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ZETA FUNCTIONS ON KRONECKER PRODUCTS OF GRAPHS

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Abstract. Ihara introduced the zeta function of a p-adic matrix group in 1966 and the idea was extended to finite graphs by Hashimoto in 1989. In her dissertation, Debra Czarneski explores the properties of graphs that are or are not determined by the zeta function. This paper defines a Kronecker product of finite graphs and explores the question: given a pair of graphs with equal zeta functions, if we take their Kronecker product with a third graph, is the equality of the zeta function preserved?

1. Definition of the Zeta Function of a Finite Graph

The definitions in this section are taken with slight changes from the introduction of Debra Czarneski's thesis [2].

In this paper, the word *graph* refers to a finite, undirected graph and may include multiple edges, loops and multiple components. A *walk* is a sequence of vertices and edges and is denoted $(v_1, e_1, v_2, e_2, \dots, v_{m-1}, e_{m-1}, v_m)$ where edge e_i connects vertex v_i and vertex v_{i+1}. If e_i is a loop, it can be traversed in either of the two possible directions, giving rise to two distinct walks. Note that for any i and j in this sequence, v_i and v_j may be equal and/or e_i and e_j may be equal. A *closed walk* C is a walk in which the first and the last vertices are the same. If no loops or multiple edges come into play, then C may be denoted $(v_1, v_2, \dots, v_{m-1}, v_m)$.

A closed walk has *backtracking* if, in the closed walk, a non-loop edge appears twice in immediate succession or if a loop is immediately followed by the same loop traveled in the opposite direction. A *tail* occurs in a closed walk without backtracking if the first edge and the last edge in the closed walk are the same non-loop edge or if the first and last edge are the same loop traveled in opposite directions. Closed walk C is *primitive* if C is not the power of another closed walk. In other words, a primitive walk C cannot be obtained by repeating another closed walk a finite number of times.

Closed walk $C = (v_1, e_1, v_2, e_2, \dots, v_{m-1}, e_{m-1}, v_m)$ and closed walk D are *equivalent* if there exists an index i such that closed walk $D = (v_i, e_i, v_{i+1}, e_{i+1}, \dots, v_{m-1}, e_{m-1}, v_1, \dots, v_{i-1}, e_{i-1}, v_i)$ for some vertex v_i in closed walk C. Let $[C]$ denote the equivalence class of all closed walks equivalent to a primitive, tail-less, backtrackless closed walk C, and let $\pi(\Gamma)$ denote the set

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of all such $[C]$ on a graph Γ. The elements of $\pi(\Gamma)$ are called "primes" in Γ. Let $deg([C])$ be the number of edges in any representative C of equivalence class $[C]$.

Definition 1.1. The *Ihara zeta function* of finite graph Γ is

$$Z_\Gamma(u) = \prod_{[C]\in\pi(\Gamma)} (1 - u^{deg([C])})^{-1}.$$

For most graphs, the zeta function is difficult to compute from this definition since most graphs will contain infinitely many primes. The following theorem due to Ihara and Hashimoto gives an equivalent formulation of the zeta function of a finite graph that in most cases is easier to compute.

Let Γ be a finite graph with vertex set V and edge set E, and let $|V|$ denote the number of vertices and $|E|$ denote the number of edges. Let $A = (a_{ij})$ be the $n \times n$ adjacency matrix of Γ. By definition, for $i \neq j$, element $a_{i,j}$ is the number of edges between vertex i and vertex j, and $a_{i,i}$ is twice the number of loops at vertex i. The degree of a vertex i will be the sum of the ith row of this matrix (i.e. the number of edges containing i with loops counted twice). Define the Q-matrix of graph Γ to be the diagonal $n \times n$ matrix $Q = (q_{ij})$ where $q_{i,i}$ is one less than the degree of vertex i and $q_{ij} = 0$ for $i \neq j$. Let I be the $n \times n$ identity matrix.

