A NEW COMPUTATION OF THE CODIMENSION SEQUENCE OF THE GRASSMANN ALGEBRA

JOEL LOUWSMA, ADILSON EDUARDO PRESOTO, AND ALAN TARR

ABSTRACT. Krakowski and Regev compute the codimension sequence of the T-ideal of polynomial identities of the Grassmann algebra using polynomial relations. We give an elementary proof of their main result and obtain some corollaries.

INTRODUCTION

The Grassmann algebra is an extremely important algebraic structure that arises in linear algebra and geometry. It has applications in many areas of mathematics as well as theoretical physics, and provides methods of understanding many topics in geometry, algebra, and analysis. The notion of Grassmann algebra is a natural generalization of that of commutative ring, and therefore the Grassmann algebra is sometimes called a "supercommutative algebra." The Grassmann algebra is the main tool in the study of superalgebras ($\mathbf{Z}_2$-graded algebras). It was also used by Kemer to obtain important results in PI theory. Thus results about the Grassmann algebra are of definite importance, and further study of them is justified.

Krakowski and Regev [4] found a basis of polynomial identities satisfied by the Grassmann algebra over a field of characteristic 0. At this time, these identities are known only for 2×2 matrices over a field of characteristic other than 2, for the Grassmann algebra over any field, for the tensor product of two Grassmann algebras in characteristic 0, and for upper-triangular matricies of any size over any field (*c.f.* [1]). Furthermore, Krakowski and Regev described the exact structure of these relations for the Grassmann algebra in terms of the symmetric group, thus giving a complete description of the multilinear structure of the polynomial relations of the Grassmann algebra. Hence they reduced the problem of determining the codimension sequence of T-ideals of polynomial identities to the computation of the rank of a specific matrix, allowing it to be solved by combinatorics and linear algebra. In this paper, we give a proof of this result of Krakowski and

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Regev without dealing with the polynomial identites since they are not needed in the statement or solution of this problem. We introduce the matricies used by Krakowski and Regev and compute their ranks, thus obtaining an elementary proof of the above result.

1. Preliminaries

Let S_n be the symmetric group on the set $\{1, 2, \ldots, n\}$. The image of a permutation $\sigma \in S_n$ is the ordered set $(\sigma(1), \sigma(2), \ldots, \sigma(n))$. We will use the notation $(i_1, \ldots, i_n)$ to denote the permutation $\begin{pmatrix} 1 & \ldots & n \\ i_1 & \ldots & i_n \end{pmatrix}$, *i.e.*, we write only the image of the elements under the permutation. For any permutation σ, define $s(\sigma)$ to be the sign of σ: 1 if σ is an even permutation and -1 if σ is an odd permutation. One simple way to compute the sign of a permutation is to count the number of inversions, *i.e.*, pairs that appear out of order, with a smaller number after a larger number. If σ has p inversions then $s(\sigma) = (-1)^p$. In the permutation $(3, 2, 5, 1, 4)$, there are five inversions:

$$(3, 2), (3, 1), (2, 1), (5, 1), (5, 4).$$

Thus $s((3, 2, 5, 1, 4)) = (-1)^5 = -1$.

Given a subset $\Omega \subseteq \{1, 2, \ldots, n\}$, we define σ_Ω to be the permutation of Ω induced by this ordering of $\{1, 2, \ldots, n\}$. In other words, σ_Ω is σ with elements not in Ω deleted. For example,

$$(3, 2, 5, 1, 4)_{\{2,4,5\}} = (2, 5, 4).$$

Now let $\sigma = (i_1, \ldots, i_{m-1}, i_m, i_{m+1}, \ldots, i_n)$. Define

$$(\sigma, n + 1) = (i_1, \ldots, i_n, n + 1) \in S_{n+1}$$

to be σ with $n + 1$ added to the end. Also, define

$$\sigma - i_m = (i_1, \ldots, i_{m-1}, i_{m+1}, \ldots, i_n) \in S_\Omega,$$

where $\Omega = \{1, \ldots, n\} \backslash \{i_m\}$. Thus $\sigma - k$ is σ with k deleted.

