' = \lambda u \tilde{f}(u) \\ \quad u(0) = 0 = u(1) \end{cases} \tag{3.1}$$

where $\tilde{f}(u)$ is a C^1 decreasing function such that $\tilde{f}(r) = 0$ for some $r > 0$, and $\tilde{f}(0)$ is a positive constant.

Using the quadrature method, we establish the following precise bifurcation diagram:
To do so we establish the following theorems:

Theorem 3.1. $\lim_{\rho \to r^-} [G(\rho)]^2 = \infty$.

Remark 3.2. *We note that* $r = \frac{a}{b}$ *when* $\tilde{f}(u) = a - bu$, *where* a *and* b *are positive constants.*

Theorem 3.3. $\lim_{\rho \to 0} [G(\rho)]^2 = \frac{\pi^2}{\tilde{f}(0)}$.

Theorem 3.4. $G'(\rho) > 0, \forall\ \rho \in \boldsymbol{S} = (0, r)$.

Remark 3.5. *It follows that for logistic type reactions, in the case of time dependent problems, for any positive initial data, the solution eventually dies out if* $\lambda < \frac{\pi^2}{\tilde{f}(0)}$, *and approaches the unique positive steady state if* $\lambda > \frac{\pi^2}{\tilde{f}(0)}$ *(unconditional persistence).*

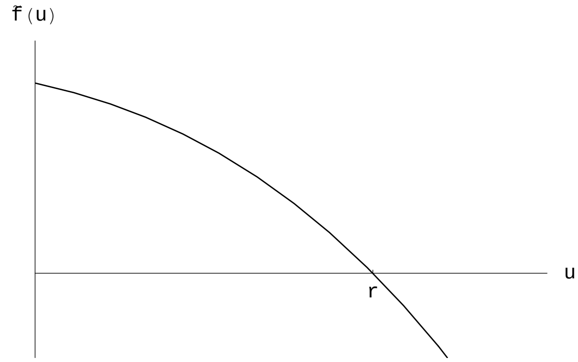

Figure 2: $\tilde{f}(u)$ vs. u

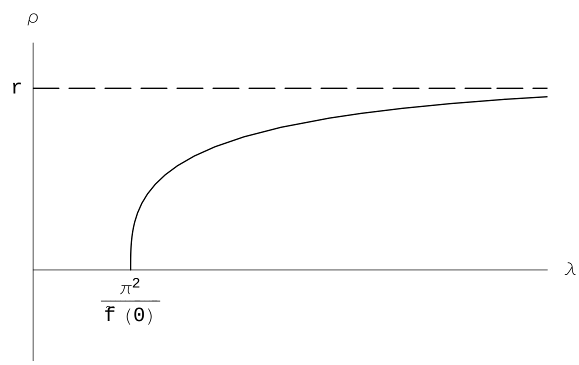

Figure 3: ρ vs. λ

3.1 Proofs of Theorems 3.1, 3.3, and 3.4

Now we prove Theorems 3.1, 3.3, and 3.4. Let $f(u) = u\tilde{f}(u)$.

Proof of Theorem 3.1. By the Mean Value Theorem, we have,

$$F(\rho) - F(s) = F'(c)(\rho - s)$$

for some $c \in [s, \rho]$.

Clearly there exists $N > 0$ such that $f(z) \leq N(r - z)\ \forall z \in (0, r)$. Hence,

$$\begin{aligned} F(\rho) - F(s) &= F'(c)(\rho - s) \\ &= f(c)(\rho - s) \\ &\leq N(r - c)(\rho - s) \\ &= N(r - s)^2 \end{aligned}$$

Therefore,

$$\frac{1}{\sqrt{F(\rho) - F(s)}} \geq \frac{n}{r - s}$$

with $n = \frac{1}{\sqrt{N}}$.

Since

$$n\sqrt{2}\int_0^\rho \frac{ds}{r-s} = -n\sqrt{2}\ln(r-s)|_0^\rho$$

approaches infinity as $\rho \to r^-$, $G(\rho)$ also approaches infinity. Thus, $[G(\rho)]^2 \to \infty$ as $\rho \to r^-$.

□

Proof of Theorem 3.3.

$$\begin{aligned}\lim_{\rho\to 0} G(\rho) &= \lim_{\rho\to 0}\sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}\\ &= \lim_{\rho\to 0}\sqrt{2}\int_0^1 \frac{\rho ds}{\sqrt{F(\rho)-F(\rho s)}}\\ &= \lim_{\rho\to 0}\sqrt{2}\int_0^1 \frac{ds}{\sqrt{\frac{F(\rho)-F(\rho s)}{\rho^2}}}\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem, the limit can be used inside the integral and

$$\begin{aligned}\lim_{\rho\to 0}\frac{F(\rho)-F(\rho s)}{\rho^2} &= \lim_{\rho\to 0}\frac{f(\rho)-sf(\rho s)}{2\rho}\\ &= \lim_{\rho\to 0}\frac{f'(\rho)-s^2 f'(\rho s)}{2}\\ &= \frac{f'(0)}{2}(1-s^2).\end{aligned}$$

Hence

$$lim_{\rho\to 0}G(\rho) = \frac{2}{\sqrt{f'(0)}}\int_0^1 \frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{\sqrt{f'(0)}} = \frac{\pi}{\sqrt{\tilde{f}(0)}},$$

and $\lim_{\rho\to 0}[G(\rho)]^2 = \frac{\pi^2}{\tilde{f}(0)}$.

□

Proof of Theorem 3.4. Recall $H(z) = F(z) - \frac{z}{2}f(z)$ (see(2.9)). Then

$$\begin{aligned}H'(z) &= \frac{1}{2}[f(z) - zf'(z)]\\ &= \frac{1}{2}(z\tilde{f}(z) - z[z\tilde{f}'(z) + \tilde{f}(z)])\\ &= -\frac{1}{2}z^2\tilde{f}'(z) > 0\end{aligned}$$

for $z > 0$, since $\tilde{f}'(z) < 0$ for logistic type per capita growth rates. Therefore $H(z) > 0$ and increasing for $z > 0$.