Theorem 1.2. *(Hashimoto)* [1] *The zeta function of finite graph* Γ *can be written*

$$Z_\Gamma(u) = \frac{(1-u^2)^{|V|-|E|}}{det(I - Au - Qu^2)}$$

2. Kronecker Product of Finite Graphs

Definition 2.1. Let Γ and Γ' be finite graphs with adjacency matrices A and A'. Then we will define the Kronecker (tensor) product of graphs, $\Gamma \otimes \Gamma'$, as the graph whose adjacency matrix is $A \otimes A'$, where $\otimes$ is the Kronecker product of matrices defined by:

$$A \otimes A' = \begin{pmatrix} a_{11}A' & a_{12}A' & a_{13}A' & \dots \\ a_{21}A' & a_{22}A' & \dots & \\ a_{31}A' & \dots & & \\ \vdots & & & \end{pmatrix}$$

where a_{ij} is the entry in the i^{th} row, j^{th} column of A.

The vertices of $\Gamma \otimes \Gamma'$ are ordered pairs, (p, q) with $p \in V(\Gamma)$ and $q \in V(\Gamma')$. The number of edges between (p, q) and (s, t) is km where k is the number of edges between p and s in Γ and m is the number of edges between q and t in Γ'. The number of loops at (p, q) is the product of the number of loops at p and the number of loops at q.

The following general theorems are original results about Kronecker products of graphs.

Theorem 2.2. *Let* Γ_1 *and* Γ_2 *be graphs. Then* $\Gamma_1 \otimes \Gamma_2$ *is isomorphic to* $\Gamma_2 \otimes \Gamma_1$*, written* $\Gamma_1 \otimes \Gamma_2 \cong \Gamma_2 \otimes \Gamma_1$.

Proof. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & \dots & & \\ \vdots & & & \\ a_{s1} & \dots & & a_{ss} \end{pmatrix}$$

be the adjacency matrix of Γ_1 and

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1t} \\ b_{21} & \dots & & \\ \vdots & & & \\ b_{t1} & \dots & & b_{tt} \end{pmatrix}$$

be the adjacency matrix of Γ_2. The elements of $A \otimes B$ are naturally written as $a_{ij}b_{kl}$ for $i, j = 1, \dots, s$ and $k, l = 1, \dots, t$ and $B \otimes A$ consists of $b_{kl}a_{ij}$ for $k, l = 1, \dots, t$ and $i, j = 1, \dots, s$. So $A \otimes B$ and $B \otimes A$ contain all of the same entries. Furthermore, in $A \otimes B$ the entry $a_{ij}b_{kl}$ occurs in row $(i-1)t+k$ and column $(j-1)t+l$. In $B \otimes A$, the entry $b_{kl}a_{ij}$ occurs in row $(k-1)s+i$ and column $(l-1)s+j$. (This is all from the definition of the Kronecker product.) Every integer between 1 and st can be written uniquely in the form $(i-1)t+k$ as well as in the form $(k-1)s+i$.

Now let π be the permutation of $1, \dots, st$ given by

$$\begin{pmatrix} 1 & 2 & \dots & (i-1)t+k & \dots & st \\ 1 & s+1 & \dots & (k-1)s+i & \dots & st \end{pmatrix}$$

Let P be the matrix obtained by applying π to the rows of $I_{st \times st}$. Then for any $st \times st$ matrix M, the product PM is the matrix obtained by applying π to the rows of M. Similarly MP^T is the matrix obtained by applying π to the columns of M. Finally, PMP^T is the matrix obtained by applying π to the rows and then to the columns of M. Furthermore, the graphs defined by the adjacency matrices M and PMP^T are isomorphic since multiplication by P on the left and P^T on the right simply re-indexes the vertices.

Consider the entry $a_{ij}b_{kl}$ in $A \otimes B$. This is the element in row $(i-1)t+k$ and column $(j-1)t+l$. In the matrix $P(A \otimes B)P^T$, this entry is now in row $\pi((i-1)t+k) = (k-1)s+i$ and in column $\pi((j-1)t+l) = (l-1)s+j$. Hence $P(A \otimes B)P^T = B \otimes A$, and we can conclude that $\Gamma_1 \cong \Gamma_2$. ∎

Let X_i denote the sum of the i^{th} row of matrix $X_{m\times n}$. We say X is *row regular* if $X_i = X_j$ for all $1 \leq i, j \leq m$.