Definition 1. *For $n \in \mathbf{N}$, let H_n be a $2^n \times n!$ matrix with rows enumerated by the subsets of $\{1, 2, \ldots, n\}$ and columns enumerated by the elements of S_n, where the entry in the Ω^{th} row and the σ^{th} column is $s(\sigma_\Omega)$.*

Example. *We compute H_2 and H_3:*

$$H_2 = \begin{array}{c} \\ \varnothing \\ \{1\} \\ \{2\} \\ \{1,2\} \end{array} \begin{array}{cc} (1,2) & (2,1) \\ \end{array} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H_3 = \begin{array}{c} \\ \varnothing \\ \{1\} \\ \{2\} \\ \{1,2\} \\ \{3\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \end{array} \begin{pmatrix} \rotatebox{90}{(1,2,3)} & \rotatebox{90}{(2,1,3)} & \rotatebox{90}{(1,3,2)} & \rotatebox{90}{(2,3,1)} & \rotatebox{90}{(3,1,2)} & \rotatebox{90}{(3,2,1)} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Unless otherwise specified, the ordering of the rows and columns of H_n is built inductively from the orderings of H_{n-1}. We order the rows of H_n by enumerating the first 2^{n-1} rows by the same subsets in the same order as those used for H_{n-1} and enumerating the last 2^{n-1} rows with these same subsets in the same order with the element n added. The columns are ordered inductively as well, such that the i^{th} section of $(n-1)!$ permutations (*i.e.*, those labeling the $(i-1)(n-1)!+1^{th}$ through $i(n-1)!^{th}$ columns) in H_n has n in the $(n-i+1)^{th}$ position, and the other $n-1$ elements are ordered as in the $(n-1)!$ columns of H_{n-1}. For example, the permutations of four elements are ordered as follows:

$$\begin{aligned} &(1,2,3,4),(2,1,3,4),\ (1,3,2,4),(2,3,1,4),\ (3,1,2,4),(3,2,1,4),\\ &(1,2,4,3),(2,1,4,3),\ (1,3,4,2),(2,3,4,1),\ (3,1,4,2),(3,2,4,1),\\ &(1,4,2,3),(2,4,1,3),\ (1,4,3,2),(2,4,3,1),\ (3,4,1,2),(3,4,2,1),\\ &(4,1,2,3),(4,2,1,3),\ (4,1,3,2),(4,2,3,1),\ (4,3,1,2),(4,3,2,1). \end{aligned}$$

Definition 2. *We define G_n to be the submatrix of H_n that consists of those columns enumerated by even permutations.*

Note that G_n consists of exactly those columns of H_n with a 1 in the last row.

2. Some Lemmas

Using the above orderings, H_n has a particular structure, which we explore in the following two lemmas.

Lemma 1.

$$H_n = \left(\begin{array}{c|c|c|c|c}
H_{n-1} & \multicolumn{3}{c|}{\cdots} & H_{n-1} \\
\hline
H_{n-1} & \begin{array}{c} H_{n-2} \\ \hline -H_{n-2} \end{array} & A_1 & \cdots & A_{n-2}
\end{array}\right)$$

Proof. The first 2^{n-1} rows of H_n are enumerated by all subsets of $\{1, \ldots , n-1\}$. Also, by construction, the i^{th} block of $(n-1)!$ columns in H_n consists of all permutations of $\{1, \ldots , n\}$ with n in the $(n-i+1)^{th}$ position. Thus, the top half of the i^{th} block of $(n-1)!$ columns is H_{n-1}, so the top half of H_n consists of n copies of H_{n-1}.

In the bottom half of H_n, the first block of $(n-1)!$ columns consists of all permutations of $\{1, \ldots , n-1\}$ with n in the last position. Since n is the largest element of $\{1, \ldots , n\}$, mapping it to itself does not create any additional inversions. This means that for all $\sigma \in S_{n-1}$, $s(\sigma) = s(\sigma, n)$. Thus, the lower-left submatrix of H_n is the same as the upper-left submatrix.

Next consider the section of the matrix enumerated by the subsets containing n but not $n-1$ and all permutations of the form $(\sigma, n, n-1)$ for $\sigma \in S_{n-2}$. Since these subsets do not contain $n-1$, we may ignore $n-1$ in these permutations, and since n is then the largest element and in the last position of these permutations, it may also be ignored. This block is therefore equivalent to a matrix with rows enumerated by all subsets of $\{1, \ldots , n-2\}$ and columns enumerated by permutations of $\{1, \ldots , n-2\}$, which is precisely H_{n-2}.

