The Riemann Surface of the Logarithm Constructed in a Geometrical Framework

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Abstract

The logarithmic Riemann surface Σ_{log} is a classical holomorphic 1-manifold. It lives into $\mathbb{R}^4$ and induces a covering space of $\mathbb{C} \setminus \{0\}$ defined by $\exp_{\mathbb{C}}$.

This paper suggests a geometric construction of it, derived as the limit of a sequence of *vector fields extending* $\exp_{\mathbb{C}}$ suitably to *embeddings of* $\mathbb{C}$ *into* $\mathbb{R}^3$, which turn to be *helicoid* surfaces living into $\mathbb{C}\times\mathbb{R}$. In the limit we obtain a bijective complex exponential on the covering space in question, and thus a well-defined complex logarithm. In addition, the *helicoids* are *diffeomorphic (not bi-holomorphic) copies* of Σ_{log} as C^∞*-realizations living into* $\mathbb{R}^3$, without obstruction.

Our approach is purely *geometrical* and *does not employ any tools provided by the complex structure*, thus *holomorphy is no longer necessary to obtain constructively this Riemann surface* Σ_{log}. Moreover, the differential geometric framework we adopt affords explicit generalization on submanifolds of $\mathbb{C}^m \times \mathbb{R}^m$ and certain corollaries are derived.

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"You can not claim you are well aware of a theorem unless you have more than one proof for it"

M. Atiyah's dictum

Introduction.

Riemann surfaces have been, at least conceptually, first introduced by Riemann in his celebrated 1851 PhD dissertation. He considered a surface spread over $\mathbb{C}$ with several sheets lying over it, in such a way that a complex multi-valued function with branches becomes a true well defined function, mapping each branch to a sheet. His posterity gave a rigorous definition of the (so called) *Concrete Riemann surfaces* as covering spaces of $\mathbb{C}$ (for an enlightening review see [18]).

Klein was the first to put away the covering space approach and adopted a differential geometric one: he studied complex functions living on a curved surface of some ambient Euclidean space. In fact, it was the first time the *Atlas of holomorphic structure* was introduced. At those times, "Riemann surface" meant compact 2-manifold with an arc-length element ds^2 and bi-holomorphic transition functions in the atlas. Several years later, adding Cantor's 2nd countability and Hausdorff's axioms (not before the 1920's), Radó reaches via triangulations the *Abstract Riemann surfaces'* definition: a *Hausdorff 2nd-countable topological surface with complex structure.*

The equivalence of *Concrete* and *Abstract* Riemann surfaces follows from a theorem of Behnke & Stein (1947−49) [2], improved by Gunning and Narasimhan (1967) [8].

In modern times, the research interest has been transferred to the study of their *Moduli (& Teichmüller) spaces*, being in effect sets of equivalence classes of distinct complex structures, modulo the action of orientation preserving diffeomorphism of the surface with certain topological structures [13], as well as to their vasty applications in *theoretical physics*, specifically *M-theory unifying the several String theories* (see e.g. [12]).

In this paper we follow a semi-classical approach, standing on the line between the *abstract* and the *concrete*. Our principal result is that *we recover in a pure differential-geometric fashion results classically obtained via the holomorphic* *structure itself, bypassing analytic continuation.*

Technically, the employed apparatus consists of a sequence of vector fields $\{\mathrm{Exp}_n\}_{n\in\mathbb{N}}$ defined on $\mathbb{C}$ and valued in (the tangent bundle of) $\mathbb{R}^3$. Each field $\mathrm{Exp}_n : \mathbb{C} \longrightarrow \mathbb{C}\setminus\{0\}\times\mathbb{R}$ (*Def.* 1.3) geometrically is in fact a smooth parametric surface of $\mathbb{R}^3$ called (exponential) helicoid (*Def.* 1.1). These Exp_n's extend (due to the extra component) the ordinary $\exp_{\mathbb{C}}$ in such a way that they become C^∞-diffeomorphisms onto their images (& embeddings of $\mathbb{C}$ into $\mathbb{C}\times\mathbb{R}$) removing $2\pi i$ periodicity.

The Exp_n-maps by construction converge to the covering space of the punctured plane $\exp_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}\setminus\{0\}$. Bijectivity is preserved in the limit allowing a well defined limit logarithm $\log_{\mathbb{C}}$ to be introduced on the direct limit surface $\varinjlim \mathrm{Exp}_n(\mathbb{C}) \cong \exp_{\mathbb{C}}(\mathbb{C})$ (in 2.7) which essentially is its *Riemann surface* Σ_{log} immersed into the punctured plane. Even though all the countably many sheets are "compressed" to a single sheet, we still have the correspondence of the $\log_{\mathbb{C}}$-branches and the sheets of the covering space via the convergence mechanism.

Notwithstanding, we show that each Exp_n is C^∞-diffeomorphic to Σ_{log}, when the latter is considered as equipped with the smooth sub-atlas of the holomorphic one coming from the ambient $\mathbb{R}^4$-space. The fact that the helicoids can only be diffeomorphic copies of Σ_{log} and not bi-holomorphic is imposed by topological obstructions due to the (general) non-imbeddability of Riemann surfaces into $\mathbb{R}^3$.

The last *Section* is devoted to high-dimensional generalizations on multi-helicoid submanifolds of Euclidean $3m$-space.

In view of the above, a natural question perhaps arises to the reader: what kind of outstanding property do the helicoids enjoy and what is so special about the Exp vector fields? The answer is that they are an *ad hoc* choice in order to provide the desired properties via convergence. There may very well exist even more privileged surfaces of $\mathbb{R}^3$, but the smooth realizations of the holomorphic Σ_{log} into 3-space are *dimension-wise* clearly *optimal.*

1. Preliminaries.

We collect a few preparatory results which will be needed in the main part of the paper, noted here for the reader's convenience.