Thus, $H(\rho) - H(\rho s) > 0$ for every $\rho > 0$ and $s \in [0, 1)$, and $G'(\rho) > 0$ for every $\rho \in (0, r)$ (by using 2.8).

□

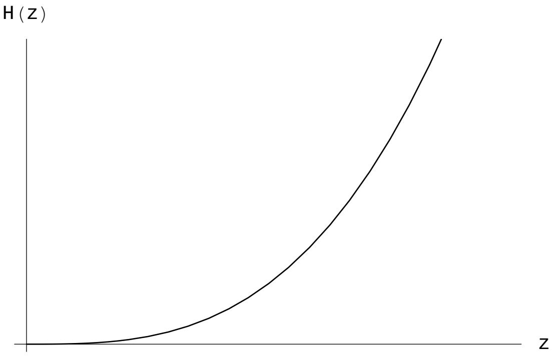

Figure 4: $H(z)$ vs. z

3.2 Logistic Type Growth Examples

In this section we present bifurcation diagrams for logistic type examples.

Here we consider a logistic type example $\tilde{f}(u) = 10 - u$.

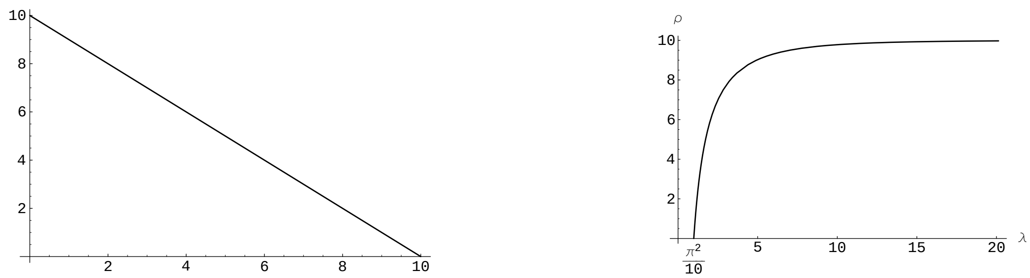

Figure 5: $\tilde{f}(u)$ vs. u for $\tilde{f}(u) = 10 - u$ and the corresponding bifurcation diagram.

Next we consider a logistic type example $\tilde{f}(u) = 8 - u^3 - u^4$.

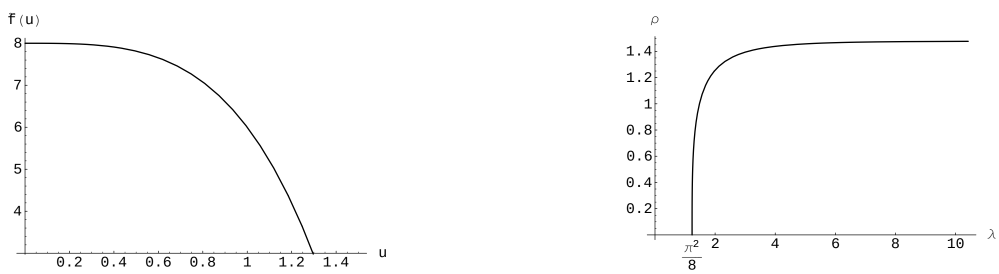

Figure 6: $\tilde{f}(u)$ vs. u for $\tilde{f}(u) = 8 - u^3 - u^4$ and the corresponding bifurcation diagram.

The following is a *predation* example with $\tilde{f}(u) = (10 - u) - \frac{0.5}{1+u}$.

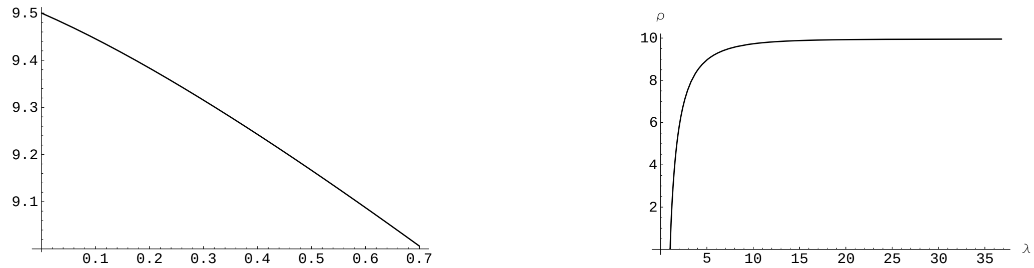

Figure 7: $\tilde{f}(u)$ vs. u for $\tilde{f}(u) = (10 - u) - \frac{0.5}{1+u}$ and the corresponding bifurcation diagram.

4 Weak Allee Effect Type Growth

In this section we consider the boundary value problem

$$\begin{cases} -u'' = \lambda u \tilde{f}(u) \\ \quad u(0) = 0 = u(1) \end{cases} \tag{4.1}$$

where $\tilde{f}(u)$ is a C^1 function increasing on $[0, \alpha)$ and decreasing on (α, r), $\tilde{f}(0)$ is positive, and $\tilde{f}(r) = 0$ for some $r > 0$.