The submatrix below this, *i.e.*, the one defined by the same columns but with subsets containing both n and $n-1$, is the same except that $n-1$ is added to every subset. In each of these permutations, $n-1$ is inverted with exactly one element, namely n, so adding $n-1$ to the subsets creates exactly one additional inversion, which switches the sign of each entry. Thus, this block is $-H_{n-2}$. □

We have shown that in the bottom half of H_n, the first $(n-1)!$ columns are an H_{n-1} block and the next $(n-2)!$ columns are a block of H_{n-2} and $-H_{n-2}$. This leaves

$$\begin{aligned}
n! - (n-1)! - (n-2)! &= (n-1)(n-1)! - (n-2)! \\
&= ((n-1)^2 - 1)(n-2)! \\
&= (n^2 - 2n)(n-2)! \\
&= (n-2)(n(n-2)!)
\end{aligned}$$

columns remaining in the bottom half , which are filled by $n-2$ matrices of size $2^{n-1} \times n(n-2)!$.

The rows of A_i are enumerated by all subsets of $\{1, \ldots, n\}$ which contain n. We may view the columns of H_n as n sections of permutations each with $n-1$ subsections, such that the i^{th} subsection of the j^{th} section consists of all $(n-2)!$ permutations σ with $n-1$ in the $(n-i)^{th}$ position of $\sigma - n$ and n in the $(n-j+1)^{th}$ position of σ. For example, the second subsection of the second section of the columns of H_4 consists of all permutations σ of $\{1,2,3,4\}$ such that 3 is in second position of $\sigma - 3$ and 4 is in the third position of σ. In this case, there are two such permutations: $(1,3,4,2)$ and $(2,3,4,1)$.

In this way we see that the first $(n-2)!$ columns of A_i are the $(i+1)^{th}$ subsection of the $(i+1)^{th}$ section. Therefore in the first $(n-2)!$ columns of A_i, $n-1$ and n are next to each other in the $(n-i-1)^{th}$ position and $(n-i)^{th}$ position respectively.

As an example, look at the ordered list of permutations of $\{1,2,3,4\}$, written earlier as 4 sections each with 3 subsections, and see that each subsection has $(4-2)! = 2$ permutations. Now look at the second subsection of the second section, which will be the first two columns of A_1, and note that they are indeed the only two permutations for which 3 is in the $(4-1-1)^{th}$ or second position and 4 is in the $(4-1)^{th}$ or third position.

By the preceding lemma, we may let B_i be $2^{n-2} \times (n-2)!$ matrices such that H_{n-1} in terms of submatrices of this size looks like

$$H_{n-1} = \left(\begin{array}{c|c|c|c} H_{n-2} & H_{n-2} & \ldots & H_{n-2} \\ \hline H_{n-2} & B_1 & \ldots & B_{n-2} \end{array} \right)$$

Lemma 2. *In terms of these B_i, the structure of A_i is*

$$A_i = \left(\begin{array}{c|c} B_i & \\ \hline H_{n-2} & \end{array} \quad * \quad \right)$$

Proof. The top half of the first $(n-2)!$ columns of A_i has rows enumerated by all 2^{n-2} subsets that contain n but do not contain $n-1$ and columns enumerated by all permutations of $\{1, \ldots, n\}$ with $n-1$ and n next to each other in the $(n-i-1)^{th}$ position and $(n-i)^{th}$ position. So if we delete $n-1$ from these permutations, they simply become all permutations of $\{1, \ldots, n-2, n\}$ such that n is in the $(n-i-1)^{th}$ position. The subsets are all subsets of $\{1, \ldots, n-2, n\}$ containing n. In this set, n acts the same as $n-1$ does in the set $\{1, \ldots, n-1\}$ with respect to the sign function, and therefore this block is identical to B_i, which consists of all subsets containing $n-1$ and all permutations with $n-1$ in the $(n-i-1)^{th}$ position.

Finally, the bottom half of the first $(n-2)!$ columns of A_i is enumerated all subsets containing both $n-1$ and n and the same permutations as above. Recalling that $n-1$ and n are adjacent and not inverted in these permutations, we see that they are both involved in the same number of inversions for any permutation restricted to any subset in this block. Indeed, they are the second largest and largest elements in any subset, so if the permutation is $(\ldots, n-1, n, i_1, i_2, \ldots, i_m)$, then $n-1$ and n are each in exactly m inversions. Therefore, for all permutations σ and subsets Ω in the block in question, we have

$$s(((\sigma_\Omega) - (n-1)) - n) = s(\sigma_\Omega)(-1)^{2m} = s(\sigma_\Omega).$$

In particular, if we simply delete $n-1$ and n from all subsets and permutations in this block, we will not change any entries. This leaves rows that are enumerated by all subsets of $\{1, \ldots, n-2\}$ and columns that are enumerated by all permutations of $\{1, \ldots, n-2\}$, which is exactly H_{n-2}. □

There are some other orderings of the rows and columns that are useful. We introduce these orderings and use them in the final lemma dealing with the structure of H_n.

