

# The Riemann Surface of the Logarithm Constructed in a Geometrical Framework

Nikolaos I. Katzourakis\*

University of Athens, Department of Mathematics

e-mail: nkatzourakis@math.uoa.gr

## Abstract

The logarithmic Riemann surface  $\Sigma_{log}$  is a classical holomorphic 1-manifold. It lives into  $\mathbb{R}^4$  and induces a covering space of  $\mathbb{C} \setminus \{0\}$  defined by  $\exp_{\mathbb{C}}$ .

This paper suggests a geometric construction of it, derived as the limit of a sequence of *vector fields extending*  $\exp_{\mathbb{C}}$  suitably to *embeddings of*  $\mathbb{C}$  into  $\mathbb{R}^3$ , which turn to be *helicoid* surfaces living into  $\mathbb{C} \times \mathbb{R}$ . In the limit we obtain a bijective complex exponential on the covering space in question, and thus a well-defined complex logarithm. In addition, the *helicoids* are *diffeomorphic (not bi-holomorphic) copies* of  $\Sigma_{log}$  as  $C^\infty$ -realizations living into  $\mathbb{R}^3$ , without obstruction.

Our approach is purely *geometrical* and *does not employ any tools provided by the complex structure*, thus *holomorphy is no longer necessary to obtain constructively this Riemann surface*  $\Sigma_{log}$ . Moreover, the differential geometric framework we adopt affords explicit generalization on submanifolds of  $\mathbb{C}^m \times \mathbb{R}^m$  and certain corollaries are derived.

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\*Address: Troias 6, 18541, Kaminia, Pireaus, Greece

*"You can not claim you are well aware of a theorem  
unless you have more than one proof for it"*

*M. Atiyah's dictum*

## Introduction.

Riemann surfaces have been, at least conceptually, first introduced by Riemann in his celebrated 1851 PhD dissertation. He considered a surface spread over  $\mathbb{C}$  with several sheets lying over it, in such a way that a complex multi-valued function with branches becomes a true well defined function, mapping each branch to a sheet. His posterity gave a rigorous definition of the (so called) *Concrete Riemann surfaces* as covering spaces of  $\mathbb{C}$  (for an enlightening review see [18]).

Klein was the first to put away the covering space approach and adopted a differential geometric one: he studied complex functions living on a curved surface of some ambient Euclidean space. In fact, it was the first time the *Atlas of holomorphic structure* was introduced. At those times, "Riemann surface" meant compact 2-manifold with an arc-length element  $ds^2$  and bi-holomorphic transition functions in the atlas. Several years later, adding Cantor's 2nd countability and Hausdorff's axioms (not before the 1920's), Radó reaches via triangulations the *Abstract Riemann surfaces'* definition: a *Hausdorff 2nd-countable topological surface with complex structure*.

The equivalence of *Concrete* and *Abstract* Riemann surfaces follows from a theorem of Behnke & Stein (1947–49) [2], improved by Gunning and Narasimhan (1967) [8].

In modern times, the research interest has been transferred to the study of their *Moduli* (*ℓ' Teichmüller*) *spaces*, being in effect sets of equivalence classes of distinct complex structures, modulo the action of orientation preserving diffeomorphism of the surface with certain topological structures [13], as well as to their vasty applications in *theoretical physics*, specifically *M-theory unifying the several String theories* (see e.g. [12]).

In this paper we follow a semi-classical approach, standing on the line between the *abstract* and the *concrete*. Our principal result is that *we recover in a pure differential-geometric fashion results classically obtained via the holomorphic*

*structure itself, bypassing analytic continuation.*

Technically, the employed apparatus consists of a sequence of vector fields  $\{\text{Exp}_n\}_{n \in \mathbb{N}}$  defined on  $\mathbb{C}$  and valued in (the tangent bundle of)  $\mathbb{R}^3$ . Each field  $\text{Exp}_n : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\} \times \mathbb{R}$  (*Def.* 1.3) geometrically is in fact a smooth parametric surface of  $\mathbb{R}^3$  called (exponential) helicoid (*Def.* 1.1). These  $\text{Exp}_n$ 's extend (due to the extra component) the ordinary  $\text{exp}_{\mathbb{C}}$  in such a way that they become  $C^\infty$ -diffeomorphisms onto their images (& embeddings of  $\mathbb{C}$  into  $\mathbb{C} \times \mathbb{R}$ ) removing  $2\pi i$  periodicity.