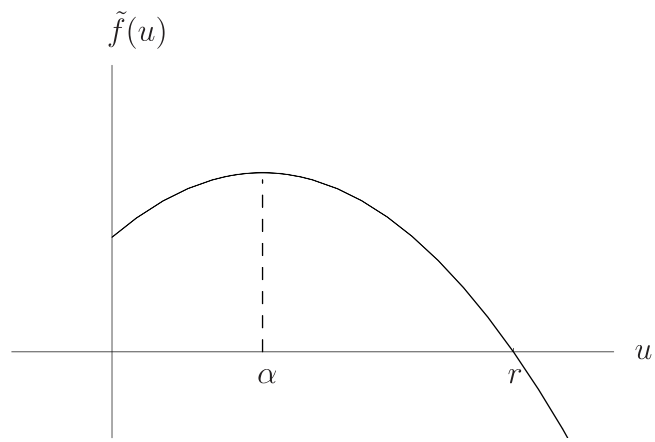

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Figure 8: $\tilde{f}(u)$ vs u

Using the quadrature method, we establish that the bifurcation diagrams look like:

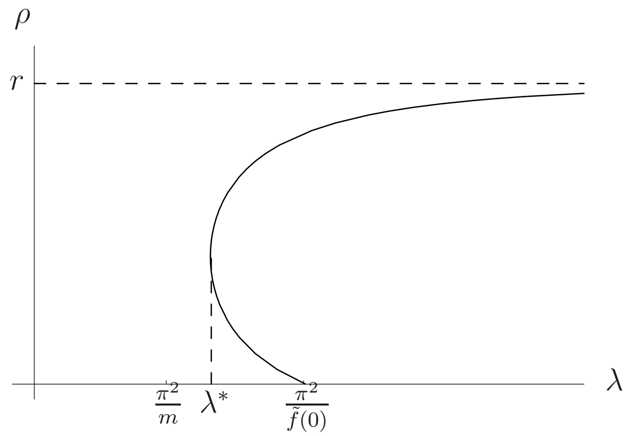


Figure 9: ρ vs λ

Namely we establish the following theorems:

Theorem 4.1. $\lim_{\rho\to r^-}[G(\rho)]^2 = \infty$.

Theorem 4.2. *If $\lambda < \frac{\pi^2}{m}$, then (4.1) has no positive solutions. (Here $m = \tilde{f}(\alpha)$)*

Remark 4.3. *We note that $r = b$ when $\tilde{f}(u) = (u+a)(b-u)$, where a and b are positive constants with $b > a$.*

Theorem 4.4. $\lim_{\rho\to 0}[G(\rho)]^2 = \frac{\pi^2}{\tilde{f}(0)}$.

Theorem 4.5. $G'(\rho) < 0$, *for ρ small.*

Theorem 4.6. $G'(\rho) > 0$, *for ρ near r.*

Remark 4.7. *It follows that for weak Allee effect type reactions, in the case of time dependent problems, for any positive initial data, the solution eventually dies out if $\lambda < \lambda^*$ (see Figure 9) and approaches the unique positive steady state if $\lambda > \frac{\pi^2}{\tilde{f}(0)}$, while if $\lambda \in (\lambda^*, \frac{\pi^2}{\tilde{f}(0)})$ the steady state of the solutions depends on the size of the initial data (conditional persistence).*

4.1 Proofs of Theorems 4.1, 4.2, 4.4, 4.5, and 4.6

Proof of Theorem 4.1. Proof follows similar arguments as in Theorem 3.1 □

Proof of Theorem 4.2. Suppose u is a positive solution. Then

$$-u'' = \lambda f(u) \leq \lambda u m$$

where $m = \tilde{f}(\alpha)$. Hence

$$\int_0^1 (-u''\phi)dx \leq \int_0^1 (\lambda u m \phi)dx$$

where $\phi = \sin(\pi x)$. Integrating, we get

$$\int_0^1 (-u''\phi)dx = \int_0^1 (u'\phi')dx$$
$$= -\int_0^1 (u\phi'')dx$$
$$= \int_0^1 (\pi^2 u\phi)dx.$$

Therefore

$$\int_0^1 (\pi^2 u\phi)dx \leq \int_0^1 (\lambda m u\phi)dx.$$

This is possible only if $\lambda \geq \frac{\pi^2}{m}$. Hence for

$$\lambda < \frac{\pi^2}{m}$$

(4.1) has no positive solutions. □

Proof of Theorem 4.4. Proof follows similar arguments as in Theorem 3.3 □

Proof of Theorems 4.5 and 4.6. Recall Equation (2.8)

$$G'(\rho) = \sqrt{2}\int_0^1 \frac{H(\rho) - H(\rho s)}{(F(\rho) - F(\rho s))^{\frac{3}{2}}}.$$

Now $H(s) = F(s) - \frac{s}{2}f(s)$ and $H'(s) = -\frac{s^2}{2}\tilde{f}'(s)$. Thus, H decreases on $(0, \alpha)$ and increases on (α, r) with $\lim_{s\to r} H(s) = F(r) > 0$. That is H has the shape:

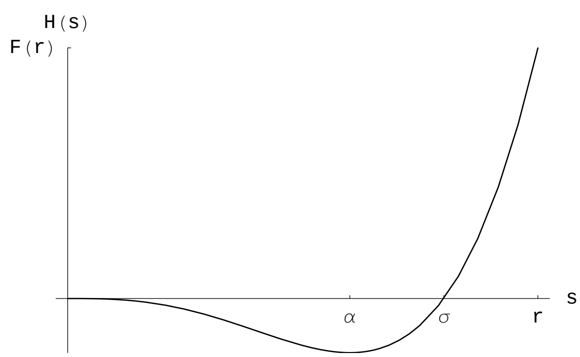


Figure 10: $H(s)$ vs s

Hence, clearly for $\rho < \alpha$, $H(\rho) - H(\rho s) < 0$ for all $s \in [0, 1)$ and for $\rho > \sigma$, $H(\rho) - H(\rho s) > 0$ for all $s \in [0, 1]$. This implies that $G'(\rho) < 0$ for $\rho < \alpha$ and $G'(\rho) > 0$ for $\rho > \sigma$. □

4.2 Weak Allee Effect Type Growth Examples

In this section we present bifurcation diagrams for weak Allee effect type examples. Here we consider a weak Allee effect type example $\tilde{f}(u) = (u + 3)(10 - u)$.