Lemma 2.3. *Let $A_{n\times s}$, $B_{m\times t}$ be matrices each having at least one non-zero row sum. Then $(A \otimes B)_{nm\times st}$ is row regular if and only if A and B are individually row regular.*

Proof. Let A and B be row regular. Fix k so that $1 \leq k \leq nm$. The sum of k^{th} row of $A \otimes B$ is a linear sum of some row (say row p) of B with coefficients coming from a row (say the q^{th}) of A.

$$a_{q1}\Sigma_{j=1}^{m}(b_{pj}) + \cdots + a_{qn}\Sigma_{j=1}^{m}(b_{pj}) = \Sigma_{i=1}^{n}(a_{qi})\Sigma_{j=1}^{m}(b_{pj}) = A_qB_p$$

Since A and B are row regular, this product is the same for any p, q and hence for any row k of $A \otimes B$. So $A \otimes B$ is row regular.

Conversely, suppose $A \otimes B = C$ is row regular. Then $C_1 = C_2 = \cdots = C_{mn}$ which implies

$$A_1B_1 = \cdots = A_1B_m = A_2B_1 = \cdots = A_2B_m = \cdots = A_nB_m$$

We know there exists one row of A, say the i^{th}, such that the row sum $A_i \neq 0$ and some row of B, say the j^{th}, such that the row sum $B_j \neq 0$. Then since $A_iB_1 = \cdots = A_iB_m$, we can cancel the A_i leaving $B_1 = \cdots = B_m$. Similarly we can cancel the B_j of $A_1B_j = \cdots = A_nB_j$ leaving $A_1 = \cdots = A_n$. So A and B are row regular.

□

Definition 2.4. A graph is *regular* if all vertices have the same degree. A graph is *bipartite* if its vertices can be partitioned into two sets X and Y (called a *bipartition*) such that there are no edges between any two vertices in X and no edges between any two vertices in Y.

Theorem 2.5. *Let Γ_1, Γ_2 be graphs with at least one edge. Then $\Gamma_1 \otimes \Gamma_2$ is regular if and only if Γ_1 and Γ_2 are both regular.*

Proof. Let A be the adjacency matrix of Γ_1 and B be the adjacency matrix of Γ_2. Since the graphs have at least one edge, there is a non-zero row in each matrix. For the forward direction, assume Γ_1, Γ_2 regular. Then for any vertices $i, j \in \Gamma_1$, $deg(i) = deg(j)$, so $A_i = A_j$ and A is row regular. By same reasoning, B is also row regular. By Lemma 2.3, $A \otimes B$ is row regular. Hence, all vertices of $\Gamma_1 \otimes \Gamma_2$ have the same degree so $\Gamma_1 \otimes \Gamma_2$ is regular.

Conversely, let $\Gamma_1 \otimes \Gamma_2$ be regular. Then $A \otimes B$ is row regular and by Lemma 2.3 A and B are both row regular. So Γ_1 and Γ_2 are regular graphs. □

Theorem 2.6. *Let β be a bipartite graph and Γ any other graph. Then $\beta \otimes \Gamma$ is bipartite.*

Proof. Let $U_1 \sqcup U_2$ be a bipartition of the vertices of β, and let V be the set of vertices of Γ. Then $U_1 \times V$ and $U_2 \times V$ form a bipartition of the vertices of $\beta \otimes \Gamma$ since for (u_i, v_k), $(u_j, v_l) \in U_1 \times V$ there are no edges between u_i and u_j in β and hence no edges between (u_i, v_k) and $(u_j, v_l) \in \beta \otimes \Gamma$. An identical argument can be used to show there are no edges between the vertices of $U_2 \times V$.