Definition 3. *Let $k \in \{1, 2, \ldots, n\}$. Let $H_n^{(k)}$ denote H_n with the rows and columns of H_n ordered by the inductive procedure, but adding elements in the order $[1, \ldots, k-1, k+1, \ldots, n, k]$ instead of the usual order.*

Lemma 3. *The top half of $H_n^{(k)}$ consists of n copies of H_{n-1}. In particular the rank of the top half of H_n in this ordering is equal to the rank of H_{n-1}.*

Proof. None of the subsets in the top half of H_n contain k, so we may delete k from all permutations when computing this submatrix. Each n^{th} of the columns now runs through all permutations of the set $\Omega = \{1, \ldots, k-1, k+1, \ldots, n\}$, while the rows are all of the subsets of Ω. The resulting matrix is exactly H_{n-1}, calculated using Ω instead of $\{1, \ldots, n-1\}$. The last statement of the lemma follows since $rank([M|M|\cdots|M]) = rank(M)$ for any matrix M where $[M|M|\cdots|M]$ is some number of copies of M put together in one matrix. □

When $k = n$, this lemma simply restates part of Lemma 1, and notes that this implies that the top 2^{n-1} rows of H_n have the same rank as H_{n-1}. But for other values of k, this lemma implies the same fact for certain other submatrices consisting of half of the rows of H_n. When the rows of H_n are in the usual order, the rows of subsets not containing k are the first 2^{k-1} rows of every block of 2^k rows.

Example. *In H_3, when $k = 2$, this lemma implies that the columns of the first $2^{k-1} = 2$ rows of each section of $2^k = 4$ rows can be rearranged to give three copies of H_2. When $k = 1$, we obtain that rows 1, 3,* *5, and 7 together as a submatrix can also be rearranged to give three copies of* H_2*.*

Definition 4. *Let* $\sigma = (i_1, \ldots, i_n)$*. Define* $d_\sigma : \{1, \ldots, n\} \to \{-1, +1\}$ *by* $d_\sigma(k) = (-1)^{\sigma(k)-k} = (-1)^{|\sigma(k)-k|}$*. Thus,* d_σ *determines whether an element is displaced an even or odd number of places by the permutation* σ*.*

Definition 5. *Define the permutation* $(\sigma - k)^{(k)}$ *to be* $\sigma - k$ *with* k *put back in its ordinary position. For example,*

$$((1,3,2,5,4) - 3)^{(3)}) = (1,2,3,5,4).$$

Lemma 4. *Let* $\sigma \in S_n$ *with* n *odd. Then*

$$s(\sigma) = \sum_{k=1}^{n} (-1)^{k-1} s(\sigma - k).$$

Proof. Letting $k = i_j$, we have

$$\begin{aligned} s(\sigma - k) \cdot d_\sigma(k) &= s(1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n) d_\sigma(k) \\ &= s((\sigma - k)^{(k)}) d_\sigma(k). \end{aligned}$$

This step uses the fact that the sign of any permutation on a set is the same as the sign of a permutation on a larger set that moves the elements of the smaller set in the same way and leaves the others fixed.

Let $\iota_1 \circ \ldots \circ \iota_{|\sigma(k)-k|}$ be the composition of $|\sigma(k) - k|$ transpositions that move k from k to $\sigma(k)$ in σ. That is, each ι_i moves k one more place from its natural position in $(\sigma - k)^{(k)}$ to its position $\sigma(k)$ in σ, so that

$$\iota_1 \circ \ldots \circ \iota_{|\sigma(k)-k|} \circ (\sigma - k)^{(k)} = \sigma.$$