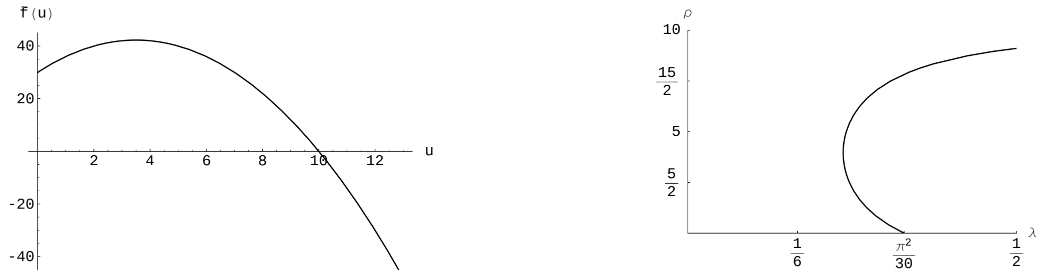


Figure 11: $\tilde{f}(u)$ vs. u for $\tilde{f}(u) = (u + 3)(10 - u)$ and the corresponding bifurcation diagram

Next we consider a weak Allee effect type example $\tilde{f}(u) = 10sin(\frac{u}{10}) + 10$.

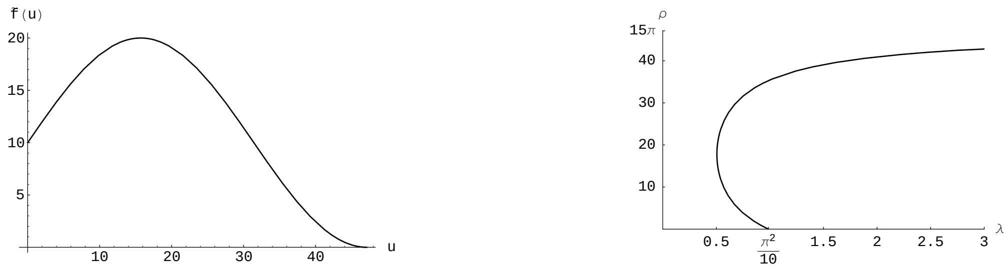


Figure 12: $\tilde{f}(u)$ vs. u for $\tilde{f}(u) = 10\sin(\frac{u}{10}) + 10$ and the corresponding bifurcation diagram

The following is a *predation* example with $\tilde{f}(u) = (10 - u) - \frac{7}{1+u}$.

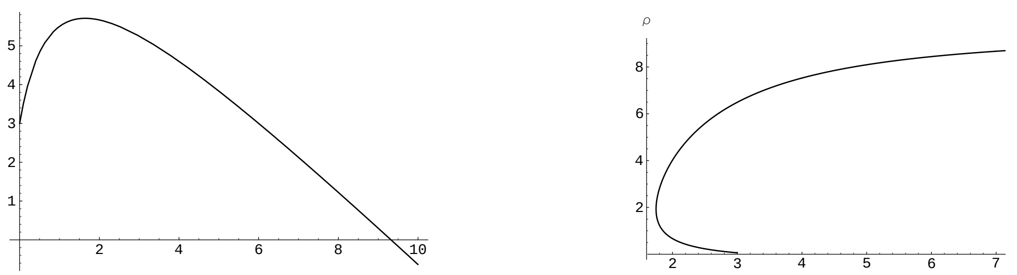


Figure 13: $\tilde{f}(u)$ vs. u for $\tilde{f}(u) = (10 - u) - \frac{7}{1+u}$ and the corresponding bifurcation diagram.

5 Nonlinear Diffusion

In this section we analyze the case of nonlinear diffusion with the types of per capita growth rates discussed in Sections 3 and 4. We consider Equation (1.1)

$$\begin{cases} -(d\phi(u)u')' = u\tilde{f}(u), & x \in (0, 1) \\ \quad u(0) = 0 = u(1) \end{cases}$$

where $\phi(u)$ is a non constant function of u. We show that nonlinear diffusion can produce conditional persistence even when $\tilde{f}$ is of the logistic type and produce unconditional persistence even when $\tilde{f}$ is of the weak Allee type. Examples of such behavior are included. For ease of analysis, we study the equivalent form of (1.1), namely

$$\begin{cases} -[\psi(u)]'' = \lambda u \tilde{f}(u), \quad x \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{5.1}$$

where

$$\psi(u) = \int_0^u \phi(z)\, dz. \tag{5.2}$$

Note that (5.1) is obtained by differentiating (5.2) and substituting

$$\frac{d}{dx}(\psi(u)) = \phi(u)u' \tag{5.3}$$

into (1.1) and letting $d = \frac{1}{\lambda}$.
Throughout in Section 5 we assume both $\tilde{f}$ and ψ are C^3 functions.

5.1 Analysis of the Nonlinear Diffusion Equation

In this section we define a function $L(u)$ which will later be used to determine whether the combined effect of $\psi(u)$ and $\tilde{f}(u)$ is of logistic or weak Allee type.

