The Riemann Surface of the Logarithm Constructed in a Geometrical Framework

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Abstract

We present a geometrical construction of the Riemann Surface corresponding to the complex logarithm function, derived as the uniform limit of the sequence of surfaces in $\mathbb{C} \times \mathbb{R}$. This approach, even though results in finite (but arbitrarily many) covers, is purely geometrical and does *not* imply any tools provided by the complex structure (as analytic continuation). Moreover, the Differential Geometry framework we adopt affords explicit generalization in arbitrary dimensions and certain corollaries are derived.

Keywords: Logarithmic Riemann Surface

Introduction

In this paper we present a purely geometrical construction of the complex logarithm's Riemann Surfaces via Differential Geometry, without any implication of the standard tools and methods provided by the complex structure (as analytic continuation), classically encountered in Riemann Surfaces. Instead, we consider elementary notions of Geometry, as smooth surfaces and manifolds, vector fields and convergence of sequences to provide and alternative approach.

The basic concept of our analysis is that, if we wish to obtain a single-valued complex logarithm defined on an appropriate geometrical object, we may instead try to turn the complex exponential function into an injective map, removing the $2\pi i$-periodicity (so that it would be inversible). The injectivity of the exponential is derived on the

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covering space of $\mathbb{C} \setminus \{0\}$, (the former produced by the complex polar coordinate map) by a simple but considerable idea: we could extend exp from a complex function to a vector field on $\mathbb{C} \times \mathbb{R}$, and seek for a proper sequence of surfaces, whereon the extended exponential would be injective, (restricted on each element of the sequence) and the limit surface of the sequence would be the covering of $\mathbb{C} \setminus \{0\}$. The requested sequence is obviously that of the helicoids and in the uniform limit (equivalently finite covering sheets on $\mathbb{C} \setminus \{0\}$) the injectivity is preserved. Considering then the inverse map, the logarithm, we create an one-to-one correspondence between each branch of the multi-valued function and the spirals of the helicoid, preserved when the convergence is uniform. This fact provides the single-valuedness on the (limit) covering space of $\mathbb{C} \setminus \{0\}$.

Our results, even if cannot be characterized novel, in the sense that we illustrate an already known object, they seem to provide this rather unexpected capability of Geometry to provide a Riemann Surface (classically obtained by analytic continuation), without analytic continuation! Although with our method we achieve to derive finite covers (but arbitrarily many), this approach stands apart from the corner-stone of Complex Analysis.

This paper is organized as follows: in section 1 we collect some elementary results that will be needed in the sequel. In section 2, we present in detail the results outlined above. In section 3 we treat immediate generalization in higher dimensions. We substitute the helicoids by a sequence of multihelicoid submanifolds of $\mathbb{C}^m \times \mathbb{R}^m$ and analogously introduce a proper vector field in this space. In the uniform limit we obtain injectivity on the covering m-manifold of $\mathbb{C}^m \setminus \{0\}$. Finally, we prove the (quite expected) result that this vector field (generalizing naturally the holomorphic exp of $\mathbb{C}$) can be extended smoothly from a local chart to a global section of the tangent bundle of a paracompact manifold.

1. Preliminaries

We collect a few elementary results which will be needed in the main part of the paper, noticed here for the reader's convenience.

Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be the total coordinate system of the helicoid, given by

$$(u, v) \longmapsto (u \cos v, u \sin v, \mathrm{a}v), \ \ \mathrm{a} > 0$$

where a is a parameter. In complex coordinates we may write

$$X : \mathbb{C} \to \mathbb{C} \times \mathbb{R} \quad X(u + iv) = (u \exp(iv), \mathrm{a}v) \equiv (P(z), \mathrm{aIm}(z)),$$

$z = u+iv$ where P is the complex function $P : \mathbb{C} \to \mathbb{C}$ with $P(z) := Re(z) \exp(i\mathrm{Im}(z))$, representing the map of polar coordinates onto $\mathbb{C}$.

