An identity of derangements

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Abstract

In this note, we present a new identity for derangements. As a corrolary, we have a combinatorial proof of the irreducibility of the standard representation of symmetric groups.

1 Introduction

A derangement is the permutation σ of $\{1, 2, \ldots, n\}$ that there is no i satisfying $\sigma(i) = i$. It is well-known that the number $d(n)$ of dereangements equals:

$$d(n) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

and satisfies the following identity (since both sides are the number of permutations on n letters)

$$\sum_{k=0}^{n} \binom{n}{k} d(k) = n!. \tag{1}$$

The Stirling set number $S(n, m)$ is the number of ways of partitioning a set of n elements into m nonempty sets. We definte $[x]_r = x(x-1)\ldots(x-r+1)$ (by convention $[x]_0 = 1$). Then (see [4])

$$x^n = \sum_{m=0}^{n} S(n, m)[x]_r. \tag{2}$$

The number of ways a set of n elements can be partitioned into nonempty subsets is called a Bell number and is denoted B_n. We use the convention that $B_0 = 1$. The integer B_n can also be define by the sum (see [3])

$$B_n = \sum_{m=0}^{n} S(n, m) \tag{3}$$

The main result of this note is the following generalization of (1).

Theorem 1 *Suppose that $n \geq m$ are two natural numbers. Then*

$$\sum_{k=0}^{n} k^m \binom{n}{k} d(n-k) = B_m n!. \tag{4}$$

We use the convention that $\binom{n}{m} = 0$ if $m < 0$ or $n < m$. Also set $d(k) = 0$ if $k < 0$ and $d(0) = 1$. Note that taking $m = 0$ in (4) implies (1) since $B_0 = 1$. Furthermore, by linearity we have the following corollary.

Corollary 1 *Suppose that $n \geq m$ are two natural numbers. Let $g(x) = a_m x^m + \ldots + a_0$ be a polynomial with integer coefficients. Then*

$$\sum_{k=0}^{n} g(k) \binom{n}{k} d(n-k) = \left\{ \sum_{i=0}^{m} a_i B_i \right\} n!. \tag{5}$$

2 Some Lemmas

We define $f_n(k)$ to be the number of permutations of $\{1, \ldots, n\}$ that fix exactly k positions. By convention, $f_n(k) = 0$ if $k < 0$ or $k > n$. We have the following recursion for $f_n(k)$.

Lemma 1 *Suppose that n, k are positive integers. Then*

$$f_{n+1}(k) = f_n(k-1) + (n-k) f_n(k) + (k+1) f_n(k+1).$$

Proof Let σ be any permutation of $\{1, \ldots, n+1\}$ which has exactly k fixed points. We have two cases.

1. Suppose that $\sigma(n+1) = n+1$. Then σ corresponds to a restricted permutation on $\{1, \ldots, n\}$ which fixes $k-1$ points of $\{1, \ldots, n\}$. This case applies to the first term in the statement of the lemma.

2. Suppose that $\sigma(n+1) = i$ for some $i \in \{1, \ldots, n\}$. Then there exists $j \in \{1, \ldots, n\}$ such that $\sigma(j) = n+1$. There are two separate subcases.

   (a) If $i = j$ then we can obtain a correspondence between σ and a permutation σ' of $\{1, \ldots, n\}$ from σ as follows: $\sigma'(i) = i$ and $\sigma'(t) = \sigma(t)$ for $t \neq i$. It is clear that σ' has $k+1$ fixed points. Conversely, for each permutation of $\{1, \ldots, n\}$ that has $k+1$ fixed points, we can choose i to be any of its fixed points and then swapping i and $n+1$ to have a permutation of $\{1, \ldots, n+1\}$ that has k fixed points. This case applies to the third term in the statement of the lemma.

   (b) If $i \neq j$ then we can obtain a correspondence between σ and a permutation σ' of $\{1, \ldots, n\}$ from σ as follows: $\sigma'(j) = i$ and $\sigma'(t) = \sigma(t)$ for $t \neq j$. It is clear that σ' has k fixed points. Conversely, for each permutation σ' of $\{1, \ldots, n\}$ that has k fixed points, we can choose any j such that $\sigma'(j) = i \neq j$, and get back a permutation σ of $\{1, \ldots, n+1\}$ that has k fixed points by letting $\sigma(t) = \sigma'(t)$ for $t \neq j, n+1, \sigma(j) = n+1$ and $\sigma(n+1) = \sigma'(j) = i$. This case applies to the second term in the statement of the lemma.