□

3. Zeta Functions of Kronecker Products of Graphs

Now that we have defined the zeta function on graphs and the Kronecker product of graphs, we turn our attention to zeta functions on these products. The primary question we ask is, given graphs Γ_1, Γ_2 with equal zeta functions and a third graph Γ', when do $\Gamma_1 \otimes \Gamma'$ and $\Gamma_2 \otimes \Gamma'$ have the same zeta function? This section proves a result for the case when all three graphs are regular.

Definition 3.1. Two graphs are *cospectral* provided their adjacency matrices have the same eigenvalues (counted with multiplicity).

Definition 3.2. A graph Γ is said to be *md2* if $2 \leq deg(v)$, for every vertex $v \in \Gamma$.

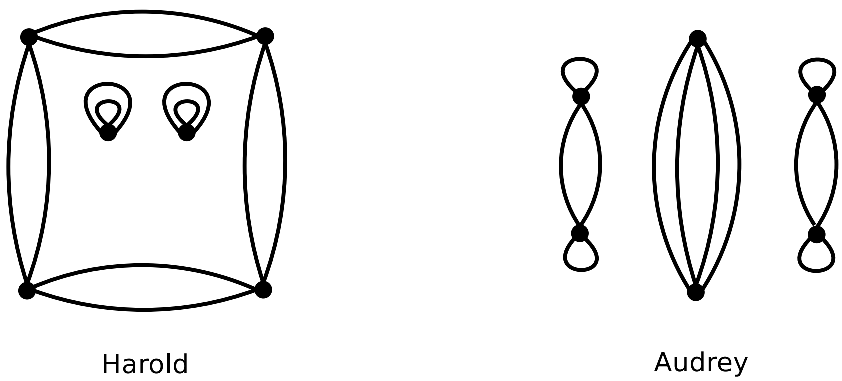
Theorem 3.3. *(Aubi Mellein)* [2] *Let Γ_1, Γ_2 be regular md2 graphs. Then $Z_{\Gamma_1}(u) = Z_{\Gamma_2}(u)$ if and only if Γ_1 and Γ_2 are cospectral.*

Corollary 3.4. *Let Γ_1, Γ_2 be regular md2 graphs with $Z_{\Gamma_1}(u) = Z_{\Gamma_2}(u)$ and let Γ' be any k-regular graph with $1 \leq k$. Then $Z_{\Gamma_1 \otimes \Gamma'}(u) = Z_{\Gamma_2 \otimes \Gamma'}(u)$.*

Proof. Since Γ_1 and Γ_2 are regular, md2, and have the same zeta function, their adjacency matrices have the same eigenvalues, $\lambda_1, \lambda_2, ...\lambda_n$. Let the eigenvalues of the adjacency matrix of Γ' be $\delta_1, \ldots, \delta_m$. By a property of the Kronecker product of matrices, the eigenvalues of the adjaceny matrix of $\Gamma_1 \otimes \Gamma'$ are $\lambda_i \delta_j$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ [4]. Similarly, the eigenvalues of the adjacency matrix of $\Gamma_2 \otimes \Gamma'$ must also be $\lambda_i \delta_j$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$. So $\Gamma_1 \otimes \Gamma'$ and $\Gamma_2 \otimes \Gamma'$ are cospectral and by Theorem 2.5, they are also regular. Furthermore, for any vertex $(p, q) \in \Gamma_1 \otimes \Gamma'$, q is connected to some t since Γ' is k-regular, $k \geq 1$, and p is connected to some s by at least two edges since Γ_1 is md2. Hence there are at least 2×1 edges between (p, q) and (s, t). The same is true for $\Gamma_2 \otimes \Gamma'$ and so both graphs are md2. Now by Theorem 3.3, $Z_{\Gamma_1 \otimes \Gamma'}(u) = Z_{\Gamma_2 \otimes \Gamma'}(u)$. □

As useful as this corollary sounds, it remains to be checked that $\Gamma_1 \otimes \Gamma' \not\cong \Gamma_2 \otimes \Gamma'$, an issue that is not as trivial as it may seem. For example, if Γ' is a single vertex and $|V(\Gamma_1)| = |V(\Gamma_2)| = n$ then it is always true that $\Gamma_1 \otimes \Gamma' \cong \Gamma_2 \otimes \Gamma'$. (Both adjacency matrices are the $n \times n$ zero matrix.)