Note that (5.2) clearly shows that $\psi(0) = 0$. Since $\psi'(u) = \phi(u) > 0$ for all $u \geq 0$ we can define the one-to-one function

$$v = \psi(u). \tag{5.4}$$

Since v is one-to-one, ψ^{-1} exists and so solving for v is equivalent to solving for u. Thus, we can rewrite (5.1) as

$$\begin{cases} -v'' = \lambda \psi^{-1}(v) \tilde{f}(\psi^{-1}(v)), \quad x \in (0,1) \\ v(0) = 0 = v(1). \end{cases} \tag{5.5}$$

Letting

$$\tilde{g}(v) = \frac{\psi^{-1}(v)\tilde{f}(\psi^{-1}(v))}{v} \tag{5.6}$$

we finally obtain

$$-v'' = \lambda v \tilde{g}(v). \tag{5.7}$$

Therefore, we can analyze solutions to (5.1) in precisely the same manner as we analyzed solutions to (1.1) when the diffusion was linear by studying $\tilde{g}(v)$. Now, since $v = \psi(u)$,

$$\tilde{g}(v) = \frac{u\tilde{f}(u)}{\psi(u)} = L(u) \text{ (say)}. \tag{5.8}$$

Since u, $\tilde{g}(v)$, and $\psi(u)$ are all differentiable, $L(u)$ is also differentiable (for $u > 0$). However, $L(u)$ is clearly not defined at $u = 0$ whereas (5.1) is. We extend $L(u)$ to be defined at $u = 0$ by letting

$$L(0) = \lim_{u \to 0} L(u). \tag{5.9}$$

Applying L'Hopitals' rule to (5.8) and evaluating at $u = 0$ we obtain

$$\lim_{u \to 0} L(u) = \frac{\tilde{f}(0)}{\psi'(0)} \tag{5.10}$$

and so

$$L(u) = \begin{cases} \frac{u\tilde{f}(u)}{\psi(u)} & u > 0 \\ \frac{\tilde{f}(0)}{\psi'(0)} & u = 0 \end{cases} \tag{5.11}$$

which is the function we study. Note that

$$\tilde{g}'(v) = L'(u)\frac{du}{dv} \tag{5.12}$$

and hence $\tilde{g}'(v)$ and $L'(u)$ have the same sign.

5.2 Logistic Type Reactions

In this section we introduce a class of $\psi(u)$ whose diffusive effects will produce a combined Allee effect type growth even with $\tilde{f}(u)$ of logistic type.

Theorem 5.1. *For a given logistic type $\tilde{f}(u)$, if $-\psi''(0)\tilde{f}(0)+2\tilde{f}'(0)\psi'(0) > 0$ then solutions to (5.1) will exhibit a combined Allee effect.*

Proof. Note that $L(u) = \frac{u\tilde{f}(u)}{\psi(u)}$. Thus

$$L'(u) = \frac{\psi(u)[u\tilde{f}'(u) + \tilde{f}(u)] - u\tilde{f}(u)\psi'(u)}{(\psi(u))^2}. \tag{5.13}$$

By continuity of $L'(u)$, we know that

$$L'(0) = \lim_{u \to 0} L'(u). \tag{5.14}$$

In order to find this limit, L'Hopital's rule is used twice on (5.13) and we obtain

$$L'(0) = \frac{-\psi''(0)\tilde{f}(0) + 2\tilde{f}'(0)\psi'(0)}{2(\psi'(0))^2}. \tag{5.15}$$

Hence if we apply our hypothesis to (5.15) we obtain

$$L'(0) > 0 \tag{5.16}$$

which implies a combined Allee effect.

Corollary 5.1. *It is necessary that $\psi''(0) < 0$ in order to produce a combined Allee effect from logistic type $\tilde{f}(u)$.*

Proof. Suppose that $\psi''(0) \geq 0$. Then since $\tilde{f}(0) > 0$, $\psi'(0) > 0$ and $\tilde{f}'(0) < 0$ we have that

$$-\psi''(0)\tilde{f}(0) + 2\tilde{f}'(0)\psi'(0) < 0 \tag{5.17}$$

which implies that $L'(0) < 0$, and so a combined Allee effect will not occur.

5.2.1 Examples

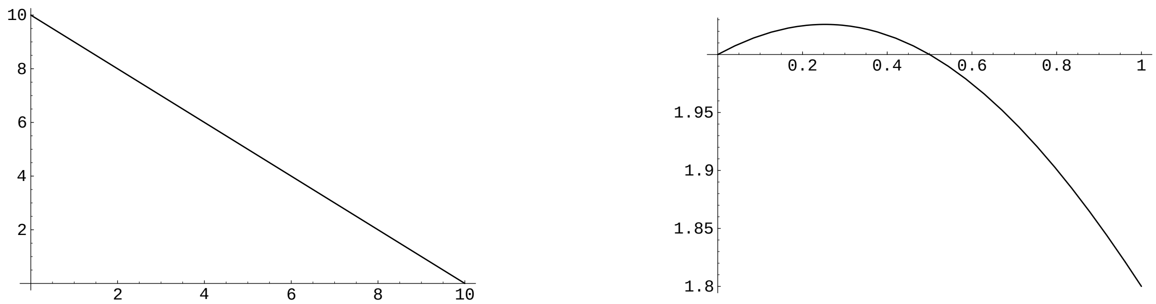
Consider the logistic growth $\tilde{f}(u) = a - bu$ and $\psi(u) = u^3 - Bu^2 + Cu$ with $B > 0$, $C > 0$, $B^2 < 3C$, and $Ba > Cb$. Then

$$\begin{aligned} & \psi(0) = 0, & & \\ & \psi'(u) = 3u^2 - 2Bu + C > 0 & & (since \quad B^2 < 3C), \\ \text{and} \quad & \psi''(u) = 6u - 2B < 0 & & if \quad u < \tfrac{B}{3}. \end{aligned} \tag{5.18}$$

Hence

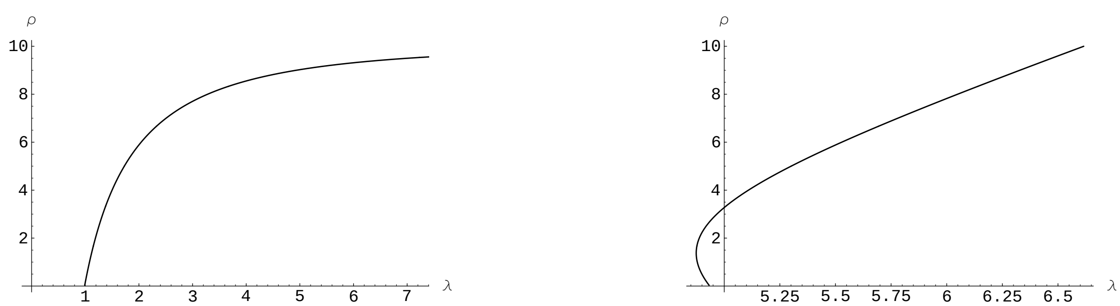
$$-\psi''(0)\tilde{f}(0) + 2\tilde{f}'(0)\psi'(0) = 2Ba - 2Cb > 0. \tag{5.19}$$

Thus $L'(0) > 0$ and a combined Allee effect is produced. Letting $a = 10$, $b = 1$, $B = 1$, and $C = 5$ we produced the following figures.