The  $\text{Exp}_n$ -maps by construction converge to the covering space of the punctured plane  $\text{exp}_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$ . Bijectivity is preserved in the limit allowing a well defined limit logarithm  $\log_{\mathbb{C}}$  to be introduced on the direct limit surface  $\lim_{\longrightarrow} \text{Exp}_n(\mathbb{C}) \cong \text{exp}_{\mathbb{C}}(\mathbb{C})$  (in 2.7) which essentially is its *Riemann surface*  $\Sigma_{\log}$  immersed into the punctured plane. Even though all the countably many sheets are "compressed" to a single sheet, we still have the correspondence of the  $\log_{\mathbb{C}}$ -branches and the sheets of the covering space via the convergence mechanism.

Notwithstanding, we show that each  $\text{Exp}_n$  is  $C^\infty$ -diffeomorphic to  $\Sigma_{\log}$ , when the latter is considered as equipped with the smooth sub-atlas of the holomorphic one coming from the ambient  $\mathbb{R}^4$ -space. The fact that the helicoids can only be diffeomorphic copies of  $\Sigma_{\log}$  and not bi-holomorphic is imposed by topological obstructions due to the (general) non-embeddability of Riemann surfaces into  $\mathbb{R}^3$ .

The last *Section* is devoted to high-dimensional generalizations on multi-helicoid submanifolds of Euclidean  $3m$ -space.

In view of the above, a natural question perhaps arises to the reader: what kind of outstanding property do the helicoids enjoy and what is so special about the  $\text{Exp}$  vector fields? The answer is that they are an *ad hoc* choice in order to provide the desired properties via convergence. There may very well exist even more privileged surfaces of  $\mathbb{R}^3$ , but the smooth realizations of the holomorphic  $\Sigma_{\log}$  into 3-space are *dimension-wise* clearly *optimal*.

## 1. Preliminaries.

We collect a few preparatory results which will be needed in the main part of the paper, noted here for the reader's convenience.

**1.1 Definition. (Helicoid)** The smooth (parametric) surface of the (*exponential*) *helicoid* (see e.g. [17], [19]), is the map  $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$  given by

$$(u, v) \longmapsto (e^u \cos v, e^u \sin v, av), \quad a > 0$$

where  $a$  is a parameter. We consider this as the (global) coordinate system of a 2-manifold imbedded in  $\mathbb{R}^3$ . In complex coordinates we may write

$$X : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R} \quad :$$

$$u + iv \longmapsto (\exp_{\mathbb{C}}(u + iv), av) \equiv (\exp_{\mathbb{C}}(z), a\text{Im}(z))$$

where  $z \equiv u + iv$ . The complex functions *Re*, *Im*,  $\exp_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$  denote the "real part", the "complex part" and the "complex exponential" of  $\mathbb{C}$  respectively.

The terminologies of imbeddings and immersions we follow are the standard ones, referring e.g. to [7], [11].

We employ the convention that "*imbedding*" stands for topological imbedding and "*embedding*" for geometrical embedding in the sense of smooth manifolds.

In this paper *smoothness* means  $C^\infty$  - *smoothness* in the usual geometric sense and diffeomorphism stands for  $C^\infty$ -diffeomorphism.

Furthermore, as usual by the term *smooth n-manifold M* (and in particular, surface) we mean a  $C^\infty$ -smooth connected, paracompact Hausdorff manifold, of real  $n$  dimensions  $\dim_{\mathbb{R}}(M) = n$  (2, in the 2nd case).