1.1 Lemma.

a) *The function P defined above has the property to be surjective, C^∞-differentiable but not holomorphic*

b) *The map $X := (P, \mathrm{aIm}) : \mathbb{C} \to \mathbb{C} \times \mathbb{R}$ is injective for every $\mathrm{a} > 0$.*

Proof. It is straightforward from the definitions and the Cauchy-Riemann equations. □

1.2 Remark. We note that if we restrict P on the subspace $\{z \in \mathbb{C}/Re(z) > 0\}$ (which from now on we shall denote by $\mathbb{C}^\# \equiv \mathbb{R} \times \mathbb{R}^+$), it may be easily verified that

$$P(\mathbb{C})/\mathbb{C}^\# = P(\mathbb{C}^\#) = \mathbb{C} \setminus \{0\}$$

and P is onto $\mathbb{C} \setminus \{0\}$.
Let $\exp : \mathbb{C} \mapsto \mathbb{C} \setminus \{0\} : z \mapsto \exp(z) \equiv e^z$ be the ordinary complex exponential function. From the point of view of Differential Geometry, this map may be considered as a vector field, tangent on the 2-dimensional plane manifold $\mathbb{C} \cong \mathbb{R}^2$, i.e.

$$\mathbb{C} \equiv \mathbb{R}^2 \ni ue_1 + ve_2 \longmapsto (e^u \cos v)e_1 + (e^u \sin v)e_2.$$

This notion introduces the idea of the following definition:

1.3 Definition. The exponential vector field in $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ is the map, $\mathrm{Exp} : \mathbb{C} \times \mathbb{R} \mapsto \mathbb{C} \times \mathbb{R}$ given by: $(z, t) \mapsto \mathrm{Exp}(z, t) := (\exp(z), \exp(t))$, $z = x + iy$ $x, y, t \in \mathbb{R}$, and in real variables:

$$xe_1 + ye_2 + te_3 \longmapsto (e^x \cos y)e_1 + (e^x \sin y)e_2 + (e^t)e_3.$$

1.4 Lemma. *The exponential field* $\mathrm{Exp} : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ *restricted to the surface of the helicoid* $X : \mathbb{C}^\# \mapsto \mathbb{C} \times \mathbb{R}$ *(imbedded in* $\mathbb{C} \times \mathbb{R}$*) is an injective map.*

Proof. The terminologies of imbedding and immersion we shall use are the well known ones, referring e.g. to [2], [5]. If $z = \alpha + \beta i$, $w = \gamma + \delta i$ with $\mathrm{Exp}(X(z)) = \mathrm{Exp}(X(w))$, then $\exp(\alpha)\ \exp(\exp(\beta i)) = \exp(\gamma)\ \exp(\exp(\delta i))$ and $\exp(\mathrm{a}\,\beta) = \exp(\mathrm{a}\,\delta)$. The injectivity of the real exponential consists that $z = w$, which implies $X(z) = X(w)$. ☐

If $\xi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a vector field on $\mathbb{R}^n$, where $x \mapsto \xi(x)$ then the obvious identification $\xi(x) \equiv (x, \xi(x))$ is the requested one in order to consider ξ valued in the tangent bundle $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$, since it is trivial. This notion and the previous terminology for the Exp-field, lead us to the following result:

1.5 Proposition.

a) *The vector field* $\mathrm{Exp} : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ *is* C^∞*-smooth and tangent on the manifold* $\mathbb{C} \times \mathbb{R}$.

b) *If* $\pi_{\mathbb{C}} : \mathbb{C} \times \mathbb{R} \mapsto \mathbb{C}$ *is the projection to the first factor, then:*

$$\pi_{\mathbb{C}} \circ \mathrm{Exp} = \mathrm{Exp} \circ \pi_{\mathbb{C}} \equiv \exp \circ \pi_{\mathbb{C}}$$

where exp *denotes the usual exponential of* $\mathbb{C}$ *and we have introduced the identification* $\mathbb{C} \cong \mathbb{C} \times \{0\}$.