Example 3.5. *Harold* and *Audrey* are non-isomorphic, regular, and have the same zeta function: $\frac{(1-u^2)^6}{1-8u-18u^2+248u^3-121u^4-1872u^5+996u^6+5616u^7-1089u^8-6696u^9-1458u^{10}+1944u^{11}+729u^{12}}$



By Corollary 3.4, $Z_{Harold \otimes \Gamma'}(u) = Z_{Audrey \otimes \Gamma'}(u)$ for any regular graph Γ'. I propose the following conjectures.

Conjecture 3.6. *Harold* $\otimes$ *n-gon* $\cong$ *Audrey* $\otimes$ *n-gon when n is even and Harold* $\otimes$ *n-gon* $\ncong$ *Audrey* $\otimes$ *n-gon when n is odd.*

I have checked that this is true up to multiplication by an octogon.

Conjecture 3.7. *For any tree T, Harold* $\otimes T \cong$ *Audrey* $\otimes T$.

I have been able to show this for all trees up to 5 vertices.

Example 3.8. Here are two cases where the Kronecker product of a graph with *Harold* and *Audrey* produces non-isomorhpic graphs (with the same zeta function).

(i) Let K_n denote the complete graph on n vertices.
Harold $\otimes K_n \ncong$ *Audrey* $\otimes K_n$ for $n \geq 3$.

The adjacency matrices of *Harold*, *Audrey* and K_n are

$$H = \begin{pmatrix} 0 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 & 1 & \dots \\ 1 & 0 & 1 & \\ 1 & 1 & 0 & \\ \vdots & & & \end{pmatrix}_{n \times n}$$

Any isomorphism between *Harold* $\otimes K_n$ and *Audrey* $\otimes K_n$ must take the component,

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \otimes K \;\; to \;\; \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \otimes K.$$

However, the first of these gives

$$\begin{pmatrix} 0 & 4 & 4 & \cdots & & & & \\ 4 & 0 & 4 & & & \bigcirc & & \\ 4 & 4 & 0 & & & & & \\ \vdots & & & & & & & \\ & & & & 0 & 4 & 4 & \cdots \\ & & \bigcirc & & 4 & 0 & 4 & \\ & & & & 4 & 4 & 0 & \\ & & & & \vdots & & & \end{pmatrix}$$

which corresponds to a graph with 2 components, each isomorphic to K_n, while the second Kronecker product is

$$\begin{pmatrix} & & & & 0 & 4 & 4 & \cdots \\ & & \bigcirc & & 4 & 0 & 4 & \\ & & & & 4 & 4 & 0 & \\ & & & & \vdots & & & \\ 0 & 4 & 4 & \cdots & & & & \\ 4 & 0 & 4 & & & & \bigcirc & \\ 4 & 4 & 0 & & & & & \\ \vdots & & & & & & & \end{pmatrix}$$

which is the adjacency matrix of a connected bipartite graph. Hence the two products are non-isomorphic. Furthermore, since K_n is regular, $Harold \otimes K_n$ and $Audrey \otimes K_n$ have the same zeta function.

(ii) $Harold \otimes L \ncong Audrey \otimes L$ where L is any graph containing a loop.

For every vertex $x_i \in L$ with l_i loops, $Harold \otimes L$ will have two vertices with $2l_i$ loops while $Audrey \otimes L$ will have four vertices with l_i loops. Note that if L is regular then $Harold \otimes L$ and $Audrey \otimes L$ will have the same zeta function.

4. More Examples of Products

In this section, I have taken several pairs of graphs $\Gamma_1,\ \Gamma_2$ with the same zeta function and multiplied them by various small graphs Γ' in order to see whether the product yields pairs with the same zeta function. My calculations suggest three classes of pairs of graphs.