**1.2 Remark.** Let now

$$\exp_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\} \quad : \quad z \longmapsto \exp_{\mathbb{C}}(z) \equiv e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

be the ordinary complex exponential function (*the notations  $\exp_{\mathbb{C}}(z)$  and  $e^z$  will be occasionally exchanged without comments*). From the differential-geometric viewpoint, this map may be considered as a vector field, tangent on the 2-dimensional (flat) manifold  $\mathbb{C} \cong \mathbb{R}^2$ , i.e.

$$\mathbb{C} \equiv \mathbb{R}^2 \ni ue_1 + ve_2 \longmapsto (e^u \cos v)e_1 + (e^u \sin v)e_2.$$

This owes to the fact that if  $\xi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$  is a vector field of  $\mathbb{R}^n$  (in the *vector calculus* sense), where  $x \longmapsto \xi(x)$ , then the obvious identification  $\xi(x) \equiv (x, \xi(x))$  is the requested one in order to consider  $\xi$  valued in the tangent bundle  $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$  (vector field in the geometric sense) by global triviality of the vector (tangent in our case) bundle ([11], [7], [20], etc).

The current task is to define a certain extension of the exponential  $\exp_{\mathbb{C}}$  from  $\mathbb{C}$  to  $\mathbb{C} \times \mathbb{R}$  that removes the  $2\pi i$ -periodicity and become injective due to extra real component.

These remarks introduce the idea of the following definition:

**1.3 Definition. (The exponential field)** The exponential vector field on  $\mathbb{C}$  is the map

$$\text{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$$

given by:

$$u + iv \longmapsto \text{Exp}_a(u + iv) := (\exp_{\mathbb{C}}(u + iv), av)$$

where  $a > 0$  is a parameter to be fixed as will in the sequel. Operationally the map can given as

$$\text{Exp}_a = (\exp_{\mathbb{C}}, a \text{Im})$$

The following technical fact shows that this vector-wise extension of the  $\exp_{\mathbb{C}}$  function on  $\mathbb{C}$  is the requested one that provides injectivity of the map, now considered as a vector field.

**1.4 Proposition. (Structural Properties of Exp)**

*The map  $\text{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$  given by 1.3 has the following properties:*

a) *It is a vector field defined on the (trivially) embedded submanifold  $\mathbb{C} \equiv \mathbb{C} \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$  and valued into the (tangent bundle of the) ambient space  $\mathbb{R}^3$ .*

b) *It is a diffeomorphism onto its image, thus an embedding of the plane into  $\mathbb{C} \times \mathbb{R}$  and, in particular, invertible.*

As our context suggests, we consider submanifolds as subsets of  $\mathbb{R}^n$ , where the inclusion map is an imbedding and the Atlas that determines differential structure and topology is the induced from  $\mathbb{R}^n$ .

*Proof.* a) We recall that a vector field defined on a *submanifold*  $M$  of  $\mathbb{R}^n$  is not necessarily tangent to the manifold in the usual differential geometric sense, but the splitting of tangent spaces

$$\mathbb{R}^n \cong T_p \mathbb{R}^n \cong T_p M \oplus T_p^\perp M, \quad p \in \mathbb{R}^n$$

implies that it can be normal, or generally valued in  $T\mathbb{R}^n \cong \mathbb{R}^{2n}$ . The very definition of  $\text{Exp}_a$  implies that it maps a point  $z$  of  $\mathbb{C}$  to a vector  $\text{Exp}_a(z)$  of  $\mathbb{C} \times \mathbb{R}$

$$\mathbb{C} \ni z \longmapsto \text{Exp}_a(z) \in \mathbb{C} \times \mathbb{R}$$

and *Remark* 1.2 implies that the map is a vector field of  $\mathbb{C}$  valued in the tangent bundle of  $\mathbb{C} \times \mathbb{R}$ .

b) If  $\text{Exp}_a(u_1 + iv_1, av_1) = \text{Exp}_a(u_2 + iv_2, av_2)$  for  $u_1, u_2, v_1, v_2 \in \mathbb{R}$ , then this amounts to

$$(\exp_{\mathbb{C}}(u_1 + iv_1), av_1) = (\exp_{\mathbb{C}}(u_2 + iv_2), av_2)$$

that is

$$\exp_{\mathbb{R}}(u_1) \exp_{\mathbb{C}}(iv_1) = \exp_{\mathbb{R}}(u_2) \exp_{\mathbb{C}}(iv_2), \quad v_1 = v_2$$

and consequently  $u_1 = u_2$  by the monotonicity of the real exponential. This shows the injectivity of  $\text{Exp}$ , thus it is bijective onto its image  $\text{Exp}_a(\mathbb{C})$ .  $C^\infty$ -differentiability of this map goes without saying, since it has smooth component functions in  $\mathbb{C} \times \mathbb{R}$ . Smoothness of the inverse map is also obvious (for the explicit expression of the inverse map see *Lemma* 2.6 of the oncoming *Section 2*). Consequently, the map is a smooth (geometric) embedding of  $\mathbb{C}$  into  $\mathbb{C} \times \mathbb{R}$ .  $\square$

