Referree's Report
Differential Geometry of Manifolds with Density

The results in the paper are interesting, but the exposition needs a lot of work to be acceptable and in general readable. I would be very hesitant to give the article to any undergraduate to read and understand, who has not been to the SMALL Geometry program and already knew about manifolds with density.

The major problem is that the abstract's goal of describing extensions of several key concepts of differential geometry to manifolds with density is incorrect. The paper states several extensions, but does not describe the extension. For instance, the curvature of a curve in a manifold with density is defined, but not described. Leaving the reader with practically no clue why that is the definition, and how it extends the intuitive geometric definition. The paper seems to be a collection of related results with no exposition linking the results to basic results in differential geometry to show the extension.

The biggest complaint I have with the paper is that it never explains what a manifold with density is and why they merit more careful study. It seems that a manifold with density is just a Riemannian manifold with the property that volume dV is weighted by a function δ, the area of hypersurface in the manifold is weighted by the same function δ and the length of a curve is weight by the same function δ. This needs to be made clear to the reader. In addition, why the function is called a density is not clear? A concrete example of a manifold with density coming from a simple application would help considerably so the reader understands the basic concepts before delving into the extensions. It would also help to keep referring to this example as the paper continues, to ground the reader. For instance, the Gauss plane/space is referred to through out the paper, but it is never formally defined and explained.

I would suggest that a concrete example, the Gauss Plane or more generally Gauss space, is given in the introduction and explained. Then as each new concept is introduced (curvature, mean curvature, Gauss curvature) return to the example and apply the concept to the example. I would also suggest that the paper be broken into more sections. Section 2 covers everything other than the introduction. For instance, Section 2 could be on curves on surfaces (the subsection on curvature of curves in surfaces with density and the subsection geodesics in planes with density), Section 3 could be on mean curvature of surfaces and hypersurfaces, and Section 4 on Gauss curvature.

Further Comments on the Introduction: The introduction needs to be an introduction to the material, not a summary of the paper. The prospective reader should be treated as an advanced undergraduate student (this is an undergraduate journal), who has had a course in the differential geometry of curves and surfaces, and maybe knows about abstract manifolds. But the reader should not be assumed to know what anything about a manifold with density. The definitions of the curvatures should not be mentioned unless they are explained in the introduction, and specific results unless they are the main results that can easily be stated and explained should not be mentioned. Stating that the curvature is this in the introduction, then restating later in the paper with no further explanation does not yield any insight into the definition. This leaves the reader with only a formula and no insight.

Comments on Section 2: Curvature The second section on the curvature continues the lack of description started in the introduction. Definitions are given and propositions are proved, with little insight given. The justifications for the definitions for curvature that are given only make sense, if the reader is reminded of the connection in the usual Riemannian sense. It is best not to assume that the readers will make the connection on their own.

It is unclear whether Definition 2.1 is the definition of a manifold with density or the definition of the curvature of a curve in a manifold with density. In particular, the definition only seems to make sense for curves in a two-dimensional surface where there is a well-defined normal for a smooth curve. The justification also needs some explanation. The proposition itself needs some rewriting as the notation is hard to follow. ds_φ appears in the proof with no explanation. It is not surprising. In addition A_φ needs to be rethought as a notation here as it is used later to denote a different quantity.

Proposition 2.3 requires a little background, as the definition of an isoperimetric curve appears in the proof. The definition should be given to motivate the result. Proposition 2.4 seems tacked on as it does not follow from previous results and is not used in future results, and is just stuck here. The rearrangement of the sections suggested above where the results on curves are consolidated makes this proposition much more important as it is used immediately following its introduction.

The mean curvature subsection has the same general problem as the curvature of a curve. The proposition 2.6 has the annoyance that you have dA_φ/dt calculated with an integral with respect to dA_varphi. This does not make sense. I hope there is a typo. The remark 2.7 needs to be made clear that $\mathbf{n}$ is the normal to the hypersurface, else the remark does not make sense.

The subsection on Gauss curvature has the same general problems as the other sections. The application of Gauss curvature for area of a circle needs more motivation. In this subsection Proposition 5.17 is mentioned, but there is no Proposition 5.17. The proofs of Proposition 2.15 and 2.16 need to be clarified at least in terms of notation. The notation $\delta = e^\varphi$, then $\delta(r) = \delta + \delta_r\, r + (1/2)\delta_{rr} r^2 + \cdots$ should be rethought. The computation also assume seem to assume that all functions δ, δ_r, δ_{rr} are constants, but they really depend on θ as you are working in polar coordinates. But that ignores that e^φ is not a constant. It works out. The result is correct, but the proof is overly simplified for a special case.

The reference to [ADLV] needs to removed or at least not treated as it is common knowledge. Technical reports for the SMALL program are not in general known outside the SMALL program.

Comments on Section on Geodesics: I found this to be the most interesting section, as it seems to be the result with the most unknown while it is elementary. However, on page 10, you should mention a conjecture as a result. This is incorrect, conjectures are not result. They can be the most important thing in a section, but as they can be wrong they are not results.

Comments on Section on Surfaces with Constant Mean Curvature: This is my second favorite section as it continues the results in the previous section from curves to surfaces. The proof of Proposition 2.25 should be done out in full detail, show that $d\phi/d\mathbf{n}$ is constant.