(1) Ihara Universal: Pairs Γ_1 and Γ_2 such that $Z_{\Gamma_1 \otimes \Gamma'}(u) = Z_{\Gamma_2 \otimes \Gamma'}(u)$ for any graph Γ'.

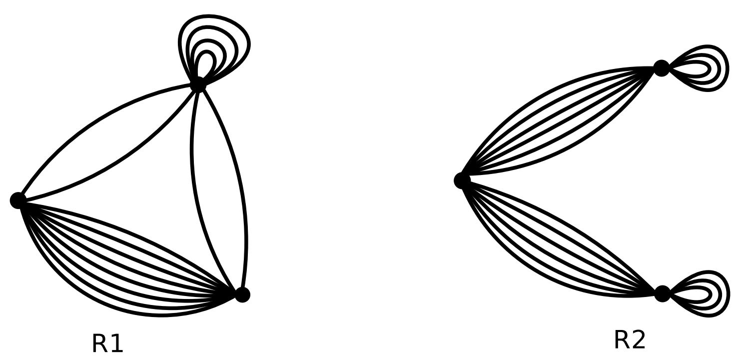
(2) Ihara Regular: Pairs Γ_1 and Γ_2 such that $Z_{\Gamma_1 \otimes \Gamma'}(u) = Z_{\Gamma_2 \otimes \Gamma'}(u)$ if Γ' is regular.

(3) Ihara Singular: Pairs Γ_1 and Γ_2 such that $Z_{\Gamma_1 \otimes \Gamma'}(u) = Z_{\Gamma_2 \otimes \Gamma'}(u)$ if Γ' consists of a disjoint union of copies of the complete graph K_2.

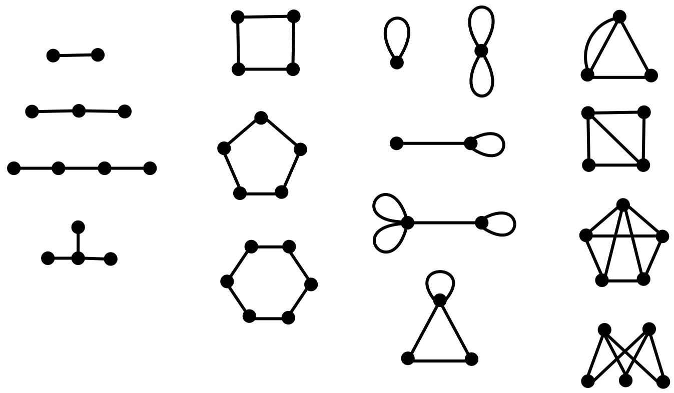
Note that $(1) \subset (2) \subset (3)$. The following conjecture suggests all pairs with the same zeta function belong to one of these classes.

Conjecture 4.1. *Let* $\Gamma_1,\ \Gamma_2$ *be graphs with* $Z_{\Gamma_1}(u) = Z_{\Gamma_2}(u)$. *Then* $Z_{\Gamma_1 \otimes K_2}(u) = Z_{\Gamma_2 \otimes K_2}(u)$.

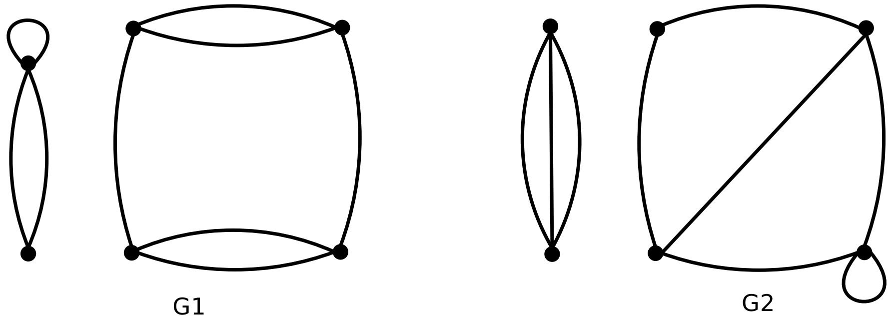
Example 4.2. The following pair of graphs, R_1 and R_2, are a candidate for being Ihara Universal. They are 12-regular, non-isomorphic and have the same zeta function.