$\tilde{f}(u)$ vs u $\qquad$ $L(u)$ vs u

Figure 14: $\tilde{f}(u)$ and $L(u)$



$\psi(u) = u$ $\qquad$ $\psi(u) = u^3 - u^2 + 5u$
$\tilde{f}(u) = 10 - u$ $\qquad$ $\tilde{f}(u) = 10 - u$

Figure 15: Bifurcations Diagrams with linear and nonlinear ψ

Note that $\tilde{f}'(u) < 0$, $\forall\ u \geq 0$ whereas $L'(u) > 0$ near $u = 0$. Therefore, in this example, the diffusive effects of ψ produced an Allee effect from logistic growth. Next, in Figure 18, we let $a = 10$, $b = 1$, $B = 1$, $C = 20$ and produce the following figure.

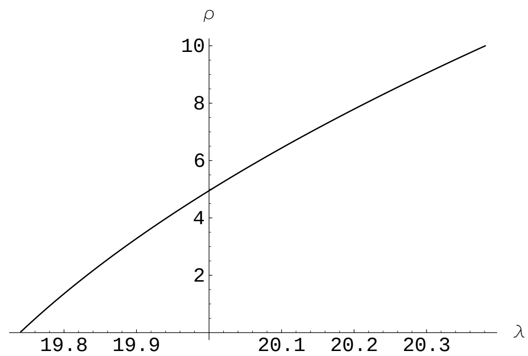


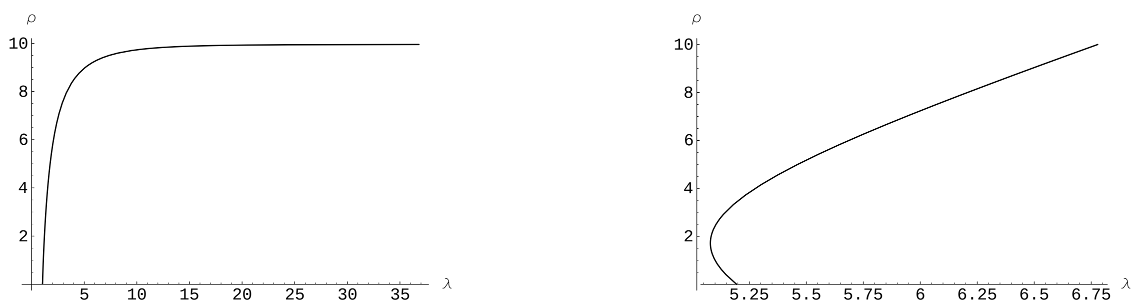
Figure 16: Bifurcation diagram for $\tilde{f}(u) = 10 - u$ with $\psi(u) = u^3 - u^2 + 20u$

Here the parameter B and C violate (5.19). Also no combined Allee effect was produced.

Next, we consider a *Logistic with predation* example of the form

$$\tilde{f}(u) = (m - u) - (\frac{K}{1+u});\ m > K > 0,\ K \leq 1.$$

In particular, we produce the following bifurcation diagrams with linear and nonlinear diffusion.



$\psi(u) = u$ $\tilde{f}(u) = (10 - u) - \frac{0.5}{1+u}$

$\psi(u) = u^3 - u^2 + 5u$ $\tilde{f}(u) = (10 - u) - \frac{0.5}{1+u}$

Figure 17: Bifurcations Diagrams with linear and nonlinear ψ

5.3 Weak Allee Type Reactions

In this section we introduce a class of $\psi(u)$ whose diffusive effects will produce a combined logistic type growth in otherwise weak Allee type $\tilde{f}(u)$. Our analysis is analogous to that of the previous section.

Theorem 5.2. *For a weak Allee type* $\tilde{f}(u)$, $-\psi''(0)\tilde{f}(0) + 2\tilde{f}'(0)\psi'(0) \leq 0$ *is a necessary condition for (5.1) to be of logistic type.*

Proof. Recall $L'(0)$ from (5.15), i.e.

$$L'(0) = \frac{-\psi''(0)\tilde{f}(0) + 2\tilde{f}'(0)\psi'(0)}{2(\psi'(0))^2}. \tag{5.20}$$

Thus if our hypothesis is not satisfied then by (5.20) we obtain

$$L'(0) > 0 \tag{5.21}$$

and the Allee effect will persist.

Corollary 5.2. *It is necessary that $\psi''(0) > 0$ in order to produce a combined logistic type growth from weak Allee type $\tilde{f}(u)$.*

Proof. Suppose that $\psi''(0) \leq 0$. Then since $\tilde{f}(0) > 0$, $\psi'(0) > 0$ and $\tilde{f}'(0) > 0$ we have that

$$-\psi''(0)\tilde{f}(0) + 2\tilde{f}'(0)\psi'(0) > 0. \tag{5.22}$$

This implies that $L'(0) > 0$ and an Allee effect will persist.