1.1 Definition. (Helicoid) The smooth (parametric) surface of the *(exponential) helicoid* (see e.g. [17], [19]), is the map $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$(u, v) \longmapsto (e^u \cos v,\ e^u \sin v,\ av) \ , \quad a > 0$$

where a is a parameter. We consider this as the (global) coordinate system of a 2-manifold imbedded in $\mathbb{R}^3$. In complex coordinates we may write

$$X\ :\ \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R} \quad :$$

$$u + iv\ \longmapsto\ (\exp_{\mathbb{C}}(u + iv),\ av)\ \equiv\ (\exp_{\mathbb{C}}(z),\ a\mathrm{Im}(z))$$

where $z \equiv u+i\,v$.The complex functions Re, Im, $\exp_{\mathbb{C}}\ :\ \mathbb{C} \longrightarrow \mathbb{C}$ denote the "*real part*", the "*complex part*" and the "*complex exponential*" of $\mathbb{C}$ respectively.

The terminologies of imbeddings and immersions we follow are the standard ones, referring e.g. to [7], [11].

We employ the convention that *"imbedding"* stands for topological imbedding and *"embedding"* for geometrical embedding in the sense of smooth manifolds.

In this paper *smoothness* means C^∞ - *smoothness* in the usual geometric sense and diffeomorphism stands for C^∞-diffeomorphism.

Furthermore, as usual by the term *smooth n-manifold M* (and in particular, surface) we mean a C^∞-smooth connected, paracompact Hausdorff manifold, of real n dimensions $\dim_{\mathbb{R}}(M) = n$ (2, in the 2nd case).

1.2 Remark. Let now

$$\exp_{\mathbb{C}}\ :\ \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\} \quad : \quad z \longmapsto \exp_{\mathbb{C}}(z) \equiv e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

be the ordinary complex exponential function (*the notations* $\exp_{\mathbb{C}}(z)$ *and* e^z *will be occasionally exchanged without comments*). From the differential-geometric wiewpoint, this map may be considered as a vector field, tangent on the 2-dimensional (flat) manifold $\mathbb{C} \cong \mathbb{R}^2$, i.e.

$$\mathbb{C} \equiv \mathbb{R}^2 \ni\ ue_1 + ve_2\ \longmapsto\ (e^u \cos v)e_1 + (e^u \sin v)e_2.$$

This owes to the fact that if $\xi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector field of $\mathbb{R}^n$ (in the *vector calculus* sense), where $x \longmapsto \xi(x)$, then the obvious identification $\xi(x) \equiv (x, \xi(x))$ is the requested one in order to consider ξ valued in the tangent bundle $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ (vector field in the geometric sense) by global triviality of the vector (tangent in our case) bundle ([11], [7], [20], etc).

The current task is to define a certain extension of the exponential $\exp_{\mathbb{C}}$ from $\mathbb{C}$ to $\mathbb{C} \times \mathbb{R}$ that removes the $2\pi i$-periodicity and become injective due to extra real component.

These remarks introduce the idea of the following definition:

1.3 Definition. (The exponential field) The exponential vector field on $\mathbb{C}$ is the map

$$\mathrm{Exp}_a \ : \ \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$$

given by:

$$u + iv \longmapsto \mathrm{Exp}_a(u + iv) \ := \ (\exp_{\mathbb{C}}(u + iv), \ av)$$

where $a > 0$ is a parameter to be fixed as will in the sequel. Operationally the map can given as

$$\mathrm{Exp}_a \ = \ (\exp_{\mathbb{C}}, \ a \ \mathrm{Im})$$

The following technical fact shows that this vector-wise extension of the $\exp_{\mathbb{C}}$ function on $\mathbb{C}$ is the requested one that provides injectivity of the map, now considered as a vector field.

1.4 Proposition. ***(Structural Properties of*** Exp***)***
The map $\mathrm{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$ *given by 1.3 has the following properties:*

a) *It is a vector field defined on the (trivially) embedded submanifold* $\mathbb{C} \equiv \mathbb{C} \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ *and valued into the (tangent bundle of the) ambient space* $\mathbb{R}^3$.

b) *It is a diffeomorphism onto its image, thus an embedding of the plane into* $\mathbb{C} \times \mathbb{R}$ *and, in particular, invertible.*

As our context suggests, we consider submanifolds as subsets of $\mathbb{R}^n$, where the inclusion map is an imbedding and the Atlas that determines differential structure and topology is the induced from $\mathbb{R}^n$.

Proof. a) We recall that a vector field defined on a *submanifold* M of $\mathbb{R}^n$ in not necessarily tangent to the manifold in the usual differential geometric sense, but the splitting of tangent spaces

$$\mathbb{R}^n \cong T_p\mathbb{R}^n \cong T_pM \bigoplus T_p^{\perp}M \ , \quad p \in \mathbb{R}^n$$

implies that it can be normal, or generally valued in $T\mathbb{R}^n \cong \mathbb{R}^{2n}$. The very definition of Exp_a implies that it maps a point z of $\mathbb{C}$ to a vector $\mathrm{Exp}_a(z)$ of $\mathbb{C}\times\mathbb{R}$

$$\mathbb{C} \ni \ z \longmapsto \mathrm{Exp}_a(z) \ \in \ \mathbb{C}\times\mathbb{R}$$

and *Remark* 1.2 implies that the map is a vector field of $\mathbb{C}$ valued in the tangent bundle of $\mathbb{C}\times\mathbb{R}$.