Consider now the standard Euclidean (Riemannian) metric  $\delta$  of  $\mathbb{R}^3$  with components the Kronecker deltas'  $\delta_{ab}$  (where  $\delta_{aa} = 1$  and 0 otherwise,  $1 \leq a \leq 3$ ) (for the standard concepts of *Riemannian geometry* employed in this paper, we refer to [5], [7], [11], [17]). Then, the (trivially) embedded surface  $\mathbb{C}$  gets a natural induced *flat* Riemannian metric, the standard inner product  $\langle \cdot, \cdot \rangle \equiv \delta$  of  $\mathbb{R}^3$  by restriction of  $\delta$  on  $\mathbb{C}$ , which can be seen as the pull-back metric via the inclusion map:

$$(\mathbb{C}, \delta|_{\mathbb{C}}) \equiv (\mathbb{C}, i^*(\delta))$$

$$i : \mathbb{C} \hookrightarrow (\mathbb{C} \times \mathbb{R}, \delta)$$

In view of these remarks, we obtain the next result giving some geometric properties of the extended object  $\text{Exp}_a$  as a vector field on the  $(\mathbb{C}, i^*(\delta))$ .

**Notation.**  $\text{angl}(\xi, \eta)(p)$  denotes the angle of 2 vector fields  $\xi, \eta$  measured at the point  $p$  of the surface (manifold), with respect to its Riemannian metric.

**1.5 Proposition. (Geometric Properties of Exp)**

a) The vector field  $\text{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$  geometrically is a smooth exponential helicoid surface of the form 1.1 and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\text{Exp}_a} & \text{Exp}_a(\mathbb{C}) \\ \downarrow \text{exp}_{\mathbb{C}} & & \downarrow i \\ \mathbb{C} \setminus \{0\} & \xleftarrow{\pi_{\mathbb{C} \setminus \{0\}}} & \mathbb{C} \setminus \{0\} \times \mathbb{R} \end{array}$$

that is,  $\text{Exp}_a$  commutes with the projection composed with the inclusion of the surface  $i : \text{Exp}_a(\mathbb{C}) \hookrightarrow \mathbb{C} \setminus \{0\} \times \mathbb{R}$  and the projection  $\pi_{\mathbb{C} \setminus \{0\}} : \mathbb{C} \setminus \{0\} \times \mathbb{R} \longrightarrow \mathbb{C} \setminus \{0\}$

$$(\pi_{\mathbb{C} \setminus \{0\}} \circ i) \circ \text{Exp}_a = \text{exp}_{\mathbb{C}}$$

b) If the tangent plane  $T_{u+iv}\mathbb{C} \cong \mathbb{C} \cong \mathbb{R}^2$  is spanned by  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$  and

$$\xi_{AB} \equiv Ae_1 + Be_2$$

is a general 2-parameter family of tangent vectors,  $A, B \in \mathbb{R}$ ,  $|A| + |B| \neq 0$ , then

$$\cos[(\Phi_{AB})(u, v)] = \frac{e^u [A \cos v + B \sin v]}{\sqrt{e^{2u} + a^2 v^2} \sqrt{A^2 + B^2}}$$

where  $\Phi_{AB}(u, v)$  is the angle of  $\text{Exp}$  with  $\xi_{AB}$  at  $u + iv$  in  $\mathbb{R}^3$ :

$$(\Phi_{AB})(u, v) \equiv \text{angl}(\text{Exp}_a, \xi_{AB})(u + iv)$$

**1.6 Remark.** The calculation of the aforementioned (non-constant) angle shows that the  $\text{Exp}_a$ -vector field is neither tangent nor normal to  $\mathbb{C}$ .

