

# A SPECIAL CASE OF SELBERG'S INTEGRAL

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ABSTRACT. We evaluate the integral

$$\int_0^\infty \cdots \int_0^\infty \prod_{1 \leq i < j \leq n} (x_j - x_i)^2 e^{-x_1} dx_1 \cdots e^{-x_n} dx_n$$

using orthogonal polynomials and techniques from linear algebra.

## 1. INTRODUCTION

The integral evaluated in this paper arose in the study of a class of invariant differential operators of a matrix argument. Briefly, these polynomials can be described by considering invariant polynomials on a cross section of diagonal matrices that is essentially  $\mathbb{R}^n$  ([BHR1] [BHR2]). The way in which the relevant measure is pushed down to this cross section can be determined by computing the integral

$$(1.1) \quad \int_0^\infty \cdots \int_0^\infty \prod_{1 \leq i < j \leq n} (x_j - x_i)^2 e^{-x_1} dx_1 \cdots e^{-x_n} dx_n.$$

Since the integrand is a polynomial, and using the fact that, for  $k \in \mathbb{N}$ , the Euler Gamma function satisfies

$$\Gamma(k + 1) = \int_0^\infty x^k e^{-x} dx = k!$$

we see that the value of this integral must be an integer that depends on  $n$ . We denote the value of the integral (1.1) by  $I(n)$ .

Somewhat later we discovered that the above integral is a special case of Selberg's integral ([AAR], hence the title of this paper) and can be evaluated using published formulae with  $\alpha = 1$  and  $\gamma = 1$ .

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Specifically,

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n x_i^{\alpha-1} \prod_{1 \leq i < j \leq n} |x_j - x_i|^{2\gamma} e^{-x_1} \cdots e^{-x_n} dx_1 \cdots dx_n$$

$$= \prod_{j=1}^n \left( \frac{\Gamma(\alpha + (j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(1+\gamma)} \right)$$

The direct computation of this integral demonstrates the use of algebraic techniques to answer a question that at first appears to be strictly analytic.

## 2. PRELIMINARIES

We first computed this integral for manageable values of  $n$  using a computer algebra system and obtained the following results:

n	$I(n)$
2	2
3	24
4	3465
5	9953280

After some consideration we realized that we could write these values as:

n	$I(n)$
2	$2!$
3	$3! (2!)^2$
4	$4! (3! 2!)^2$
5	$5! (4! 3! 2!)^2$

This led us to conjecture that, for  $n \in \mathbb{N}$

$$I(n) = n![(n-1)!(n-2)! \cdots (2)(1)]^2.$$

It was then suggested that we consider Laguerre polynomials. The Laguerre polynomial in the variable  $x$  of degree  $k$  can be defined using Rodrigues' formula by

$$L_k(x) = e^x \frac{d^k}{dx^k} (x^k e^{-x}).$$

Note that the leading term will be  $x^k$ . Direct computation using integration by parts shows that

$$\int_0^\infty L_k(x) L_m(x) e^{-x} dx = \begin{cases} (k!)^2 & k=m \\ 0 & k \neq m \end{cases}$$

so that the Laguerre polynomials are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)e^{-x} dx.$$

It is well known that the Laguerre polynomials are an orthogonal basis for the vector space of polynomials in  $x$ .

Note also that products of Laguerre polynomials in the variable  $x$  and Laguerre polynomials in the variable  $y$  will be a basis for the vector space of polynomials in the two variables  $x$  and  $y$ , orthogonal with respect to the inner product given by

$$\langle f, g \rangle = \int_0^\infty \int_0^\infty f(x, y) g(x, y) e^{-x} e^{-y} dx dy.$$

In particular, by Fubini's Theorem we have

$$\langle L_n(x)L_m(y), L_n(x)L_m(y) \rangle = \langle L_n(x), L_n(x) \rangle \langle L_m(y), L_m(y) \rangle = (n!)^2(m!)^2.$$

This idea extends to polynomials in  $n$  variables. The products of Laguerre polynomials in the variables  $x_1 \cdots x_n$  form a basis for the vector space of polynomials in these variables, orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_0^\infty \cdots \int_0^\infty f(x_1, \cdots, x_n)g(x_1, \cdots, x_n)e^{-x_1} \cdots e^{-x_n} dx_1 \cdots dx_n$$

with

$$\begin{aligned} & \langle L_{k_1}(x_1) \cdots L_{k_n}(x_n), L_{k_1}(x_1) \cdots L_{k_n}(x_n) \rangle \\ &= \langle L_{k_1}(x_1), L_{k_1}(x_1) \rangle \cdots \langle L_{k_n}(x_n), L_{k_n}(x_n) \rangle \\ &= (k_1!)^2 \cdots (k_n!)^2. \end{aligned}$$

Our first step will be to expand the integrand of (1.1) using Laguerre polynomials.