b) If $\mathrm{Exp}_a(u_1+iv_1,\ av_1) = \mathrm{Exp}_a(u_2+iv_2,\ av_2)$ for $u_1, u_2, v_1, v_2 \in \mathbb{R}$, then this amounts to

$$(\exp_{\mathbb{C}}(u_1+iv_1),\ av_1) = (\exp_{\mathbb{C}}(u_2+iv_2),\ av_2)$$

that is

$$\exp_{\mathbb{R}}(u_1)\exp_{\mathbb{C}}(iv_1) \ = \ \exp_{\mathbb{R}}(u_2)\exp_{\mathbb{C}}(iv_2) \ , \qquad v_1 = v_2$$

and consequently $u_1 = u_2$ by the monotonicity of the real exponential. This shows the injectivity of Exp, thus it is bijective onto its image $\mathrm{Exp}_a(\mathbb{C})$. C^∞-differentiability of this map goes without saying, since it has smooth component functions in $\mathbb{C}\times\mathbb{R}$. Smoothness of the inverse map is also obvious (for the explicit expression of the inverse map see *Lemma* 2.6 of the oncoming *Section 2*). Consequently, the map is a smooth (geometric) embedding of $\mathbb{C}$ into $\mathbb{C}\times\mathbb{R}$. □

Consider now the standard Euclidean (Riemannian) metric δ of $\mathbb{R}^3$ with components the Kronecker deltas' δ_{ab} (where $\delta_{aa} = 1$ and 0 otherwise, $1 \leq a \leq 3$) (for the standard concepts of *Riemannian geometry* employed in this paper, we refer to [5], [7], [11], [17]). Then, the (trivially) embedded surface $\mathbb{C}$ gets a natural induced *flat* Riemannian metric, the standard inner product $< \cdot,\cdot > \equiv \ \delta$ of $\mathbb{R}^3$ by restriction of δ on $\mathbb{C}$, which can be seen as the pull-back metric via the inclusion map:

$$(\mathbb{C},\ \delta|_{\mathbb{C}}) \ \equiv \ (\mathbb{C},\ i^*(\delta))$$

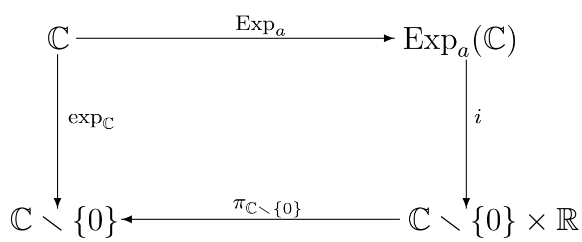
$$i \;:\; \mathbb{C} \;\hookrightarrow\; (\mathbb{C}\times\mathbb{R},\ \delta)$$

In view of theses remarks, we obtain the next result giving some geometric properties of the extended object Exp_a as a vector field on the $(\mathbb{C},\ i^*(\delta))$.

Notation. $angl(\xi,\eta)(p)$ denotes the angle of 2 vector fields ξ, η measured at the point p of the surface (manifold), with respect to its Riemannian metric.

1.5 Proposition. ***(Geometric Properties of*** Exp***)***

a) *The vector field* $\mathrm{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C}\times\mathbb{R}$ *geometrically is a smooth exponential helicoid surface of the form 1.1 and the following diagram commutes:*



that is, Exp_a *commutes with the projection composed with the inclusion of the surface* $i : \mathrm{Exp}_a(\mathbb{C}) \hookrightarrow \mathbb{C}\setminus\{0\}\times\mathbb{R}$ *and the projection* $\pi_{\mathbb{C}\setminus\{0\}} \;:\; \mathbb{C}\setminus\{0\}\times\mathbb{R} \longrightarrow \mathbb{C}\setminus\{0\}$

$$(\ \pi_{\mathbb{C}\setminus\{0\}} \circ\ i\ )\ \circ\ \mathrm{Exp}_a \;=\; \exp_{\mathbb{C}}$$

b) *If the tangent plane* $T_{u+iv}\mathbb{C} \cong \mathbb{C} \equiv \mathbb{R}^2$ *is spanned by* $e_1 = (1,0)$ *and* $e_2 = (0,1)$ *and*

$$\xi_{AB} \equiv Ae_1 + Be_2$$

is a general 2*-parameter family of tangent vectors,* $A,\ B \in \mathbb{R}$, $|A|+|B| \neq 0$, *then*

$$\cos[(\Phi_{AB})(u,v)] = \frac{e^u\,[A\cos v + B\sin v]}{\sqrt{e^{2u}+a^2v^2}\ \sqrt{A^2+B^2}}$$

where $\Phi_{AB}(u,v)$ *is the angle of* Exp *with* ξ_{AB} *at* $u+iv$ *in* $\mathbb{R}^3$*:*

$$(\Phi_{AB})(u,v) \equiv angl(\mathrm{Exp}_a,\ \xi_{AB})(u+iv)$$

1.6 Remark. The calculation of the aforementioned (non-constant) angle shows that the Exp_a-vector field is neither tangent nor normal to $\mathbb{C}$.

Proof. a) The first claim goes without saying, just by comparing the very definitions 1.1 and 1.3. Thus, the map Exp_a is a vector field of $\mathbb{R}^3$ defined on $\mathbb{C}$ and simultaneously its image constitutes a smoothly embedded surface of $\mathbb{R}^3$, an exponential helicoid diffeomorphic to $\mathbb{C}$.

Commutativity is a simple consequence of the form of the Exp_a-field:

$$(\pi_{\mathbb{C}\setminus\{0\}} \circ \ i\ ) \ \circ \ \mathrm{Exp}_a(z) \ = \ (\pi_{\mathbb{C}\setminus\{0\}} \ \circ \ (\exp_{\mathbb{C}}, \ a\, Im))(z) \ = \ \exp_{\mathbb{C}}(z)$$

b) We recall the familiar Euclidean formula giving the angle of 2 vectors in $(\mathbb{R}^3, \delta) = (\mathbb{R}^3, \ < \cdot, \cdot >)$

$$\cos\,[angl(\mathrm{Exp}_a, \xi_{AB})(u,v)] \ = \ (\frac{< \mathrm{Exp}_a, \xi_{AB} >}{||\mathrm{Exp}_a||\ ||\xi_{AB}||})(u+iv)$$