*Proof.* a) The first claim goes without saying, just by comparing the very definitions 1.1 and 1.3. Thus, the map  $\text{Exp}_a$  is a vector field of  $\mathbb{R}^3$  defined on  $\mathbb{C}$  and simultaneously its image constitutes a smoothly embedded surface of  $\mathbb{R}^3$ , an exponential helicoid diffeomorphic to  $\mathbb{C}$ .

Commutativity is a simple consequence of the form of the  $\text{Exp}_a$ -field:

$$(\pi_{\mathbb{C} \setminus \{0\}} \circ i) \circ \text{Exp}_a(z) = (\pi_{\mathbb{C} \setminus \{0\}} \circ (\exp_{\mathbb{C}}, a \text{Im}))(z) = \exp_{\mathbb{C}}(z)$$

b) We recall the familiar Euclidean formula giving the angle of 2 vectors in  $(\mathbb{R}^3, \delta) = (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$

$$\cos [\text{angl}(\text{Exp}_a, \xi_{AB})(u, v)] = \left( \frac{\langle \text{Exp}_a, \xi_{AB} \rangle}{\|\text{Exp}_a\| \|\xi_{AB}\|} \right)(u + iv)$$

The 2-parameter vector (describing a general tangent vector) is given due to the inclusion  $\mathbb{C} \times \{0\} \hookrightarrow \mathbb{C} \times \mathbb{R}$  by the formula

$$\xi_{AB} = Ae_1 + Be_2 + 0e_3 = A(1, 0, 0) + B(0, 1, 0) + (0, 0, 0) = (A, B, 0)$$

and provided that  $\text{Exp}_a = (e^u \cos v, e^u \sin v, av)$  we have

$$\cos[(\Phi_{AB})(u, v)] = \frac{A e^u \cos v + B e^u \sin v}{\sqrt{e^{2u} + a^2 v^2} \sqrt{A^2 + B^2}}$$

which gives the requested formula and this completes the proof.  $\square$

## 2. The Logarithmic Riemann surface.

### (Part I) The Convergence Constructions.

The *infinite-sheeted Riemann surface*  $\Sigma_{\log}$  of the logarithm  $\log_{\mathbb{C}}$  (see [1], [9], [22]), considered as a holomorphic manifold of (complex) dimension  $\dim_{\mathbb{C}}(\Sigma_{\log}) = 1$ , lives into  $\mathbb{C}^2 \cong \mathbb{R}^4$  and can be given as the graph of the function  $e^z = w$

$$\Sigma_{\log} := \{z \in \mathbb{C} / e^z = w\} \subseteq \mathbb{R}^4$$

or by the parametric equations

$$\mathbb{C} \ni z \longmapsto (z, \exp_{\mathbb{C}}(z)) \in \mathbb{C} \times \mathbb{C} \cong \mathbb{C}^2$$



$\Sigma_{log}^{\mathcal{O}}$  denotes the holomorphic manifold  $(\Sigma_{log}, \mathcal{A}_{\Sigma_{log}}^{\mathcal{O}}) \hookrightarrow (\mathbb{R}^4, id)$  with the induced analytic atlas coming from  $\mathbb{R}^4$ . Classically, it is obtained by analytic continuation, extending holomorphically  $\exp_{\mathbb{C}}$  to "small" discs along a disk  $\mathbb{D}(0, 1)$  of  $\mathbb{C}$ . Since the last extension that overlaps with the first does not coincide with it, we take a copy of  $\mathbb{C}$  and proceed continuation to a next sheet lying over the initial one. Continuing ad infinitum, we obtain a covering space of  $\mathbb{C} \setminus \{0\}$  bi-holomorphic to this surface.

Projecting to the second factor, we obtain the covering space map on the punctured plane  $\mathbb{C} \setminus \{0\}$  (equivalence of *concrete* & *abstract* approach, [2], [8]):

$$\pi : \Sigma_{log}^{\mathcal{O}} \subseteq \mathbb{C}^2 \longrightarrow \mathbb{C} \setminus \{0\} : (z, \exp_{\mathbb{C}}(z)) \longmapsto \exp_{\mathbb{C}}(z)$$

In general there is no way to holomorphically imbed this 2-manifold into  $\mathbb{R}^3$ , unless we allow self-intersections, which is no longer a homeomorphism onto its image but may still be an immersion.