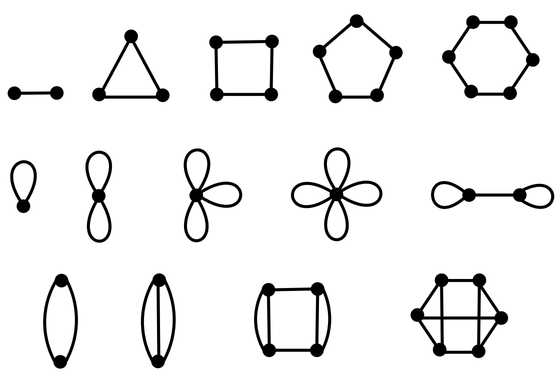
I have checked that their zeta functions remain the same when multiplied by K_3, K_4, K_{10} as well as all of the following graphs It should be noted that these products do produce non-isomorphic graphs.



Example 4.3. The following pair of graphs, G_1 and G_2, are a candidate for being Ihara Regular, but not Ihara Universal.

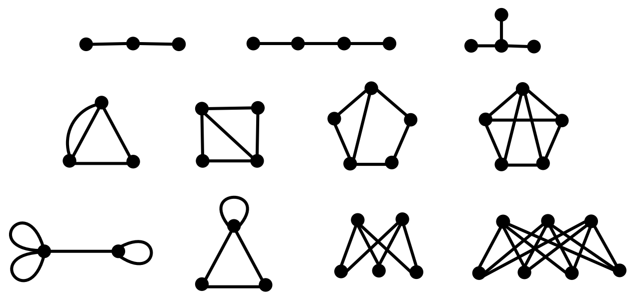


Their zeta functions are equal after multiplication by any of the following graphs:



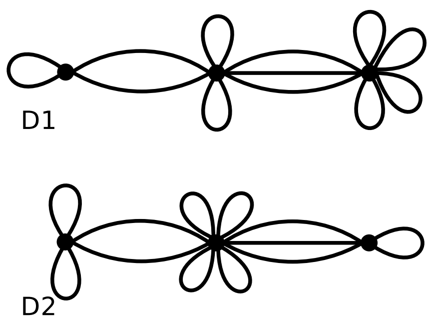
as well as K_4, K_5, $K_{3,3}$, and $K_{4,4}$.

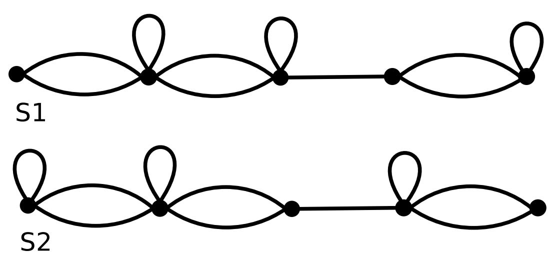
However, their zeta functions are not equal after multiplication by these:



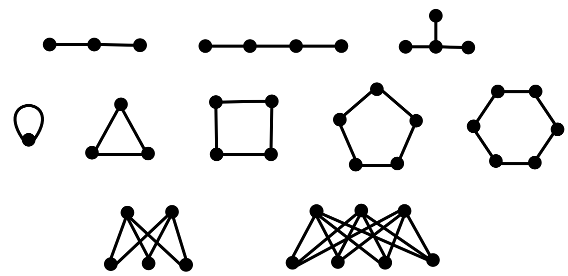
Again, in all cases the products yield non-isomorphic graphs.

Example 4.4. These two pairs are candidates for being Ihara Singular, but not Ihara Regular.





For both pairs, the zeta functions are equal when multiplied by K_2 and not equal when multiplied by any of the following graphs: as well as K_4 and K_5.



5. Further Research

Further research goals in this area include discovering what properties of a graph define each class and proving that these are the only classes. Related work on the topic of zeta functions and graph products can be found in the paper, "Defining a Zeta Function for Cell Products of Graphs" by Zuhair Khandler, another student at the LSU REU.

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