The 2-parameter vector (describing a general tangent vector) is given due to the inclusion $\mathbb{C} \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R}$ by the formula

$$\xi_{AB} \ = \ Ae_1 + Be_2 + 0e_3 \ = \ A\,(1,0,0) + B\,(0,1,0) + (0,0,0) \ = \ (A, B, 0)$$

and provided that $\mathrm{Exp}_a = (e^u \cos v, e^u \sin v, av)$ we have

$$\cos[(\Phi_{AB})(u,v)] = \frac{A\ e^u \cos v + B\ e^u \sin v}{\sqrt{e^{2u} + a^2v^2}\ \sqrt{A^2+B^2}}$$

which gives the requested formula and this completes the proof. □

2. The Logarithmic Riemann surface.

(Part I) **<u>The Convergence Constructions</u>.**

The *infinite-sheeted Riemann surface* Σ_{log} *of the logarithm* $\log_{\mathbb{C}}$ (see [1], [9], [22]), considered as a holomorphic manifold of (complex) dimension $\dim_{\mathbb{C}}(\Sigma_{log}) = 1$, lives into $\mathbb{C}^2 \cong \mathbb{R}^4$ and can be given as the graph of the function $e^z = w$

$$\Sigma_{log} \ := \ \{z \in \mathbb{C} \ / \ e^z = w\} \ \subseteq \ \mathbb{R}^4$$

or by the parametric equations

$$\mathbb{C} \ni \ z \ \longmapsto \ (z, \exp_{\mathbb{C}}(z)) \ \in \mathbb{C} \times \mathbb{C} \ \cong \ \mathbb{C}^2$$

$\Sigma_{log}^{\mathcal{O}}$ denotes the holomorphic manifold $(\Sigma_{log}, \mathcal{A}^{\mathcal{O}}_{\Sigma_{log}}) \hookrightarrow (\mathbb{R}^4, id)$ with the induced analytic atlas coming from $\mathbb{R}^4$. Classically, it is obtained by analytic continuation, extending holomorphically $\exp_{\mathbb{C}}$ to "small" discs along a disk $\mathbb{D}(0,1)$ of $\mathbb{C}$. Since the last extension that overlaps with the first does not coincide with it, we take a copy of $\mathbb{C}$ and proceed continuation to a next sheet lying over the initial one. Continuing ad infinitum, we obtain a covering space of $\mathbb{C} \setminus \{0\}$ bi-holomorphic to this surface.

Projecting to the second factor, we obtain the covering space map on the punctured plane $\mathbb{C} \setminus \{0\}$ (equivalence of *concrete* & *abstract* approach, [2], [8]):

$$\pi \ : \ \Sigma_{log}^{\mathcal{O}} \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C} \setminus \{0\} \quad : \quad (z, \exp_{\mathbb{C}}(z)) \longmapsto \exp_{\mathbb{C}}(z)$$

In general there is no way to holomorphically imbed this 2-manifold into $\mathbb{R}^3$, unless we allow self-intersections, which is no longer a homeomorphism onto its image but may still be an immersion.

Our construction allows, as we shall see, to introduce a well-defined complex logarithm. Note that *all this apparatus manages to bypass the holomorphic structure of this complex manifold and no analytic continuation is anywhere used.*

In this section we give the *geometric* construction of Σ_{log}. As stated in the *Introduction*, it is obtained as a special covering manifold of $\mathbb{C} \setminus \{0\}$ in the limit of a sequence obtained by the exponential Exp_a images of $\mathbb{C}$.

Hence, we recall the map $\mathrm{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$ given by 1.3 and substitute the positive parameter by the sequence $a_n = 1/n$, $n \in \mathbb{N}$. Thus, we obtain a sequence of vector fields (and surfaces of $\mathbb{R}^3$ as well) $\{\mathrm{Exp}_n\}_{n \in \mathbb{N}}$, where

$$\mathrm{Exp}_n \ : \ \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R} \quad : \quad z \longmapsto \big(\exp_{\mathbb{C}}(z), \frac{1}{n} Im(z))\big) \ , \qquad n \in \mathbb{N}$$

The following result is a well-known fact and can be found an any elementary textbook, e.g. [15]. For the definition of *covering manifolds* we refer to [7], [21].

2.1 Lemma. *The pair* $(\exp_{\mathbb{C}}, \mathbb{C})$*, constitutes a covering manifold of* $\mathbb{C} \setminus \{0\}$ *with countably infinite covering sheets.*

We are now in position to prove the main result of this section. The sequence of maps $\{\mathrm{Exp}_n : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}\}_{n \in \mathbb{N}}$ is a diffeomorphism onto its image, by *Proposition* 1.4, for $a = 1/n$, $n \in \mathbb{N}$ and consequently a bijection.

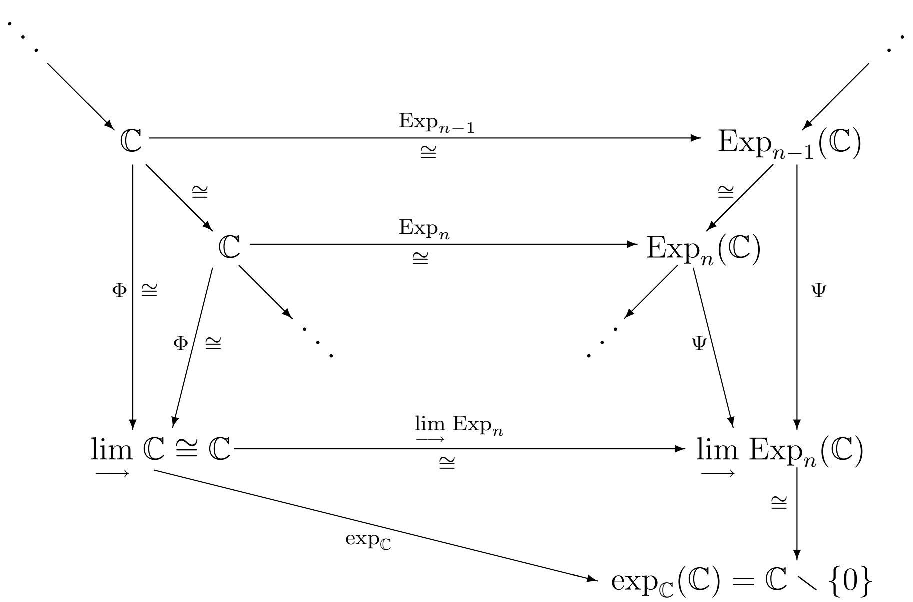
We shall presently see that this property is preserved in the limit $n \longrightarrow \infty$. The limit surface is the covering space produced by $\exp_{\mathbb{C}} \; : \; \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$ (*Lemma* 2.1), and introducing the natural inclusion $\mathbb{C} \setminus \{0\} \hookrightarrow \mathbb{C} \setminus \{0\} \times \mathbb{R}$ we may identify $\lim_{n \to \infty} (\mathrm{Exp}_n)$ on $\mathbb{C}$ with the usual complex exponential, $\exp_{\mathbb{C}}$. Thus, we construct an exponential injective on $\mathbb{C}$.