Proof.

a) By definition 1.2 the field has smooth components in $\mathbb{R}^3$ and it is obviously smooth. Identifying $\mathrm{Exp}(z, x) \equiv \big((z, x), \mathrm{Exp}(z, x)\big)$ (that is, the map with its graph) we may consider $\mathrm{Exp} : \mathbb{R}^3 \mapsto T\mathbb{R}^3 \cong \mathbb{R}^6$ as tangent vector field onto $\mathbb{R}^3$ with $\pi \circ \mathrm{Exp} = Id_{\mathbb{R}^3}$.

b) Let $z \in \mathbb{C}$ and consider the natural immersion $\mathbb{C} \hookrightarrow \mathbb{C} \times \mathbb{R}$, i.e. $z \mapsto (z, 0)$. then, we have for every $(z, x) \in \mathbb{C} \times \mathbb{R}$, that $(\pi_{\mathbb{C}} \circ \mathrm{Exp})(z, x) = \pi_{\mathbb{C}}\big(\mathrm{Exp}(z, x)\big) = \pi_{\mathbb{C}}(e^z, e^x) = e^z = \exp\big(\pi_{\mathbb{C}}(z, x)\big) = (\exp \circ \pi_{\mathbb{C}})(z, x)$, that is

$$\pi_{\mathbb{C}} \circ \mathrm{Exp} = \exp \circ \pi_{\mathbb{C}}.$$

Also, $(\mathrm{Exp} \circ \pi_{\mathbb{C}})(z, x) = \mathrm{Exp}(z) = \exp(z) = (\exp \circ \pi_{\mathbb{C}})(z, x)$. Consequently,

$$\pi_{\mathbb{C}} \circ \mathrm{Exp} = \mathrm{Exp} \circ \pi_{\mathbb{C}} = \exp \circ \pi_{\mathbb{C}}$$

which completes the proof.

☐

2. Construction of the Riemann surface

As mentioned in the Introduction, the basic object we shall introduce is an appropriate helicoid sequence, in order to provide a special covering manifold of $\mathbb{C} \setminus \{0\}$ in the uniform limit. Hence, we recall the surface of the helicoid $X : \mathbb{C}^{\#} \mapsto \mathbb{C} \times \mathbb{R}$, $X = (P, \mathrm{a}\,\mathrm{Im})$ and substitute a by the sequence $\mathrm{a}_n = 1/n$ $n \in \mathbb{N}$. Thus, we obtain a sequence of surfaces

$$X_n : \mathbb{C}^{\#} \longmapsto \mathbb{C} \times \mathbb{R} : z \longmapsto \left(P(z), \frac{1}{n}\mathrm{Im}(z)\right).$$

The following result is provided by virtue of Lemma 1.1 a). For the definition of covering manifolds we refer to [8].

2.1 Lemma. *The pair $(P, \mathbb{C}^{\#})$ constitutes a covering manifold of $\mathbb{C} \setminus \{0\}$ with infinite covering sheets. (P is the complex polar coordinate map).*

Proof. We have already seen that $P(\mathbb{C}^{\#}) = \mathbb{C} \setminus \{0\}$ and P is onto $\mathbb{C} \setminus \{0\}$. Our assertion can be easily verified considering the inversed image of a point in $\mathbb{C} \setminus \{0\}$, that is, if $z_0 \in P^{-1}(z)$, $P^{-1}(z) = \{z_0 + 2k\pi i, k \in \mathbb{Z}\}$ and the cardinal of $\mathbb{Z}$ is infinite. □

It is quite evident that the limit surface of the sequence $X_n : \mathbb{C}^{\#} \mapsto \mathbb{C} \setminus \{0\} \times \mathbb{R}$ is $(P, 0)$.

The next result correlates the helicoid sequence and the covering space in the uniform limit.

2.2 Proposition. *The sequence of helicoids $(X_n)_{n \in \mathbb{N}}$ converges uniformly to the covering manifold of $\mathbb{C} \setminus \{0\}$, determined by $(P, \mathbb{C}^{\#})$ on the strip of $\mathbb{C}^{\#}$ with bounded imaginary part.*

Proof. Recalling the natural immersion of $\mathbb{C}$ into $\mathbb{C} \times \mathbb{R}$ mentioned above, we may identify the limit of X_n with P. Namely,

$$X_n \xrightarrow{n \mapsto \infty} (P, 0) \equiv P, \quad \mathbb{C} \hookrightarrow \mathbb{C} \times \mathbb{R}.$$