Thus

$$\begin{aligned} s(\sigma) &= s(\iota_1 \circ \ldots \circ \iota_{|\sigma(k)-k|} \circ (\sigma - k)^{(k)}) \\ &= s(\iota_1) \cdot \ldots \cdot s(\iota_{|\sigma(k)-k|}) \cdot s((\sigma - k)^{(k)}) \\ &= (-1)^{|\sigma(k)-k|} s((\sigma - k)^{(k)}) \\ &= d_\sigma(k) s(\sigma - k), \end{aligned}$$

and therefore we have

$$\begin{aligned} \sum_{k=1}^{n} (-1)^{k-1} s(\sigma - k) &= \sum_{k=1}^{n} (-1)^{k-1} d_\sigma(k) s(\sigma) \\ &= \left(\sum_{k=1}^{n} (-1)^{k-1} d_\sigma(k) \right) s(\sigma). \end{aligned}$$

To obtain the desired result, we now need only to show that

$$\sum_{k=1}^{n} (-1)^{k-1} d_\sigma(k) = 1$$

for all permutations of odd length. Let $I \in S_n$ be the trivial permutation for odd n. Clearly $d_I(k) = (-1)^0 = 1$ for all $k \in \{1, \dots, n\}$. Thus

$$\begin{aligned}\sum_{k=1}^{n} (-1)^{k-1} d_I(k) &= \sum_{k=1}^{n} (-1)^{k-1} \\ &= 1 - 1 + \dots + 1 \\ &= 1 + 0 + \dots + 0 \\ &= 1.\end{aligned}$$

Now we prove inductively that this relation is true for all permutations on a set of odd size. We assume it is true for some permutation $\sigma = (i_1, \dots, i_n)$ and show that it is therefore also true for $\tau \circ \sigma$, where τ is any transposition. Let x and y be the two elements transposed by τ. Note that $d_{\tau\circ\sigma} = d_\sigma$ for all elements except x and y, so we need only show that the part of the sum involving x and y stays the same after applying τ, *i.e.*,

(1)
$$(-1)^{x-1} d_{\tau\circ\sigma}(x) + (-1)^{y-1} d_{\tau\circ\sigma}(y) = (-1)^{x-1} d_\sigma(x) + (-1)^{y-1} d_\sigma(y).$$

Since x and y are switched under τ, they move the same number of places, but in opposite directions. There are two cases to consider: $d_\tau(x) = d_\tau(y)$ is either 1 or -1.

(I) Suppose x and y both move j places, with j even. Then

$$d_{\tau\circ\sigma}(x) = (-1)^j d_\sigma(x) = d_\sigma(x),$$

and similarly for y. Thus the two terms on the left of (1) are the same as the two on the right, so the relation holds.

(II) Instead suppose that x and y move j places with j odd. Then both terms on the left side of (1) change sign, so we show that these terms initially had opposite signs so that both sides of (1) are zero. There are two subcases:

(a) If x and y are both odd or both even, say both even, then $(-1)^{x-1} = (-1)^{y-1}$. Thus we must show that $d_\sigma(x) = -d_\sigma(y)$. In I, x and y must have been an even number of spaces apart since they are both even numbers, but τ moved them an odd number of places, so that in σ they must be an odd number of places apart. The only way this can occur is if one of x and y is displaced an odd amount by σ

and the other is displaced an even amount, which means exactly that $d_\sigma(x) = -d_\sigma(y)$, as desired.

(b) Without loss of generality, say x is odd and y is even. Then $(-1)^{x-1} = -(-1)^{y-1}$, so we need to show that $d_\sigma(x) = d_\sigma(y)$. In I, x and y are an odd number of places apart, so if they are still an odd distance apart in σ, they must both have had an odd displacement or both have had an even displacement. This means $d_\sigma(x) = d_\sigma(y)$, as required. □

Corollary. *For n odd, the last row of H_n is a linear combination of the other rows of H_n.*

Proof. By the above lemma, each entry in the last row of H_n (the row given by the subset $\{1, \dots, n\}$) is equal to the alternating sum of the entries in the same column in rows labeled by subsets of all but one element. Therefore, the last row is equal to the row labeled by $\{2, \dots, n\}$, minus the row labeled by $\{1, 3, \dots, n\}$, $\dots$, plus the row labeled by $\{1, \dots, n-1\}$. □

Example. *In H_3, the row labeled by $\{1, 2, 3\}$ is equal to the row labeled by $\{2, 3\}$ minus the row labeled by $\{1, 3\}$ plus the row labeled by $\{1, 2\}$.*

3. Main Theorem

Definition 6. *H_n is said to possess the property of half-maximal rank if the first 2^{n-1} rows of $H_n^{(k)}$ have rank 2^{n-2} for all k.*

Theorem. *The rank of H_n is 2^{n-1}.*

Proof. We prove the theorem by induction. The result is immediately verified for H_1 and H_2, and it can also be fairly easily checked for H_3.