This situation resembles and simultaneously extends the analogous case in elementary complex analysis (see e.g. [1]), where $\exp_{\mathbb{C}}$ is injective and invertible on the strips of $2\pi i$-width only.

2.2 Theorem. ***(Convergence to Injective Complex Exponential)***
The sequence of diffeomorphisms $\{\mathrm{Exp}_n : \mathbb{C} \xrightarrow{\cong} (\mathrm{Exp}_n)(\mathbb{C}) \subseteq \mathbb{C} \setminus \{0\} \times \mathbb{R}\}_{n \in \mathbb{N}}$ *converges (point-wise) to a diffeomorphism* $\lim_{n \to \infty} \mathrm{Exp}_n$ *on the covering surface of* $\mathbb{C} \setminus \{0\}$ *produced by* $\exp_{\mathbb{C}} \; : \; \mathbb{C} \; \longrightarrow \; \mathbb{C} \setminus \{0\}$*:*

$$\mathrm{Exp}_n \;\xrightarrow[pw]{n \longrightarrow \infty}\; \exp_{\mathbb{C}} \times \{0\} \;\equiv\; \exp_{\mathbb{C}}$$

and the lower triangle of the following diagram defines (in the limit) a bijective complex exponential $\exp_{\mathbb{C}}$.



2.3 Remarks. Before giving the proof of this result which essentially contains the requested construction, some explanatory comments are necessary. First of all, the direct (inductive) limits "$\varinjlim$" (for their definition in a general *Category* we refer to the classical textbook [14]) are justified as follows:

The constant sequence of smooth manifolds

$$\mathbb{C} \xrightarrow[\cong]{id} \mathbb{C} \xrightarrow[\cong]{id} \mathbb{C} \longrightarrow \cdots$$

defines in a completely trivial way a direct limit set which is $\mathbb{C}$, since $id \circ id = id$. In addition,

$$\varinjlim \mathbb{C} \;=\; \lim_{n\to\infty} \mathbb{C} \;=\; \mathbb{C}$$

that is, the direct limit coincides with the usual sequence convergence. Similarly, we have the sequence

$$\mathrm{Exp}_n(\mathbb{C}) \xrightarrow[\cong]{\Theta_{n,n+1}} \mathrm{Exp}_{n+1}(\mathbb{C}) \xrightarrow[\cong]{\Theta_{n+1,n+2}} \mathrm{Exp}_{n+2}(\mathbb{C}) \longrightarrow \cdots$$

where the diffeomorphisms $\Theta_{n,m}$, $n,\ m \in \mathbb{N}$ are defined as

$$\Theta_{n,m} \;:\; \mathrm{Exp}_n(\mathbb{C}) \longrightarrow \mathrm{Exp}_m(\mathbb{C})$$

by the formula

$$\begin{aligned}(\exp_{\mathbb{C}}(z),\ \frac{1}{n}Im(z)) \longmapsto\ & \Theta_{n,m}\big(\exp_{\mathbb{C}}(z),\ \frac{1}{n}Im(z)\big) \\ &:= \ (\exp_{\mathbb{C}}(z),\ [\frac{n}{m}]\frac{1}{n}Im(z))\end{aligned}$$

The direct limit properties can be easily verified, since by construction of the Θ-maps we get

$$\Theta_{n,m} \circ \Theta_{m,k} \;=\; \Theta_{m,k}$$

as well as

$$\Theta_{n,n} = id_{\mathrm{Exp}_n(\mathbb{C})}$$

This implies that

$$\varinjlim \mathrm{Exp}_n(\mathbb{C}) \;=\; \lim_{n\to\infty} \mathrm{Exp}_n(\mathbb{C}) \;=\; \exp_{\mathbb{C}}(\mathbb{C}) \times \{0\} \;\cong\; \mathbb{C} \smallsetminus \{0\}$$

the Φ-maps pictured in the diagram are the identity (up to diffeomorphism) and the Ψ-maps are the projections on $\mathbb{C} \smallsetminus \{0\}$ composed with the inclusions of $\mathrm{Exp}_n(\mathbb{C})$ into $\mathbb{C} \times \mathbb{R}$:

$$\Psi \;:=\; \pi_{\mathbb{C}\smallsetminus\{0\}} \circ i$$

Proof. Proposition 1.4 implies that Exp_n is injective, $n \in \mathbb{N}$. Consequently, we have

$$\text{Exp}_n(z) = \text{Exp}_n(w) \implies z = w$$

Our task is to prove the following condition:

$$\exp_{\mathbb{C}}(z) \;=\; \exp_{\mathbb{C}}(w) \implies z = w$$

when z, w live into $\mathbb{C}$ and by *a)* of *Proposition* 1.5 we obtain

$$\text{Exp}_n \xrightarrow{n \longrightarrow \infty} \exp_{\mathbb{C}} \times \{0\} \equiv \exp_{\mathbb{C}}$$

having used that $(1/n) \xrightarrow{n \longrightarrow \infty} 0$.
The well-known triangle inequality gives

$$\left\|\text{Exp}_n(z) - \text{Exp}_n(w)\right\| \;\leq\; \left\|\text{Exp}_n(z) - (\exp_{\mathbb{C}}(z), 0)\right\| +$$

$$+ \left\|\text{Exp}_n(w) - (\exp_{\mathbb{C}}(w), 0)\right\| \;+\; |\exp_{\mathbb{C}}(w) - \exp_{\mathbb{C}}(z)|$$