Hence $f_{n+1}(k) = f_n(k-1) + (n-k)f_n(k) + (k+1)f_n(k+1)$ for all n, k. This concludes the proof. □

Lemma 1 can be applied to obtain the following identity for $f_n(k)$ (Note that $f_n(k) = 0$ whenever $k < 0$ or $k > n$).

Lemma 2 *Suppose that n, k, t are integers, $t \geq -1$. Then*

$$\sum_{k=0}^{n} [k]_{t+1} f_n(k) = \begin{cases} n! & \text{if } n \geqslant t+1, \\ 0 & \text{otherwise} \end{cases}$$

Proof We prove this using a double induction. The outer induction is on t and the inner one is on n. By convention, $[k]_0 = 1$. Also we have $\sum_k f_n(k) = n!$ which is trivial from the definition of $f_n(k)$. Hence the claim holds for $t = -1$. Next, suppose that the claim holds for $t-1$. We prove that it holds for t. Define

$$F(n,t) := \sum_{k=0}^{n} [k]_{t+1} f_n(k) = \sum_{k=0}^{n} k(k-1)\dots(k-t) f_n(k).$$

Suppose that $n \leq t$. If $f_n(k) \neq 0$ then $0 \leqslant k \leqslant n \leqslant t$. But this implies that $k(k-1)\dots(k-t) = 0$. Hence $F(n,t) = 0$ if $n \leqslant t$.

Suppose that $n = t+1$. Then

$$F(n,t) = \sum_{k=0}^{n} k(k-1)\dots(k-(n-1)) f_n(k) = n! f_n(n) = n!$$

since all but the last term of the sum equal zero. Hence the claim holds for $n = t+1$. For the inner induction, suppose that $F(n,t) = n!$ for some $n \geq t+1$. We will show that $F(n+1,t) = (n+1)!$. From Lemma 1, we have

$$\begin{aligned} F(n+1,t) &= \sum_{k=0}^{n+1} k(k-1)\dots(k-t) f_{n+1}(k) \\ &= \sum_{k=0}^{n+1} k(k-1)\dots(k-t) \left[f_n(k-1) + (n-k)f_n(k) + (k+1)f_n(k+1)\right] \\ &= nF(n,t) + \sum_{k=0}^{n+1} k(k-1)\dots(k-t) \left[f_n(k-1) - kf_n(k) + (k+1)f_n(k+1)\right] \end{aligned}$$

Since $f_n(-1) = 0$, we have

$$\sum_{k=0}^{n+1} k(k-1)\dots(k-t) f_n(k-1) = \sum_{k=0}^{n} (k+1)k\dots(k-t+1) f_n(k).$$

Similarly $f_n(n+1) = 0$ implies that

$$\sum_{k=0}^{n+1} k(k-1)\dots(k-t)kf_n(k) = \sum_{k=0}^{n} k(k-1)\dots(k-t)kf_n(k).$$

And $f_n(n+1) = f_n(n+2) = 0$ implies that

$$\sum_{k=0}^{n+1} k(k-1)\dots(k-t)(k+1)f_n(k+1) = \sum_{k=0}^{n}(k-1)\dots(k-t-1)kf_n(k).$$

Therefore, we have

$$\begin{aligned} F(n+1,t) &= nF(n,t) \\ &\quad + \sum_{k=0}^{n} k(k-1)\dots(k-t+1)[(k+1)-(k-t)k+(k-t)(k-t-1)]f_n(k) \\ &= nF(n,t) + \sum_{k=0}^{n} k(k-1)\dots(k-t+1)[(k+1)-(k-t)(t+1)]f_n(k) \\ &= nF(n,t) + \sum_{k=0}^{n} k(k-1)\dots(k-t+1)[(t+1)-(k-t)t]f_n(k) \\ &= nF(n,t) + (t+1)F(n,t-1) - tF(n,t) \\ &= nn! + (t+1)n! - tn! \qquad (6) \\ &= (n+1)!. \end{aligned}$$

To see (6), note that the claim is true for $t-1$ by the outer induction. So $F(n,t-1) = n!$. Also $F(n,t) = n!$ by the inner inductive hypothesis. Hence the claim holds for $n+1$. Therefore, it holds for every n, t. This concludes the proof of the lemma. □