Let $n \in \mathbf{N}$ and assume that $rank(H_i) = 2^{i-1}$ for all $i < n$. By Lemma 3, the rank of the first 2^{n-1} rows of $H_n^{(k)}$ is equal to the rank of H_{n-1}. By induction, the rank of H_{n-1} is 2^{n-2}, so H_n possesses the property of half-maximal rank. When H_n is ordered in the usual manner, this means that the first half of the rows have half-maximal rank, that the first and third quarters together have half-maximal rank, that the first, third, fifth, and seventh eights together have half-maximal rank, *etc.* All operations subsequently performed on H_n preserve these properties.

By Lemma 1, we know that the top half of H_n consists of copies of H_{n-1} and that there is another copy of H_{n-1} at the left side of the bottom half. Therefore, we subtract the top half of the rows of H_n from the bottom half of the rows, canceling out the H_{n-1} in the lower left corner, and obtain some $2^{n-1} \times (n-1)(n-1)!$ matrix, which we will call R_n, in the lower right of H_n. We then cancel all but the leftmost copy of H_{n-1} in the top half by subtracting the leftmost copy from the others. We are left with the following matrix:

$$\left(\begin{array}{c|c} H_{n-1} & 0 \\ \hline 0 & R_n \end{array}\right)$$

Now we look at R_n more closely. We know that the leftmost $(n-2)!$ columns of H_{n-1} consist of two copies of H_{n-2} on top of each other, and, by Lemma 2, these same columns in the bottom half of H_n consist of a copy of H_{n-2} on top of a copy of $-H_{n-2}$. Therefore, the top half of the leftmost $(n-2)!$ columns of R_n is the block $H_{n-2} - H_{n-2} = \mathbf{0}$ and the bottom half of these columns is the block $-2H_{n-2}$.

As for the rest of R_n, we are only interested in some parts, for example the second $(n-2)!$ columns of R_n, which correspond to the first $(n-2)!$ columns of A_1 minus the second $(n-2)!$ columns of H_{n-1}. By Lemma 2, we see that the top half of this submatrix is therefore $B_1 - H_{n-2}$ and the bottom half is $H_{n-2} - B_1$.

In general, since each A_i is exactly $(n-2)!$ columns larger than H_{n-1}, the first $(n-2)!$ columns of A_i are aligned with the $(i+1)^{th}$ set of $(n-2)!$ columns of the $(i+1)^{th}$ copy of H_{n-1} appearing in the top half of H_n. The first $(n-2)!$ columns of A_i consist of a B_i above an H_{n-2}, and the $(i+1)^{th}$ set of $(n-2)!$ columns of H_{n-1} consist of an H_{n-2} above a B_i. We therefore conclude that this section of R_n has $B_i - H_{n-2}$ in the top half and $-(B_i - H_{n-2})$ in the bottom half.

We now examine R_{n-1}, which is the matrix remaining in the bottom right of H_{n-1} when we subtract the top half from the bottom half. In terms of the B_i, its i^{th} block is simply $B_i - H_{n-2}$. So, moving the columns of R_n such that all blocks with this same matrix on top and its negative on bottom are all next to each other, the first $(n-2)(n-2)!$ columns of R_n become an R_{n-1} above a $-R_{n-1}$.

$$R_n = \left(\begin{array}{c|c|c} 0 & R_{n-1} & \multirow{2}{*}{$*$} \\ \cline{1-2} -2H_{n-2} & -R_{n-1} & \end{array}\right)$$

Returning to H_n in its current state, we add the third quarter of rows to the fourth quarter of rows and divide the fourth quarter of rows by -2. By the above, this will cancel the R_{n-1} and the $-R_{n-1}$. This leaves us with $[\ \mathbf{0}\ \ |R_{n-1}|\ \ *\ ]$ in the third quarter of the matrix, with a $\mathbf{0}$ above and below the R_{n-1}.