When $n \longrightarrow \infty$, we have

$$\left\|\text{Exp}_n(z) - (\exp_{\mathbb{C}}(z), 0)\right\| \xrightarrow{n \longrightarrow \infty} 0$$

as well as

$$\left\|\text{Exp}_n(w) - (\exp_{\mathbb{C}}(w), 0)\right\| \xrightarrow{n \longrightarrow \infty} 0$$

$$\left\|\text{Exp}_n(z) - \text{Exp}_n(w)\right\| \;\leq\; |\exp_{\mathbb{C}}(z) - \exp_{\mathbb{C}}(w)| \xrightarrow{n \longrightarrow \infty} 0$$

which means that in the limit $z = w$ by the injectivity assured by 1.4. This completes the proof. □

2.4 Corollary. *The convergence is uniform on the bounded strips of* $\mathbb{C}$ *with bounded imaginary part* $B \;\equiv\; \{z \in \mathbb{C} \;/\; Im(z) < M,\; M > 0\}$*, that is, for finite spirals of the* Exp_n*-helicoids:*

$$\text{Exp}_n|_B \;\xrightarrow[U]{n \longrightarrow \infty}\; \exp_{\mathbb{C}} \times \{0\}|_B \;\equiv\; \exp_{\mathbb{C}}|_B$$

for any $M > 0$.

2.5 Remark. The bounded strips of the form of B corresponds in the limit to a finite-sheeted covering space of $\mathbb{C} \setminus \{0\}$:

$$\exp_{\mathbb{C}}|_B \;:\; B \longrightarrow \mathbb{C} \setminus \{0\}$$

produced by the complex exponential.

Proof. In order to see when the convergence of Exp_n to $\exp_{\mathbb{C}}$ is uniform, let $\varepsilon > 0$. Then, *Th.* 2.2 implies (using *a)* of *Prop.* 1.5)

$$\big\|\mathrm{Exp}_n(z) - (\exp_{\mathbb{C}}(z),\ 0)\big\| \;=\; \Big\|\Big(\exp_{\mathbb{C}}(z),\ \frac{1}{n}Im(z)\Big) - (\exp_{\mathbb{C}}(z),\ 0)\Big\| = \;\frac{1}{n}|Im(z)|$$

Imposing $|Im(z)| < M$ bounded, we obtain uniform convergence as claimed. □

The injectivity of the Exp provided by *Proposition* 1.4 gives the ability to introduce the inverse map (which we shall denote by Log) as a well-defined generalization of the complex logarithm:

$$\mathrm{Log}_n \quad : \quad \mathrm{Exp}_n(\mathbb{C}) \;\subseteq \mathbb{C} \setminus \{0\} \times \mathbb{R} \;\longrightarrow\; \mathbb{C}$$

where

$$\mathrm{Log}_n \;:=\; \mathrm{Exp}_n^{-1}\,, \qquad n \in \mathbb{N}$$

2.6 Lemma. *The inverse* Log_n *maps are given by*

$$(K, L) \;\longmapsto\; \mathrm{Log}_n(K,\ L) \;=\; \ln(|K|) \;+\; inL$$

(ln *denotes the real Napierian logarithm).*

Proof. The proof is a calculation. Set

$$\exp_{\mathbb{C}}(z) \;\equiv\; K \;\;, \qquad \frac{1}{n}Im(z) \;\equiv\; L$$

Then,

$$z \;=\; \log_{\mathbb{C}}(K) = \ln(|K|) + i\arg(K)$$

Comparing the last relation with the formula $\log_{\mathbb{C}}(K) = \ln(|Re(z)|) + inL$ we conclude to the requested relation. □

We are now in position to introduce a well-defined complex logarithm on the limit covering space produced by the exponential.

Notational convention: In the sequel we adopt the following convention: $\Sigma_{log}^{C^\infty}$ denotes Σ_{log} equipped with the induced smooth sub-atlas of the holomorphic one coming from $\mathbb{R}^4$:

$$\Sigma_{log}^{C^\infty} \;\equiv\; (\Sigma_{log},\, \mathcal{A}_{\Sigma_{log}}^{C^\infty}) \;\hookrightarrow\; (\Sigma_{log},\, \mathcal{A}_{\Sigma_{log}}^{\mathcal{O}}) \;\subseteq\; (\mathbb{R}^4, Id)$$

2.7 Corollary. ***(Convergence to a well-defined*** $\log_{\mathbb{C}}$ ***on its Riemann Surface)***

The sequence of maps $(\mathrm{Log}_n)_{n\in\mathbb{N}}$ *of 2.6 converges point-wise to a well-defined complex logarithm and uniformly on the subsets of* $\mathrm{Exp}_n(\mathbb{C}) \subseteq \mathbb{C}\smallsetminus\{0\}\times\mathbb{R}$ *of the form* $\mathrm{Exp}_n(\mathbb{C})\bigcap \mathbb{C}\smallsetminus\{0\}\times(a,b)$*:*

$$\mathrm{Log}_n \xrightarrow{n\longrightarrow\infty} (\exp_{\mathbb{C}} \times\{0\})^{-1} \;\equiv\; \log_{\mathbb{C}}|_{\exp_{\mathbb{C}}(\mathbb{C})}$$

and the limit surface is diffeomorphic to the (immersed into $\mathbb{C}\smallsetminus\{0\}$*) Riemann Surface of the logarithm:*

$$\pi_{\mathbb{C}}(\Sigma_{log}^{C^\infty}) \;\cong\; \varinjlim\,(\mathrm{Exp}_n(\mathbb{C})) \;=\; \exp_{\mathbb{C}}(\mathbb{C}) \;\equiv\; \mathbb{C}\smallsetminus\{0\}$$