Our construction allows, as we shall see, to introduce a well-defined complex logarithm. Note that *all this apparatus manages to bypass the holomorphic structure of this complex manifold and no analytic continuation is anywhere used.*

In this section we give the *geometric* construction of  $\Sigma_{log}$ . As stated in the *Introduction*, it is obtained as a special covering manifold of  $\mathbb{C} \setminus \{0\}$  in the limit of a sequence obtained by the exponential  $\text{Exp}_a$  images of  $\mathbb{C}$ .

Hence, we recall the map  $\text{Exp}_a : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}$  given by 1.3 and substitute the positive parameter by the sequence  $a_n = 1/n$ ,  $n \in \mathbb{N}$ . Thus, we obtain a sequence of vector fields (and surfaces of  $\mathbb{R}^3$  as well)  $\{\text{Exp}_n\}_{n \in \mathbb{N}}$ , where

$$\text{Exp}_n : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R} : z \longmapsto \left( \exp_{\mathbb{C}}(z), \frac{1}{n} \text{Im}(z) \right), \quad n \in \mathbb{N}$$

The following result is a well-known fact and can be found in any elementary textbook, e.g. [15]. For the definition of *covering manifolds* we refer to [7], [21].

**2.1 Lemma.** *The pair  $(\exp_{\mathbb{C}}, \mathbb{C})$ , constitutes a covering manifold of  $\mathbb{C} \setminus \{0\}$  with countably infinite covering sheets.*

We are now in position to prove the main result of this section. The sequence of maps  $\{\text{Exp}_n : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{R}\}_{n \in \mathbb{N}}$  is a diffeomorphism onto its image, by *Proposition* 1.4, for  $a = 1/n$ ,  $n \in \mathbb{N}$  and consequently a bijection.

We shall presently see that this property is preserved in the limit  $n \rightarrow \infty$ . The limit surface is the covering space produced by  $\exp_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$  (Lemma 2.1), and introducing the natural inclusion  $\mathbb{C} \setminus \{0\} \hookrightarrow \mathbb{C} \setminus \{0\} \times \mathbb{R}$  we may identify  $\varinjlim_{n \rightarrow \infty} (\text{Exp}_n)$  on  $\mathbb{C}$  with the usual complex exponential,  $\exp_{\mathbb{C}}$ . Thus, we construct an exponential injective on  $\mathbb{C}$ .

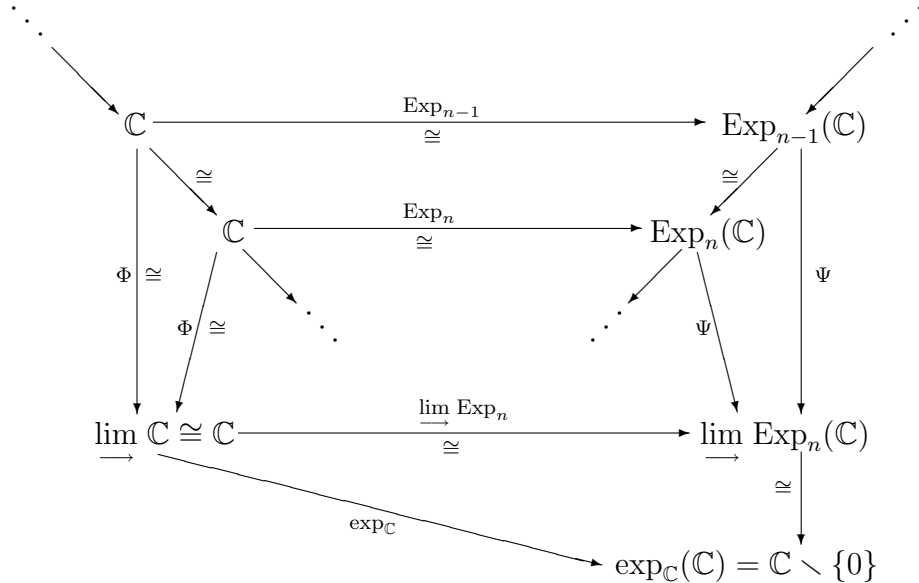
This situation resembles and simultaneously extends the analogous case in elementary complex analysis (see e.g. [1]), where  $\exp_{\mathbb{C}}$  is injective and invertible on the strips of  $2\pi i$ -width only.