We now use an argument that will be utilized multiple times to eliminate the $*$ to the left of the R_{n-1} in the matrix $[\ \mathbf{0}\ \ |R_{n-1}|\ \ *\ ]$. First, we note that this is the third quarter of a modified H_n which still possesses the property of half-maximal rank. Therefore the first and third quarters together have half-maximal rank, and since the first half is simply H_{n-1}, which possesses the property of half-maximal rank, the first quarter alone has half-maximal rank, being the first half of

H_{n-1}. Since the H_{n-1} and the $[R_{n-1}| \quad * \quad]$ are in different blocks, we conclude that the third quarter of the matrix has half-maximal rank, which in this case is equal to $((2^n)/4)/2 = 2^{n-3}$.

Consider R_{n-1}. Since we can reduce H_{n-1} to an H_{n-2} in the upper right and an R_{n-1} in the lower right with zeros everywhere else, and since we know by induction that $rank(H_{n-1}) = 2^{n-2}$ and $rank(H_{n-2}) = 2^{n-3}$, we conclude that

$$rank(R_{n-1}) = rank(H_{n-1}) - rank(H_{n-2}) = 2^{n-3}.$$

Thus we have that $rank([\quad \mathbf{0} \quad |R_{n-1}| \quad * \quad]) = rank(R_{n-1})$, and therefore the columns of $[*]$ must be linear combinations of the columns of R_{n-1}.

Using this fact, we cancel the entries to the right of the R_{n-1} using the columns that contain it. Noting that the entries in these columns are zeros above and below the R_{n-1}, we are left with the following matrix:

$$\left(\begin{array}{c|ccc} H_{n-1} & & 0 & \\ \hline \multirow{2}{*}{0} & 0 & R_{n-1} & 0 \\ \cline{2-4} & H_{n-2} & 0 & * \end{array}\right)$$

Recalling that to get to this point we subtracted the top half of H_n from the bottom half and then added the third quarter to the fourth quarter, we perform these operations on H_3 as an example. First we subtract row one from row five, row two from row six, row three from row seven, and row four from row eight. Then we add the resulting row five to row seven and row six to row eight, obtaining the following:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & -2 & 0 & -2 & -2 \\ 0 & 0 & -2 & 0 & -2 & -2 \end{pmatrix}.$$

Next we cancel the extra two copies of H_{n-1} in the top half by subtracting column one from columns three and five and column two from columns four and six. Finally, we calculate $R_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$, and note that as proved above, the block directly to the right of this can be canceled with column operations. We are now left with

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & -2 \\ 0 & 0 & -2 & 0 & -2 & -2 \end{pmatrix}.$$

Note that this has H_2 in the upper left, $-2H_1$ in the bottom quarter next to a zero matrix, R_2 above and to the right of this with zero below it, and some unknown matrix in the lower right. In this case, the $[*]$ matrix in the lower right easily cancels with $-2H_1$, but in general this does not happen.

Now we perform the same operations on the H_{n-2} matrix in the bottom quarter as we did on H_n to put it in the form of the matrix above. The bottom quarter of the matrix now becomes

$$\left(\begin{array}{c|cc|c} H_{n-3} & \multicolumn{2}{c|}{\mathbf{0}} & \\ \hline & 0 & R_{n-3} & * \\ \cline{2-3} \mathbf{0} & H_{n-4} & 0 & \end{array}\right)$$

We repeat the argument used above to cancel the top half and then the third quarter of the $[*]$ in the above matrix. The top half of the matrix above is also the seventh eighth of the whole H_n, and we know that the first, third, fifth, and seventh eighths of the matrix together have half-maximal rank. We also know that the first and third quarter of the whole matrix have half-maximal rank since H_{n-1} possesses the property of half-maximal rank. Thus the fifth and seventh eighths together have half-maximal rank. The fifth eighth is now simply the top half of R_{n-1}. By the structure of H_{n-1} after it has been reduced to H_{n-2} and R_{n-1}, the fact that the first third quarters of H_{n-1} have half-maximal rank, and the fact that the first quarter of H_{n-1} alone has half-maximal rank (since it is the top half of H_{n-2}), we conclude that the fifth eighth of the entire matrix has half-maximal rank. Therefore, since the nonzero columns of the seventh eighth of the matrix do not overlap with those of any of the other sections mentioned (first, third and fifth eighths), we conclude that the seventh eighth of the matrix alone has half-maximal rank.

The seventh eighth of H_n is now $[\ \mathbf{0}\ \ |H_{n-3}|\ \ *\ ]$, and we know by induction that H_{n-3} has half-maximal rank, so we conclude that we can use column operations to eliminate this $[*]$ in the seventh eighth.