Let $\varepsilon > 0$. We shall prove that there exists an $n_0 \in \mathbb{N}$ so that for every $n \geq n_0$, the inequality

$$\left\| X_n(z) - \big(P(z), 0\big) \right\| < \varepsilon$$

holds true (for every z, where n_0 depends only on ε. Indeed, if $z = u + iv$, then

$$\begin{aligned} \left\| X_n(z) - \big(P(z), 0\big) \right\| &= \left\| \left(P(z), \frac{1}{n}\mathrm{Im}(z)\right) - \big(P(z), 0\big) \right\| \\ &= \left\| \left(0, \frac{1}{n}\mathrm{Im}(z)\right) \right\| \leq \frac{1}{n}|\mathrm{Im}(z)|. \end{aligned}$$

Provided that we preserve the imaginary part of z bounded (arbitrarily), that is $|\mathrm{Im}(z)| < M$, for fixed $M > 0$, we define $n_0 = n_0(\varepsilon)$ so that $n_0 > M/\varepsilon$. This completes our proof in an evident mode. $\square$

We are now in position to prove the principal result of this section. As we have already claimed, the restriction of the Exp-field on the imbedded images $X_n(\mathbb{C}^{\#})$, induces a sequence of vector fields on the elements of the sequence:

$$\mathrm{Exp}_n := \mathrm{Exp}/X_n(\mathbb{C}^{\#})$$

This sequences of maps in injective on every surface $X_n(\mathbb{C}^{\#})$, in virtue of Lemma 1.3, for a $= 1/n$, $n \in \mathbb{N}$. We are about to prove that this property is preserved in the uniform limit, when $n \mapsto \infty$. The limit surface is the covering space $P : \mathbb{C}^{\#} \mapsto \mathbb{C} \setminus \{0\}$ (Lemma 2.1), and introducing the natural immersion $\mathbb{C} \hookrightarrow \mathbb{C} \setminus \{0\}$ we may identify Exp on $P(\mathbb{C}^{\#})$ with the usual complex exponential function, exp. In other words, we construct an exponential injective on $P(\mathbb{C}^{\#}) = \mathbb{C} \setminus \{0\}$ (Lemma 1.1), presupposing only that we restrict ourselves on those z's that belong to horizontal strips of $\mathbb{C}^{\#}$, arbitrarily wide. This fact resembles and simultaneously generalizes the analogous case in elementary Complex Analysis, where the exp is injective and inversible on the strips of $2\pi i$-width only.

2.3 Theorem. *Let us consider the sequence of vector fields*

$$\mathrm{Exp}_n := \mathrm{Exp}/X_n(\mathbb{C}^{\#}) : X_n(\mathbb{C}^{\#}) \longmapsto \mathbb{C} \setminus \{0\} \times \mathbb{R}$$

on the images on the helicoid sequence $(X_n)_{n\in\mathbb{N}}$. *Then:*

$$\mathrm{Exp}_n/X_n(B^+) \xrightarrow{n\mapsto\infty} \mathrm{Exp}/(P(B^+), 0) \equiv \exp/P(B^+)$$

and $\exp/P(B^+)$ *is an one-to-one function, provided that* $B^+ := \{z \in \mathbb{C}^{\#} : |\mathrm{Im}(z)| < M, M > 0\}$, *for arbitrarily fixed positive constant* M.

Proof. Lemma 1.1 consists that X_n is an injective map for every $n \in \mathbb{N}$ and Lemma 1.3 provides that $\mathrm{Exp}/X_n(\mathbb{C})$ is also injective. Consequently, we have

$$\mathrm{Exp}\big(X(z)\big) = \mathrm{Exp}\big(X(w)\big) \Longleftrightarrow z = w \Longleftrightarrow X(z) = X(w).$$