5.3.1 Examples

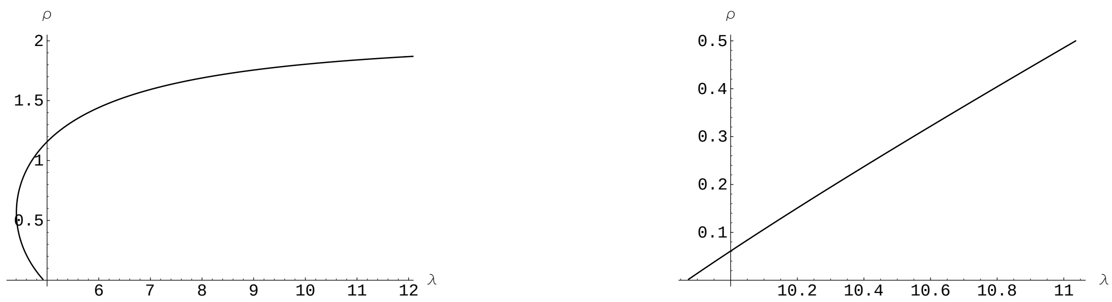
Consider $\tilde{f}(u) = (u + a)(b - u)$ with $b > a > 0$ and $\psi(u) = e^{\alpha u} - 1$ with $\alpha > \frac{2(b-a)}{ab}$. Note that

$$\begin{aligned} \psi(0) &= 0, \\ \psi'(u) &= \alpha e^{\alpha u} > 0 \end{aligned} \tag{5.23}$$

and

$$-\psi''(0)\tilde{f}(0) + 2\tilde{f}'(0)\psi'(0) = -\alpha^2(ab) + 2\alpha(b - a) < 0. \tag{5.24}$$

Hence the conditions of Theorem (5.2) are satisfied. If we set $\alpha = 2$, $a = 1$, $b = 2$ then (5.24) will be satisfied. We produced the following figures:



$$\psi(u) = u \qquad\qquad \psi(u) = e^{2u} - 1$$
$$\tilde{f}(u) = (u + 1)(2 - u) \qquad\qquad \tilde{f}(u) = (u + 1)(2 - u)$$

Figure 18: Bifurcation diagrams with linear and nonlinear ψ

It can also be shown that the Allee effect is not only reversed locally near zero, but also globally. To do so, we show that $L'(u) < 0$ for $0 < u < 2$ (note that for $u > 2$, $f(u) = u(u + 1)(2 - u) < 0$ and so $\rho \notin S$). The following Mathematica plots of $L'(u)$ for this particular example show that $L'(u) < 0$ on $[0, 2]$.

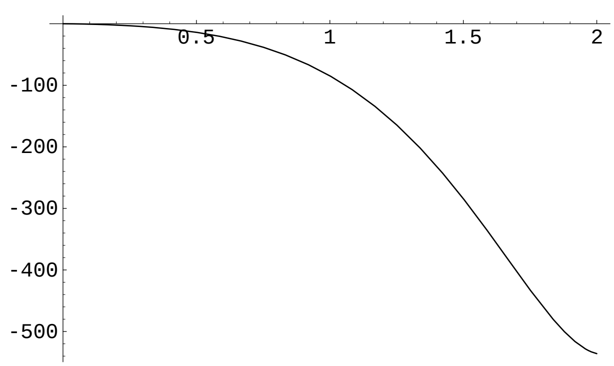
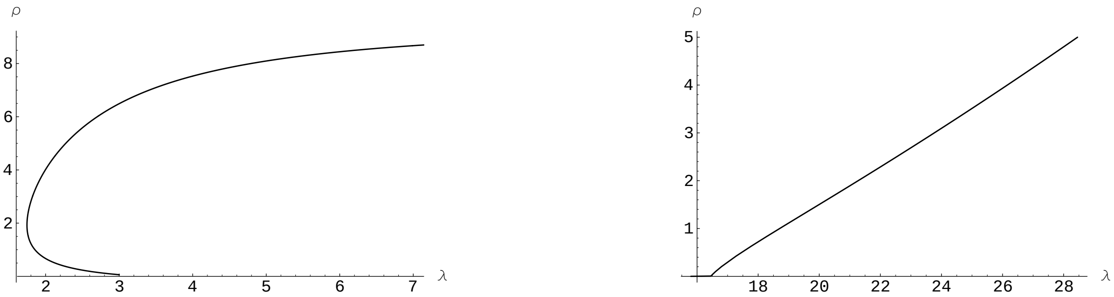


Figure 19: $L'(u)$ for $u \in [0, 2]$

Finally, we consider a *weak Allee with predation* example of the form

$$\tilde{f}(u) = (m - u) - (\frac{K}{1 + u}); \; m > K > 0, \; K > 1.$$

In particular, we produce the following bifurcation diagrams with linear and nonlinear diffusion.



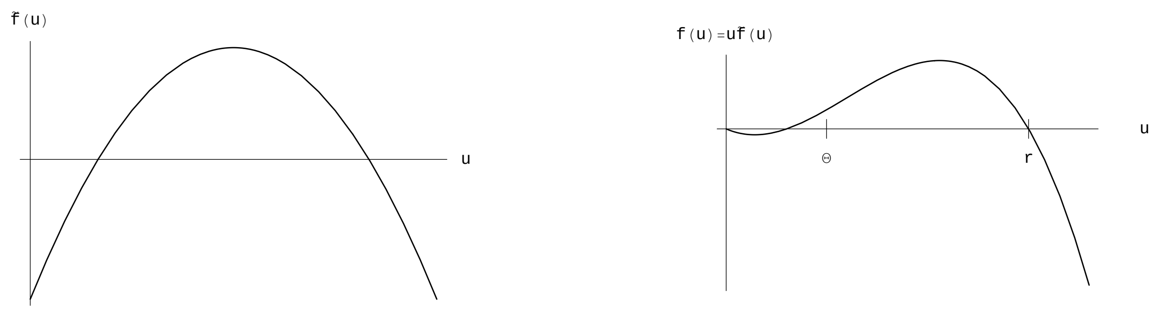
$\psi(u) = u$ $\quad$ $\psi(u) = e^{5u} - 1$

$\tilde{f}(u) = (10 - u) - \frac{7}{1+u}$ $\quad$ $\tilde{f}(u) = (10 - u) - \frac{7}{1+u}$

Figure 20: Bifurcations Diagrams with linear and nonlinear ψ

5.4 Strong Allee Effect

For a strong Allee effect type per capita growth rate (i.e. $\tilde{f}(0) < 0$), $f(u) = u\tilde{f}(u)$ is negative near $u = 0$ (see figure 14).