### 3. EXPANSION OF THE INTEGRAND USING LAGUERRE POLYNOMIALS

Making further investigations using a computer algebra system, we expanded  $\prod_{1 \leq i < j \leq n} (x_j - x_i)$  for  $n = 3$  and then subtracted off successive Laguerre polynomials in  $x_1$ ,  $x_2$ , and  $x_3$  by matching the highest order terms. For example, we have

$$(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) = x_2x_3^2 - x_2^2x_3 + x_2^2x_1 - x_1x_3^2 + x_1^2x_3 - x_1^2x_2.$$

Since the first term in the left hand side is  $x_2x_3^2$ , we first subtract  $L_1(x_2)L_2(x_3)$ . We then continue in this manner until the result is zero

(this is guaranteed to occur since the products of Laguerre polynomials form a basis) and obtain

$$\begin{aligned} (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) &= L_1(x_2)L_2(x_3) - L_1(x_3)L_2(x_2) \\ &\quad + L_1(x_1)L_2(x_2) - L_1(x_1)L_2(x_3) \\ &\quad + L_1(x_3)L_2(x_1) - L_1(x_2)L_2(x_1). \end{aligned}$$

This led us to conjecture the following formula:

$$(3.1) \quad \prod_{1 \leq i < j \leq n} (x_j - x_i) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)})$$

where  $S_n$  is the symmetric group of degree  $n$  and  $\text{sgn}(\sigma) = \pm 1$ , depending on whether the permutation  $\sigma$  is even or odd. We remark that the left hand side is the well known Vandermonde determinant.

To prove (3.1), we first show that the polynomial defined by

$$P(x_1, x_2, \dots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) L_2(x_{\sigma(3)}) \cdots L_{n-1}(x_{\sigma(n)}).$$

is alternating. That is, we show that the polynomial changes sign whenever two variables are interchanged. Let  $(i, j)$  be a transposition in  $S_n$  and let  $(i, j).P(x_1, x_2, \dots, x_n)$  denote the action of this transposition that interchanges the  $i^{\text{th}}$  and  $j^{\text{th}}$  variables in  $P(x_1, x_2, \dots, x_n)$ . Then we have

$$(i, j).P(x_1, x_2, \dots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) L_0(x_{(i,j)\sigma(1)}) L_1(x_{(i,j)\sigma(2)}) \cdots L_{n-1}(x_{(i,j)\sigma(n)}).$$

Now let  $\tau = (i, j)\sigma$ , so that  $\text{sgn}(\sigma) = -\text{sgn}(\tau)$ . Then we have

$$\begin{aligned} (i, j).P(x_1, x_2, \dots, x_n) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)}) \\ &= \sum_{\tau \in S_n} -\text{sgn}(\tau) L_0(x_{\tau(1)}) L_1(x_{\tau(2)}) \cdots L_{n-1}(x_{\tau(n)}) \\ &= -P(x_1, x_2, \dots, x_n). \end{aligned}$$

Therefore,  $P$  is alternating.

Now set  $P_n(x_n) = P(x_1, x_2, \dots, x_n)$ . That is, consider  $P_n$  as a polynomial in  $x_n$ , with the coefficients being polynomials in  $x_1, x_2, \dots, x_{n-1}$ . Since  $P$  is alternating, we have  $P_n(x_i) = 0$  for  $i \neq n$ . Thus, by the Fundamental Theorem of Algebra, each  $x_1, x_2, \dots, x_{n-1}$  is a root of

$P_n(x_n)$  and so for each  $i \neq n$ ,  $(x_n - x_i)$  is a factor of  $P_n(x_n)$ . Therefore

$$(3.2) \quad \begin{aligned} P_n(x_n) &= (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_1) A_{n-1}(x_1, \dots, x_{n-1}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) A_{n-1}(x_1, \dots, x_{n-1}) \end{aligned}$$

for some polynomial  $A_{n-1}(x_1, \dots, x_{n-1})$ . Note that  $A_{n-1}(x_1, \dots, x_{n-1})$  is the coefficient of  $x_n^{n-1}$  in  $P_n(x_n)$ .