Our task is to prove the following condition:

$$\mathrm{Exp}\big(P(z)\big) \equiv \exp\big(P(z)\big) = \exp\big(P(w)\big) \equiv \mathrm{Exp}\big(P(w)\big)$$

implies

$$P(z) = P(w),$$

when z, w satisfy the property $|\mathrm{Im}(z)| + |\mathrm{Im}(w)| < M$ and $X_n \mapsto (P, 0) \equiv P$ uniformly. Since the Exp-field is continuous (as smooth, proved in 1.4) and the convergence of X_n to P is uniform, by virtue of Proposition 2.2, we obtain that if $X_n \mapsto P$, then

$$\mathrm{Exp}/X_n \xrightarrow{n \mapsto \infty} \mathrm{Exp}/(P, 0) \equiv \exp/P$$

and the last identification comes from the natural immersion $\mathbb{C} \hookrightarrow \mathbb{C} \times \mathbb{R}$ (see 1.4. b)). The well-known triangle inequality consists that

$$\begin{aligned} &\big\|\mathrm{Exp}\big(X_n(z)\big) - \mathrm{Exp}\big(X_n(w)\big)\big\| \le \big\|\mathrm{Exp}\big(P(z)\big) - \mathrm{Exp}\big(P(w)\big)\big\| \\ &+ \big\|\mathrm{Exp}\big(X_n(z)\big) - \mathrm{Exp}\big(P(z)\big)\big\| + \big\|\mathrm{Exp}\big(X_n(w)\big) - \mathrm{Exp}\big(P(w)\big)\big\|. \end{aligned}$$

When the convergence of X_n to P is uniform we have

$$\big\|\mathrm{Exp}\big(X_n(z)\big) - \mathrm{Exp}\big(P(z)\big)\big\| \longmapsto 0$$

as well as

$$\big\|\mathrm{Exp}\big(X_n(w)\big) - \mathrm{Exp}\big(P(w)\big)\big\| \longmapsto 0$$

and supposing that $\exp\big(P(z)\big) = \exp\big(P(w)\big)$, we obtain

$$\big\|\mathrm{Exp}\big(X_n(z)\big) - \mathrm{Exp}\big(X_n(w)\big)\big\| \longmapsto 0.$$

The smoothness of all the implicating mapping insures that Lemma 1.3 may be slightly modified in order to apply it in this case. Namely, simply substituting equalities with convergence in the proof, we conclude that the last relation implies

$$\|X_n(z) - X_n(w)\| \longmapsto 0$$

and in the same sense of thinking, Lemma 1.1 implies

$$\|z - w\| \longmapsto 0$$

thus $X_n(z) = X_n(w)$ and $z = w$. Finally, for $z \mapsto w$,

$$\|P(z) - P(w)\| \longmapsto 0.$$

And this notion completes the proof. □

The injectivity of the Exponential Field provided by Lemma 1.3 gives the ability to introduce the inverse map (which we shall denote by Log) as a well-defined generalization of the complex logarithm:

$$\mathrm{Log}/\mathrm{Exp}\big(X_n(\mathbb{C}^{\#})\big) : \mathrm{Exp}\big(X_n(\mathbb{C}^{\#})\big) \longmapsto X_n(\mathbb{C}^{\#})$$

where

$$\mathrm{Log} \equiv \mathrm{Exp}^{-1}.$$

The inverse map is given by

$$\mathrm{Exp}^{-1}(xe^{i\theta}, y) = \big(\ln x + i(\theta + 2k\pi), \ln y\big)$$

where the k-branch (of the projection to the complex factor) corresponds to y-spiral of the (exponential image of the) helicoid as follows:

$$\theta + 2k\pi = \begin{cases} \ln x \, \tan(n \, \ln y), & n \, \ln y \neq 2k\pi \pm \dfrac{\pi}{2} \\ 0, & n \, \ln y = 2k\pi \pm \dfrac{\pi}{2}. \end{cases}$$

We are now in position to introduce a well-defined complex logarithm on the exponential image of the limit covering space:

2.4 Corollary. *The sequence of maps*

$$\mathrm{Log}_n := \mathrm{Exp}^{-1}/\mathrm{Exp}\big(X_n(\mathbb{C}^{\#})\big), \quad n \in \mathbb{N}$$

converges uniformly to a well-defined complex logarithm

$$\mathrm{Log}_n/\mathrm{Exp}\big(X_n(B^+)\big) \xrightarrow{n \mapsto \infty} \mathrm{Exp}^{-1}/\mathrm{Exp}\big(P(B^+), 0\big) \equiv \log / \exp\,\big(P(B^+)\big)$$

where

$$\log / \exp\,\big(P(B^+)\big) \equiv \exp^{-1} / \exp\,\big(P(B^+)\big)$$

and the Riemann Surface of the logarithm is the limit covering space:

$$(\exp \circ P)(B^+) = \{\exp(ue^{iv}) \in \mathbb{C}/u > 0, |v| < M, M > 0\}. \qquad \square$$

3. The exponential field in higher dimensions

In this section we generalize the formalism and the basic concepts outlined in the previous section, in arbitrarily high dimensions. We shall consider geometrical objects as linear spaces, smooth manifolds and vector fields to obtain analogous corollaries. Never the less, this framework implies that the results will not stand any more in the region of Riemann Surfaces, but instead within (Classical) Differential Geometry.

In the sequel, we shall adopt the Einstein summation convention; so, if V_n is an n-dimensional vector space with basis $\{e_1, \dots, e_n\}$, then for any $u \in V_n$ we write

$$u = \sum_{j=1}^{n} u^j e_j \equiv u^j e_j, \quad u^j \in \mathbb{R}.$$

Furthermore, by the term smooth manifold we mean a real, C^∞-smooth (connected), Hausdorff and second countable manifold, of finite dimensions.

We are primarily concerned in generalizing the exponential onto a vector space, since the definition onto a manifold (locally) will imply the transport from $\mathbb{R}^n$ to M via a local chart. The definition of the field onto a submanifold of $\mathbb{R}^n$ will occur by a simple restriction.

The terminology of submanifolds followed here may be found i.e. in [5]. We shall consider the submanifold as a subset of $\mathbb{R}^n$, where the inclusion map is an imbedding and the Atlas that determines differential structure and topology is the induced from the Euclidean Space.

3.1 Definition. Let V_n be an n-dimensional vector space over the $\mathbb{R}$-field. If $u = u^j e_j \in V_n$ $(1 \leq j \leq n)$, we define

$$\begin{aligned} \mathrm{Exp} : & V_n \longmapsto V_n : \\ u \longmapsto \mathrm{Exp}(u) := & \sum_{h=1}^{k} \big[\exp(u^{2h-1})\cos(u^{2h})\big] e_{2h-1} \\ & + \sum_{h=1}^{k} \big[\exp(u^{2h-1})\sin(u^{2h})\big] e_{2h} \\ & + \sum_{i=1}^{m} \big(\exp(u^i)\big) e_i \end{aligned}$$

where $n = 2k + m$, for some prefixed integers k, m.

It will be convenient to express the field in complex coordinates. Considering the canonical isomorphism $V_n \cong \mathbb{C}^k \times \mathbb{R}^m$ and the usual bases of $\mathbb{C}^k$ and $\mathbb{R}^m$

$$\mathbb{C}^k = \langle \{\xi_h / 1 \leq h \leq k\} \rangle, \quad \mathbb{R}^m = \langle \{e_i, 1 \leq i \leq m\} \rangle$$

then we may write

$$\mathrm{Exp}(u) = \sum_{h=1}^{k} \exp(u^h + i u^{h+1}) \xi_h + \sum_{i=1}^{m} \exp(u^i) e_i.$$

3.2 Definition. Let M be an n-dimensional smooth manifold, $p \in M$ and (U, φ) local chart at p with $x^j \equiv \pi^j \circ \varphi$ the coordinate maps. We define the vector field $\mathrm{Exp} : M \mapsto TM$ as

$$p \longmapsto \mathrm{Exp}(p) := (\mathrm{Exp} \circ \varphi^{-1})\big(\varphi(p)\big) = (\mathrm{Exp} \circ \varphi^{-1})\big(x^j(p)e_j\big).$$

The field is considered as valued on TM with the aid of the local trivializations on the tangent bundle TM. If $\pi : TM \mapsto M$ is the surjective submersion, then the local trivialization provides in a domain $\pi^{-1}(U)$ (of TM) the local structure of a Cartesian product

$$(\varphi, d\varphi) \circ t_p : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^n \longmapsto \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}.$$