$\tilde{f}(u)$ vs u $\quad$ $f(u)$ vs. u

Figure 21: General type of strong Allee $\tilde{f}$ and f

This produces conditional persistence for λ large. In fact, the typical bifurcation diagram for

$$\begin{cases} -u'' = \lambda u\tilde{f}(u) \\ u(0) = 0 = u(1) \end{cases} \tag{5.25}$$

is as follows:

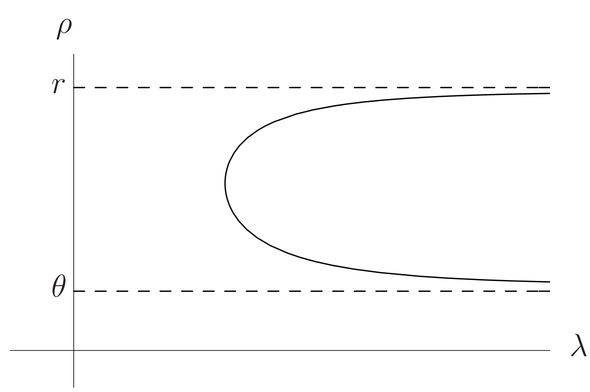


Figure 22: Typical bifurcation diagram for Strong Allee type $f(u)$

Here $F(\theta) = 0$ where $F(z) = \int_0^z f(t)\,dt$. Now, even with nonlinear diffusion $\psi(u)$ which is equivalent to studying

$$\begin{cases} -v'' = \lambda\psi^{-1}(v)\tilde{f}(\psi^{-1}(v)) = \lambda g(v) \\ \quad v(0) = 0 = v(1), \end{cases} \tag{5.26}$$

$g(v)$ still remains negative for v small. Thus a similar bifurcation diagram persists. In particular, conditional persistence persists for λ large and cannot be altered by the presence of nonlinear diffusion.

6 Constant Yield Harvesting

In recent literature there has been considerable interest in the effects of constant yield harvesting in population dynamics (see [5], [7] and [8]). In this section we first consider a logistic type growth model with linear diffusion $\psi(u) = u$ and constant yield harvesting.

$$\begin{cases} -u'' = \lambda[10u - u^2 - c] \\ u(0) = 0 = u(1). \end{cases} \tag{6.1}$$

By using the quadrature method described in Section 2, we obtain the bifurcation diagrams (ρ vs. λ) as c increases in $(0, 18.75)$ (Note that for $\mathbf{S}_c$ to be non-empty, $c < 18.75$)(see Figure 23). Next we consider the effects produced by the nonlinear diffusion $\psi(u) = u^3 - u^2 + 5u$, by studying the bifurcation diagram for

$$\begin{cases} -(u^3 - u^2 + 5u)'' = \lambda[10u - u^2 - c] \\ u(0) = 0 = u(1) \end{cases} \tag{6.2}$$

as c varies, and compare the corresponding bifurcation diagrams (see Figure 24). We note that with harvesting present, even in the case of logistic type growth, ranges of λ exist where there is multiplicity of positive steady states. This persists even in the case of these classes of nonlinear diffusion.

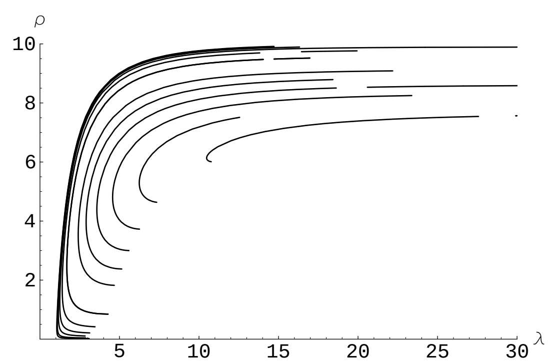


Figure 23: Bifurcation Diagrams of ρ vs. λ for $f(u) = 10u - u^2 - c$ with $0 < c < 18.75$

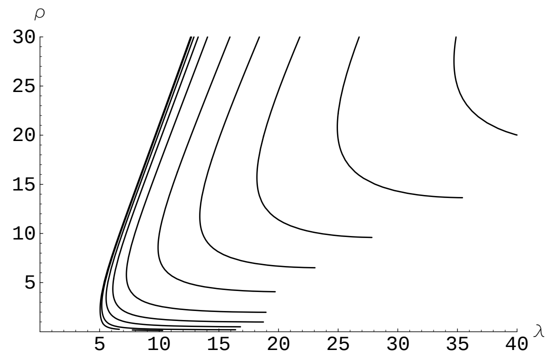


Figure 24: Bifurcation Diagrams of ρ vs. λ for $f(u) = 10u - u^2 - c$ with $0 < c < 18.75$ and $\psi(u) = u^3 - u^2 + 5u$

7 Conclusions

In conclusion, we used a (quadrature) method for analyzing solutions to equation (1.1) and have shown that solutions will exhibit either logistic or Allee type reactions, depending on both the growth rate and diffusion. We have established that the combined effects of $\tilde{f}$ and ψ can produce both Allee and logistic type reactions from both logistic and weak Allee type $\tilde{f}$. Furthermore, we obtained results on the effects of constant yield harvesting on a model with logistic growth and nonlinear diffusion.

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