Next we split the sum (3.1) over those elements of  $S_n$  which leave  $n$  fixed (a subgroup of  $S_n$  isomorphic to  $S_{n-1}$ ), and those which do not.

$$\begin{aligned} P_n(x_n) &= \sum_{\sigma(n)=n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)}) \\ &\quad + \sum_{\sigma(n) \neq n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)}) \\ &= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-2}(x_{\sigma(n-1)}) L_{n-1}(x_n) \\ &\quad + (\text{lower degree terms}) \\ &= L_{n-1}(x_n) \left[ \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-2}(x_{\sigma(n-1)}) \right] \\ &\quad + (\text{lower degree terms}). \end{aligned}$$

Equating the leading coefficient of  $x_n^{n-1}$  above with that in (3.2) we see that

$$\begin{aligned} A_{n-1}(x_1, x_2, \dots, x_{n-1}) &= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-2}(x_{\sigma(n-1)}) \\ &= P_{n-1}(x_{n-1}). \end{aligned}$$

Combining these results we have

$$\begin{aligned} P_n(x_n) &= (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_1) A_{n-1}(x_1, \dots, x_{n-1}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) A_{n-1}(x_1, \dots, x_{n-1}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) P_{n-1}(x_{n-1}). \end{aligned}$$

Now by the same argument used earlier, we have that  $x_1, x_2, \dots, x_{n-2}$  are roots of  $P_{n-1}(x_{n-1})$ , so

$$\begin{aligned} P_n(x_n) &= \prod_{i=1}^{n-1} (x_n - x_i) P_{n-1}(x_{n-1}) A_{n-2}(x_1, x_2, \dots, x_{n-2}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) \prod_{j=1}^{n-2} (x_{n-1} - x_j) A_{n-2}(x_1, x_2, \dots, x_{n-2}) \end{aligned}$$

where  $A_{n-2}$  is the coefficient of  $x_{n-1}^{n-2}$ .

Now, for  $n = 2$  we can show that

$$\sum_{\sigma \in S_2} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) = L_0(x_1) L_1(x_2) - L_0(x_2) L_1(x_1) = x_2 - x_1$$

and so the result follows by induction. Namely,

$$\prod_{0 \leq i < j \leq n} (x_j - x_i) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)})$$

as required.

#### 4. EVALUATING THE INTEGRAL AS AN INNER PRODUCT

Recall that we have an inner product on the vector space of polynomials on  $\mathbb{R}^n$  given by

$$\langle f, g \rangle = \int_0^\infty \cdots \int_0^\infty f(x_1, \dots, x_n) g(x_1, \dots, x_n) e^{-x_1} dx_1 \cdots e^{-x_n} dx_n.$$

Therefore we can treat our integral

$$(4.1) \quad I(n) = \int_0^\infty \cdots \int_0^\infty \prod_{1 \leq i < j \leq n} (x_j - x_i)^2 e^{-x_1} dx_1 \cdots e^{-x_n} dx_n$$

as the inner product

$$I(n) = \left\langle \prod_{1 \leq i < j \leq n} (x_j - x_i), \prod_{1 \leq i < j \leq n} (x_j - x_i) \right\rangle.$$

By (3.1), we have

$$\begin{aligned} I(n) &= \\ &\left\langle \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}), \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}) \right\rangle. \end{aligned}$$

But each of the terms in the above sums are mutually orthogonal, *i.e.*

$$\begin{aligned} & \langle L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}), L_0(x_{\tau(1)}) \cdots L_{n-1}(x_{\tau(n)}) \rangle \\ &= \begin{cases} (0!)^2(1!)^2(2!)^2 \cdots (n-1!)^2 & \sigma = \tau, \\ 0 & \sigma \neq \tau. \end{cases} \end{aligned}$$

Evaluating our integral in this context yields the required result:

$$\begin{aligned} I(n) &= \\ & \left\langle \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}), \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}) \right\rangle \\ &= \sum_{\sigma \in S_n} \langle \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}), \operatorname{sgn}(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}) \rangle \\ &= \sum_{\sigma \in S_n} (0!)^2(1!)^2(2!)^2 \cdots (n-1!)^2 \\ &= n![(n-1)! \cdots (2)!(1)!]^2. \end{aligned}$$

#### REFERENCES

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