**2.2 Theorem. (Convergence to Injective Complex Exponential)**

The sequence of diffeomorphisms  $\{\text{Exp}_n : \mathbb{C} \xrightarrow{\cong} (\text{Exp}_n)(\mathbb{C}) \subseteq \mathbb{C} \setminus \{0\} \times \mathbb{R}\}_{n \in \mathbb{N}}$  converges (point-wise) to a diffeomorphism  $\varinjlim_{n \rightarrow \infty} \text{Exp}_n$  on the covering surface of  $\mathbb{C} \setminus \{0\}$  produced by  $\exp_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$ :

$$\text{Exp}_n \xrightarrow[pw]{n \rightarrow \infty} \exp_{\mathbb{C}} \times \{0\} \equiv \exp_{\mathbb{C}}$$

and the lower triangle of the following diagram defines (in the limit) a bijective complex exponential  $\exp_{\mathbb{C}}$ .



**2.3 Remarks.** Before giving the proof of this result which essentially contains the requested construction, some explanatory comments are necessary. First of all, the direct (inductive) limits " $\varinjlim$ " (for their definition in a general *Category* we refer to the classical textbook [14]) are justified as follows:

The constant sequence of smooth manifolds

$$\mathbb{C} \xrightarrow[\cong]{id} \mathbb{C} \xrightarrow[\cong]{id} \mathbb{C} \longrightarrow \dots$$

defines in a completely trivial way a direct limit set which is  $\mathbb{C}$ , since  $id \circ id = id$ . In addition,

$$\lim_{\longrightarrow} \mathbb{C} = \lim_{n \rightarrow \infty} \mathbb{C} = \mathbb{C}$$

that is, the direct limit coincides with the usual sequence convergence. Similarly, we have the sequence

$$\text{Exp}_n(\mathbb{C}) \xrightarrow[\cong]{\Theta_{n,n+1}} \text{Exp}_{n+1}(\mathbb{C}) \xrightarrow[\cong]{\Theta_{n+1,n+2}} \text{Exp}_{n+2}(\mathbb{C}) \longrightarrow \dots$$

where the diffeomorphisms  $\Theta_{n,m}$ ,  $n, m \in \mathbb{N}$  are defined as

$$\Theta_{n,m} : \text{Exp}_n(\mathbb{C}) \longrightarrow \text{Exp}_m(\mathbb{C})$$

by the formula

$$\begin{aligned} (\exp_{\mathbb{C}}(z), \frac{1}{n}Im(z)) &\longmapsto \Theta_{n,m}(\exp_{\mathbb{C}}(z), \frac{1}{n}Im(z)) \\ &:= (\exp_{\mathbb{C}}(z), [\frac{n}{m}] \frac{1}{n}Im(z)) \end{aligned}$$

The direct limit properties can be easily verified, since by construction of the  $\Theta$ -maps we get

$$\Theta_{n,m} \circ \Theta_{m,k} = \Theta_{n,k}$$

as well as

$$\Theta_{n,n} = id_{\text{Exp}_n(\mathbb{C})}$$

This implies that

$$\lim_{\longrightarrow} \text{Exp}_n(\mathbb{C}) = \lim_{n \rightarrow \infty} \text{Exp}_n(\mathbb{C}) = \exp_{\mathbb{C}}(\mathbb{C}) \times \{0\} \cong \mathbb{C} \setminus \{0\}$$

the  $\Phi$ -maps pictured in the diagram are the identity (up to diffeomorphism) and the  $\Psi$ -maps are the projections on  $\mathbb{C} \setminus \{0\}$  composed with the inclusions of  $\text{Exp}_n(\mathbb{C})$  into  $\mathbb{C} \times \mathbb{R}$ :