By the same (only somewhat longer) argument, the rank of the fifteenth sixteenth of the matrix, which is now $[\ \mathbf{0}\ \ |R_{n-3}|\ \ *\ ]$ has half-maximal rank, as does R_{n-3} alone, so we may cancel this part of the unknown matrix as well.

We have now reduced H_n to the point where the rank of the first fifteen sixteenths is known by inductive hypothesis to be

$$\begin{aligned} &rank(H_{n-1}) + rank(R_{n-1}) + rank(H_{n-3}) + rank(R_{n-3}) \\ &\qquad = 2^{n-2} + 2^{n-3} + 2^{n-4} + 2^{n-5}. \end{aligned}$$

Furthermore, in the last sixteenth, which is $[\ \mathbf{0}\ \ |H_{n-4}|\ \ *\ ]$, we are left with the same situation that we had in the last quarter. We proceed as above, reducing the rank of H_n to the sum of ranks of known matrices. In each step, we determine the rank of the top three quarters of the matrix and leave the bottom quarter to the next step. This process terminates with either one row or two rows remaining (the only powers of two not divisible by four). Thus, we have two cases to consider: when n is even and when n is odd.

(I) Suppose n is even. At each step we reduce the matrix to its bottom quarter, and eventually are left with the last one row with unknown rank. This row must be of the form $[\ \mathbf{0}\ \ |H_0|\ \ *\ ]$, by the nature of our algorithm, and $H_0 = [1]$, so the rank of this last row must be 1. Therefore we have

$$\begin{aligned} rank(H_n) &= rank(H_{n-1}) + rank(R_{n-1}) + rank(H_{n-3}) \\ &\quad + \ldots + rank(H_1) + rank(R_1) + rank(H_0) \\ &= 2^{n-2} + 2^{n-3} + \ldots + +2^1 + 2^0 + 1 \\ &= 2^{n-1}, \end{aligned}$$

as required.

(II) Suppose n is odd. Then we are left with two rows of unknown rank, and last two rows have the form $[\ \mathbf{0}\ \ |H_1|\ \ *\ ]$. Now, we know $H_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so we can subtract the top row from the bottom row, and obtain $[\ \mathbf{0}\ \ |\ \ *\ ]$ in the last row of the matrix. Since all the entries above this $*$ are zeros, we know that if $[*] \neq \mathbf{0}$, the last row must be linearly independent from the others, since we never canceled other rows with the last row. But, by the corollary to Lemma 4, the last row of H_n is a linear combination of the other rows for odd n, so we must have that this $*$ is the zero matrix. Therefore we have

$$\begin{aligned} rank(H_n) &= rank(H_{n-1}) + rank(R_{n-1}) + rank(H_{n-3}) \\ &\quad + rank(R_{n-3}) + \ldots + rank(H_2) + rank(R_2) + 1 \\ &= 2^{n-2} + 2^{n-3} + \ldots + 2^1 + 2^0 + 1 \\ &= 2^{n-1}, \end{aligned}$$

as required. □

Corollary. *For $n \geq 2$, $rank(G_n) \leq 2^{n-1} - 1$.*

Proof. First note that in the last step of the above theorem, we showed that the last row is linearly independent from all the others for even n, and the second to last row is linearly independent from all rows above it for odd n. Consider G_n in each case. For even n, the last row is now a row of 1s, and therefore cancels with any row corresponding to a subset of one element or the empty set. In particular, the last row is now linearly dependent on the others. For odd n, we know by the corollary to Lemma 4 that the alternating sum of rows enumerated by subsets of all but one element is equal to the last row of H_n. Since G_n consists of a subset of columns of H_n, it also has the property that if we take this alternating sum of rows, we obtain the last row, which is a row of 1s. This again cancels with the empty set row, so the second to last row is dependent on just those above it.

In either case, the last two rows of G_n are linearly dependent on the others, while in H_n exactly one of the last two rows is not dependent on the others. Thus, we have that

$$rank(G_n) \leq rank(H_n) - 1 = 2^{n-1} - 1,$$

by our main result. □

This bound is in fact optimal, as it is known by other methods that $rank(G_n) = 2^{n-1} - 1$ [3].

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University of Michigan, Ann Arbor, Michigan, USA
E-mail address: jlouwsma@umich.edu

Universidade Federal de São Carlos, São Carlos, São Paulo, Brasil
E-mail address: adilson.presoto@bol.com.br

Pomona College, Claremont, California, USA
E-mail address: alan.tarr@pomona.edu