And as a consequence Exp/U may be identified with its graph.
For relevant comments in the same sense and for a detailed analysis in this direction we refer to [7].
We wish to consider Exp as defined on an imbedded submanifold of $\mathbb{R}^k$. Consequently if $x = x^j e_j$, $(1 \leq j \leq k)$ and $x \in M \cap \mathbb{R}^k$, we write $(\dim M = n)$

$$M \cap \mathbb{R}^k \ni x \longmapsto \widetilde{\mathrm{Exp}}(x) := \mathrm{Exp}(x)/M \cap \mathbb{R}^k.$$

The expression through the intrinsic coordinates of the manifold are obtained as follows: if (U, φ) is chart at $x_0 \in U \cap \mathbb{R}^k$, then for $x^j = x^j(y)$, $y = y^i e_i \in \mathbb{R}^n$ and $x^j \equiv \pi^j \circ \varphi^{-1}$ $(1 \leq j \leq k)$, we have

$$(\pi^i \circ \varphi^{-1})(y)e_i \longmapsto \mathrm{Exp}\Big(\big(\pi^i \circ \varphi^{-1}(y)\big)e_i\Big).$$

3.3 Remark. The exponential field is evidently C^∞-smooth in all the occasions outlined above. In the case of $\mathbb{R}^k$ and its n-dimensional submanifold it is obtained automatically and in the case of arbitrary manifolds it is quite obvious after considering a local expression and the usual notions (for the independence of the local chart under consideration).

3.4 Lemma. *Let M be an n-manifold. The aforementioned* Exp *field can be extended smoothly from a local chart to the whole M, namely*

$$\mathrm{Exp}/U \in \varGamma(TU) \Longrightarrow \mathrm{Exp} \in \varGamma(TM)$$

Proof. Our manifold M is considered paracompact, (since it is second countable) and consequently admits partitions of unity. Let $\{U_a\}_{a \in I}$ be an open cover of M with a partition of unity $\{f_a\}$ with respect to it, where $\sup p(f_a) \subseteq U_a$, $a \in I$.

Then the properties of the partitions constitute that the field

$$\overline{\mathrm{E}}\mathrm{xp} := \sum_a \left[(f_a \circ \varphi_a^{-1})\,(\mathrm{Exp} \circ \varphi_a^{-1})\right](\pi^j \circ \varphi_a e_j)$$

is a C^∞-section (global) of the tangent (vector) bundle of M, with respect to the differential Atlas $\{(U_a, \varphi_a\}_{a\in I}$. □

We aim particularly at extending the results of Theorem 2.3 from $\mathbb{C}$ to $\mathbb{C}^m$. For this purpose we have to consider a special type of submanifold of $\mathbb{R}^s \cong \mathbb{C}^m \times \mathbb{R}^n$ that the relevant sequence will have as the limit manifold the covering m-manifold of $\mathbb{C}^m \setminus \{0\}$ that appears normally by Cartesian product. Consequently, we consider the sequence of multi-helicoid submanifolds of $\mathbb{C}^m \times \mathbb{R}^m$

$$X_n : \mathbb{C}^{\#m} \longmapsto \mathbb{C}^m \setminus \{0\} \times \mathbb{R}^m \cong \prod^m (\mathbb{C} \setminus \{0\} \times \mathbb{R})$$

where if $z = z^a \xi_a = \sum_{a=1}^m (u^a + iv^a)\xi_a \in \mathbb{C}^{\#m}$, $x = x^a e_a \in \mathbb{R}^m$, $u^a, v^a, x^a \in \mathbb{R}$, we define

$$z^a\xi_a \longmapsto \sum_{a=1}^m \big(u^a \exp(iv^a)\big)\xi_a + \left(\frac{1}{n}v^a\right) e_a.$$

The following is the obvious generalization of the results 2.1 and 2.2 presented in the previous section.