3 Proof of Theorem 1

Suppose that $n \geq m$ are two natural numbers. From (2), we have

$$\begin{aligned} \sum_{k=0}^{n} k^m f_n(k) &= \sum_{k=0}^{n}\sum_{j=0}^{m} S(m,j)[k]_j f_n(k) \\ &= \sum_{j=0}^{m} S(m,j)\left(\sum_{k=0}^{n}[k]_j f_n(k)\right) \\ &= \sum_{j=0}^{m} S(m,j)F(n,j-1). \qquad (7) \end{aligned}$$

From Lemma 2, $F(n, j-1) = n!$ for all $0 \leq j \leq n$. Aslo, from (3) $B_m = \sum_{j=0}^{m} S(m,j)$. Thus, (7) implies that

$$\sum_{k=0}^{n} k^m f_n(k) = \sum_{j=0}^{m} S(m,j) n! \\ = B_m n!. \tag{8}$$

To have a permutation with exactly k fixed points, we can first choose k fixed points in $\binom{n}{k}$ ways. Then for each set of k fixed points, we have $d(n-k)$ ways to arrange the $n-k$ remaining numbers such that we have no more fixed points. Hence

$$f_n(k) = \binom{n}{k} d(n-k). \tag{9}$$

Substituting (9) into (8), we obtain (4). This concludes the proof of the theorem.

4 An application

In this section, we will apply Theorem 1 to prove the irreducibility of the standard representation of symmetric groups. Let $G = S_n$ be the symmetric group on $X = \{1, \ldots, n\}$. Let $\mathbb{C}$ denote the complex numbers. Let $GL(d)$ stand for the group of all $d \times d$ complex matrices that are invertible with respect to multiplication.

Definition 1 *A matrix representation of a group G is a group homomorphism*

$$\rho : G \to GL(d).$$

Equivalently, to each $g \in G$ is assigned $\rho(g) \in GL(d)$ such that

1. *$\rho(1) = I$, the identity matrix,*
2. *$\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$.*

The parameter d is called the *degree* or *dimension* of the representation and is denoted by $\deg(\rho)$. All groups have the trivial representation of degree 1 which sends every $g \in G$ to the matrix (1). We denote the trivial representation by 1. An important representation of the symmetric group S_n is the permutation representation π, which is of degree n. If $\delta \in S_n$ then we let $\pi(\delta) = (r_{i,j})_{n \times n}$ where

$$r_{i,j} = \begin{cases} 1 & \text{if } \delta(j) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2 *Let G be a finite group and let ρ be a matrix representation of G. Then the character of ρ is*

$$\chi_\rho(g) = tr\ \rho(g),$$

where tr denotes the trace of a matrix.

It is clear from Definition 2 that if $\delta \in S_n$ then

$$\chi_1(\delta) = 1,$$
$$\chi_\pi(\delta) = \text{number of fixed points of } \delta.$$

Definition 3 *Let χ and ϕ be characters of a finite group G. Then*

$$\langle \chi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\phi(g^{-1}).$$

A matrix representation ρ of a group is called irreducible if $\langle \chi_\rho, \chi_\rho \rangle = 1$. Maschke's Theorem (see [2, 5]) states that every representation of a finite group having positive dimension can be written as a direct sum of irreducible representations. The permutation representation π can be written as a direct sum of the trivial representation 1 and another representation σ. The representation σ is called the *standard representation* of S_n. We have $\chi_\pi = \chi_1 + \chi_\sigma$ since for any $\delta \in S_n$ then $\pi(\delta) = 1(\delta) \oplus \sigma(\delta)$. Thus, for all $\delta \in S_n$ then

$$\chi_\sigma(\delta) = (\text{number of fixed points of } \delta) - 1.$$

Now we want to prove that σ is irreducible. In other words, we need to show $\langle \chi_\sigma, \chi_\sigma \rangle = 1$, which is equivalent to

$$\sum_{k=0}^{n} (k-1)^2 f_n(k) = n! \tag{10}$$

Identity (10) can be obtained easily from Corollary 1 as follows.

$$\begin{aligned} \sum_{k=0}^{n} (k-1)^2 f_n(k) &= \sum_{k=0}^{n} (k^2 - 2k + 1) f_n(k) \\ &= (2 - 2 + 1)n! = n! \end{aligned}$$

since $B_0 = B_1 = 1$ and $B_2 = 2$. This implies the irreducibility of standard representation of symmetric groups.

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