Proof. Every set contained into $\mathrm{Exp}_n(\mathbb{C})$ (for any $n\in\mathbb{N}$) of this form with bounded height (that is, $\mathbb{R}$-component) is contained into a sub-helicoid $\mathrm{Exp}_n(B)\subseteq \mathrm{Exp}_n(\mathbb{C})$ of finite height, thus under its inversed image via Exp_n into a bounded strip B of $\mathbb{C}$, for some $M>0$.

The diffeomorphism comes from the previous *Th.* 2.2 and the construction of injective complex exponential on $\exp_{\mathbb{C}}(\mathbb{C})$, as well as the remarks in the beginning of the present *Section* concerning the covering space approach. We note that the covering space map coincides with the immersion map into $\mathbb{C}\smallsetminus\{0\}$. □

(Part II) The Realization Theorem.

We have not so far shown the explicit relation between the surfaces $\mathrm{Exp}_a(\mathbb{C})$ and Σ_{log}.

We recall that we manage to construct an injective $\exp_{\mathbb{C}}$ and thus a well-defined $\log_{\mathbb{C}}$ on $\mathrm{Exp}_a(\mathbb{C}) \subseteq \mathbb{R}^3$ via convergence of helicoids. Taking into consideration the initial *"defining characterization" of B. Riemann himself* on the surfaces of multi-valued holomorphic functions (as the one of $\log_{\mathbb{C}}$), *"surface onto which the function considered becomes single-valued"* (see also [18]), we conclude that $\mathrm{Exp}_a(\mathbb{C})$ and Σ_{log} must necessarily be different representations of the "same" object, but the former living into $\mathbb{R}^3$ instead of the latter which lives into $\mathbb{R}^4$.

This indeed turns to be the case, where "same" is expounded due to the dimensional reduction to a realizable level (of 3-dimensions) as follows:

2.8 Theorem. ***(The Realization of Σ_{log} into $\mathbb{R}^3$.)*** *The following diffeomorphism holds true*

$$\Xi \;:\; \mathrm{Exp}_a(\mathbb{C}) \xrightarrow{\;\cong\;} \Sigma_{log}^{C^\infty}$$

Thus, the (exponential) helicoid is C^∞- diffeomorphic to the Logarithmic Riemann surface, when the latter is equipped with the induced smooth sub-atlas $\mathcal{A}_{\Sigma_{log}}^{C^\infty}$ of the holomorphic $\mathcal{A}_{\Sigma_{log}}^{\mathcal{O}}$ coming from $\mathbb{R}^4$.

The diffeomorphism Ξ will be constructed in the proof. As a consequence, 2.8 implies that we can equip the helicoid surface $\mathrm{Exp}_a(\mathbb{C})$ with a holomorphic structure via the bijective correspondence of it with Σ_{log}, that is, if (U, ϕ) is a holomorphic chart of $\mathcal{A}_{\Sigma_{log}}^{\mathcal{O}}$, then define an atlas of $\mathrm{Exp}_a(\mathbb{C})$ as follows

$$\mathcal{A}_{\mathrm{Exp}_a(\mathbb{C})}^{\mathcal{O}} := \{(\Xi^{-1}(U), \Xi^{-1} \circ \phi) \;/\; (U, \phi) \in \mathcal{A}_{\Sigma_{log}}^{\mathcal{O}}\}$$

Of course, this holomorphic atlas of the helicoid does <u>not</u> coincide with its induced smooth atlas coming from the ambient 3- space.

Proof. $\mathrm{Exp}_a(\mathbb{C})$ is given as the following subset of $\mathbb{R}^3$:

$$\{(e^u \cos v, e^u \sin v, av) \;/\; u,\; v \in \mathbb{R}\}$$

and Σ_{log} as the subset of $\mathbb{R}^4$

$$\{(u, v, e^u \cos v, e^u \sin v) \;/\; u,\; v \in \mathbb{R}\}$$

Define a map Ξ as

$$\Xi \;:\; (e^u \cos v, e^u \sin v, av) \longmapsto (u, v, e^u \cos v, e^u \sin v)$$

Simple algebraic manipulations show that this map is a bijection with inverse the projection onto $\mathbb{R}^3$ composed with a rigid motion

$$\Omega \equiv \Xi^{-1} \;:\; (u, v, e^u \cos v, e^u \sin v) \longmapsto (e^u \cos v, e^u \sin v, av)$$

both maps can be easily seen to have smooth components and this completes the proof. □

3. Multi-dimensional Generalizations.

In this section we present the reasonable high-dimensional analogues of the results presented in the previous *Sections*. Notwithstanding, the new context suggests that the results will no longer stand in the region of Riemann Surfaces, but will be of a more general differential-geometric nature.
The following definitions arise naturally from the respective 1.1, 1.3 of the *Section 1.*

3.1 Definition. (Multi-Exponential field) The *(exponential) multi-helicoid* is (in complex coordinates) the map ($a_1, \ldots, a_m > 0$ parameters):

$$\mathrm{Exp}_{a_1,\ldots,a_m} \;:\quad \mathbb{C}^m \longrightarrow (\mathbb{C} \setminus \{0\})^m \times \mathbb{R}^m \quad :$$

defined as

$$(z_1, \ldots, z_m) \longmapsto (\exp_{\mathbb{C}}(z_1), \ldots, \exp_{\mathbb{C}}(z_m);\; a_1 Im(z_1), \ldots, a_m Im(z_m))$$

that is, (up to a natural identitification $\mathbb{C}^m \times \mathbb{R}^m \cong (\mathbb{C} \times \mathbb{R})^m$)

$$\mathrm{Exp}_{a_1,\ldots,a_m} := \prod_{1\leq k\leq m} \mathrm{Exp}_{a_k}$$

This is a vector field by 1.4 (used component-wise), defined on the (flat) trivially embedded submanifold $\mathbb{C}^m$ of $\mathbb{R}^{3m}$ and valued in (the tangent bundle of) the ambient $\mathbb{R}^{3m}$ by 1.2.