3.5 Proposition. *The pair $(P, \mathbb{C}^{\#m})$ is a covering m-manifold of $\mathbb{C}^m \setminus \{0\}$ with infinite covering sheets, where $P := \pi_{\mathbb{C}^m} \circ X_n$ (projection onto $\mathbb{C}^m$) and X_n converges uniformly to P for those multistrips of $\mathbb{C}^{\#m}$ with bounded imaginary part.*

Proof. We simply apply 2.1 and 2.2 in every component $\mathbb{C} \times \mathbb{R}$ of $\mathbb{C}^m \times \mathbb{R}^m$. This forms a proof of our assertion is a trivial way, just by noting that $|\mathrm{Im}(z^a)| < M \Leftrightarrow \sum_a |\mathrm{Im}(z^a)| < \overline{M}$, where $M, \overline{M} > 0$. □

Recalling the definition 3.1 of the Exp-field and its restriction on an m-submanifold, we obtain a sequence of vector field of $\mathbb{C}^m \times \mathbb{R}^m$ where the submanifold sequence is the one determined by the multi-helicoids:

$$\mathrm{Exp}_n := \mathrm{Exp}/X_n(\mathbb{C}^{\#m}) : X_n(\mathbb{C}^{\#m}) \longmapsto \mathbb{R}^{6m}$$

through the canonical identifications

$$T(\mathbb{C}^m \times \mathbb{R}^m) \cong T(\mathbb{R}^{3m}) \cong \mathbb{R}^{6m}.$$

We are now in position to prove the analogous result of Theorem 2.3.

3.6 Theorem. *The limit map of the sequence of vector fields noted above is an injective map on the covering m-manifold $P : \mathbb{C}^{\#m} \mapsto \mathbb{C}^m \smallsetminus \{0\}$ for those z's on the bounded multistrips of $C^{\#m}$, that is*

$$\mathrm{Exp}_n/X_n(B^{+m}) \xrightarrow{n\mapsto\infty} \mathrm{Exp}/\big(P(B^{+m}),0\big) \equiv \mathrm{Exp}/P(B^{+m})$$

$$B^{+m} := \left\{ z \in \mathbb{C}^{\#m} : \sum_{a=1}^{m} |\mathrm{Im}(z^a)| < M, M > 0 \right\}.$$

Proof. The Exp-field restricted onto $X_n(\mathbb{C}^{\#m})$ has the expression:

$$\mathrm{Exp}\big(X_n(z)\big) = \sum_{a=1}^{m} \left\{ \exp(u^a) \exp\big(\exp(iv^a)\big)\xi_a + \exp\left(\frac{1}{n}v^a\right) e_a \right\}.$$

Linear independence of the complex and real directions ξ_a and e_a implies that we may apply Theorem 2.3 in each component $\mathbb{C} \times \mathbb{R}$ to obtain the requested results. Indeed, the a-component of the multi helicoid converges uniformly in the bounded strips of $\mathbb{C} \smallsetminus \{0\}$

$$\big(Re(z^a)e^{i\mathrm{Im}(z^a)}\big)\xi_a + \left(\frac{1}{n}\mathrm{Im}(z^a)\right) e_a \xrightarrow[\cong]{n\mapsto\infty} Re(z^a)e^{i\mathrm{Im}(z^a)}$$

through the natural immersion, in virtue of Proposition 2.2. Besides, the a-component of the (induced) sequence of vector fields is injective on the aforementioned (limit) covering space of $\mathbb{C} \smallsetminus \{0\}$, by virtue of Theorem 2.3:

$$\lim_{n\to\infty} \big(\pi_{\mathbb{C}_a\times\mathbb{R}_a} \circ \big(\mathrm{Exp}/X(B^{+m})\big)\big) = \pi_{\mathbb{C}_a\times\mathbb{R}_a} \circ \big(\mathrm{Exp}/P(B^{+m})\big)$$

and now the conclusion is a simple consequence of the analysis in the previous section. □

3.7 Remark. The method we expounded for this specific construction raises the question if and how it might be modified and applied in other Riemann Surfaces, incorporated in the framework of a General Theory apart from analytic methods.
The problem is focused in the quest of an appropriate sequence, whereon the complex function (properly extended) will become single valued, and the surface will be obtained as a uniform limit. This attractive concept would be of great importance in the finite cover surface of algebraic functions.

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