Using 1.5, $\mathrm{Exp}_{a_1,\ldots,a_m}(\mathbb{C}^m)$ is a globally coordinated smooth submanifold of (real) dimension $\dim_{\mathbb{R}}(\mathrm{Exp}_{a_1,\ldots,a_m}(\mathbb{C}^m)) = 2m$. In fact, a **multi-helicoid**, in complete analogy with 1.1.

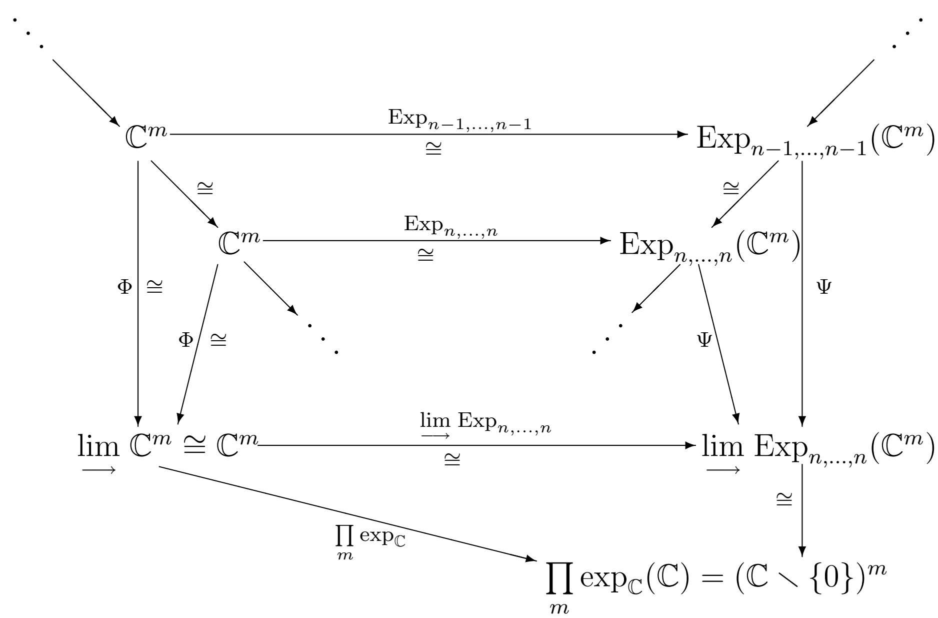
The proofs of the following results go without saying using the respective results of the previous sections, so we will not bother showing anything explicitly. A component-wise argument suffices in all cases, due to the product form of the definition 3.1.

If we set $a_1 = \cdots = a_m = 1/n$, we obtain a multi-sequence of vector fields and in turn of multi-helicoid submanifolds of $\mathbb{R}^{3m}$:

$$\mathrm{Exp}_{n,\ldots,n}(z_1,\ldots,z_m) \;:=\; (\exp_{\mathbb{C}}(z_1),\ldots,\exp_{\mathbb{C}}(z_m);\; \frac{1}{n}Im(z_1),\ldots,\frac{1}{n}Im(z_1))$$

We are now in position to present the analogous result of Theorem 2.2.

3.2 Theorem. ***(Convergence to Injective multi-Exponential)***
The sequence of exponential vector fields $(\mathrm{Exp}_{n,\ldots,n})_{n\in\mathbb{N}}$ *converges* *point-wise* *to a* C^∞*- diffeomorphism* $\prod_m \exp_{\mathbb{C}} \;:\; \mathbb{C}^m \longrightarrow (\mathbb{C}\setminus\{0\})^m$ *:*



The convergence is *uniform* *on the bounded multi-strips* $B^m \equiv \{z_k \in \mathbb{C} \;:\; |Im(z_k)| < M_k \;/\; M_k > 0,\; 1 \le k \le m\} \;\subseteq\; \mathbb{C}^m$.

$$\mathrm{Exp}_{n,\ldots,n}|_{B^m} \;\xrightarrow[U]{n\longrightarrow\infty}\; \prod_m \exp_{\mathbb{C}} \times\{0\}|_B \;\equiv\; \prod_m \exp_{\mathbb{C}}|_B$$

3.3 Remark. The methods we expounded for these specific construction raise the question if and how they may be extended and applied to other surfaces, in the framework of a general approach that would, at least, be in position to give back the already known facts concerning the classical Riemann surfaces, e.g. those of $\log_{\mathbb{C}}(z)$ and $\sqrt[n]{z}$ which, the latter, more or less constituted the beginnings of the subject of the study of holomorphic 1-manifolds.
Our problem is focused in the quest of an appropriate sequence of surfaces in $\mathbb{R}^3$ (if any), whereon the complex function in question (properly extended as a vector field) will become single valued, and the covering surface will be obtained in a uniform limit.

The above attractive concept might be of significant importance in the finite-sheeted Riemann surfaces of algebraic functions satisfying the general formula $a_n(z)[f(z)]^n + a_{n-1}(z)[f(z)]^{n-1} + \cdots + a_0(z) = 0$.

In a nutshell, recovering via differential geometry pure analytic information concerning complex functions through their Riemann surfaces would give a measure of the amount that (the latter) in fact depend on the holomorphic structure of the plane and, more generally, of every complex manifold modeled upon it.

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