$$\Psi := \pi_{\mathbb{C} \setminus \{0\}} \circ i$$

*Proof.* *Proposition 1.4* implies that  $\text{Exp}_n$  is injective,  $n \in \mathbb{N}$ . Consequently, we have

$$\text{Exp}_n(z) = \text{Exp}_n(w) \implies z = w$$

Our task is to prove the following condition:

$$\exp_{\mathbb{C}}(z) = \exp_{\mathbb{C}}(w) \implies z = w$$

when  $z, w$  live into  $\mathbb{C}$  and by *a)* of *Proposition 1.5* we obtain

$$\text{Exp}_n \xrightarrow{n \rightarrow \infty} \exp_{\mathbb{C}} \times \{0\} \equiv \exp_{\mathbb{C}}$$

having used that  $(1/n) \xrightarrow{n \rightarrow \infty} 0$ .

The well-known triangle inequality gives

$$\begin{aligned} \|\text{Exp}_n(z) - \text{Exp}_n(w)\| &\leq \|\text{Exp}_n(z) - (\exp_{\mathbb{C}}(z), 0)\| + \\ &+ \|\text{Exp}_n(w) - (\exp_{\mathbb{C}}(w), 0)\| + |\exp_{\mathbb{C}}(w) - \exp_{\mathbb{C}}(z)| \end{aligned}$$

When  $n \rightarrow \infty$ , we have

$$\|\text{Exp}_n(z) - (\exp_{\mathbb{C}}(z), 0)\| \xrightarrow{n \rightarrow \infty} 0$$

as well as

$$\|\text{Exp}_n(w) - (\exp_{\mathbb{C}}(w), 0)\| \xrightarrow{n \rightarrow \infty} 0$$

$$\|\text{Exp}_n(z) - \text{Exp}_n(w)\| \leq |\exp_{\mathbb{C}}(z) - \exp_{\mathbb{C}}(w)| \xrightarrow{n \rightarrow \infty} 0$$

which means that in the limit  $z = w$  by the injectivity assured by 1.4. This completes the proof.  $\square$

**2.4 Corollary.** *The convergence is uniform on the bounded strips of  $\mathbb{C}$  with bounded imaginary part  $B \equiv \{z \in \mathbb{C} / \text{Im}(z) < M, M > 0\}$ , that is, for finite spirals of the  $\text{Exp}_n$ -helicoids:*

$$\text{Exp}_n|_B \xrightarrow[U]{n \rightarrow \infty} \exp_{\mathbb{C}} \times \{0\}|_B \equiv \exp_{\mathbb{C}}|_B$$

for any  $M > 0$ .

**2.5 Remark.** The bounded strips of the form of  $B$  corresponds in the limit to a finite-sheeted covering space of  $\mathbb{C} \setminus \{0\}$ :

$$\exp_{\mathbb{C}}|_B : B \longrightarrow \mathbb{C} \setminus \{0\}$$

produced by the complex exponential.

*Proof.* In order to see when the convergence of  $\text{Exp}_n$  to  $\exp_{\mathbb{C}}$  is uniform, let  $\varepsilon > 0$ . Then, *Th.* 2.2 implies (using *a*) of *Prop.* 1.5)

$$\|\text{Exp}_n(z) - (\exp_{\mathbb{C}}(z), 0)\| = \|(\exp_{\mathbb{C}}(z), \frac{1}{n}\text{Im}(z)) - (\exp_{\mathbb{C}}(z), 0)\| = \frac{1}{n}|\text{Im}(z)|$$

Imposing  $|\text{Im}(z)| < M$  bounded, we obtain uniform convergence as claimed.  $\square$

The injectivity of the  $\text{Exp}$  provided by *Proposition* 1.4 gives the ability to introduce the inverse map (which we shall denote by  $\text{Log}$ ) as a well-defined generalization of the complex logarithm:

$$\text{Log}_n : \text{Exp}_n(\mathbb{C}) \subseteq \mathbb{C} \setminus \{0\} \times \mathbb{R} \longrightarrow \mathbb{C}$$

where

$$\text{Log}_n := \text{Exp}_n^{-1}, \quad n \in \mathbb{N}$$

**2.6 Lemma.** *The inverse  $\text{Log}_n$  maps are given by*

$$(K, L) \longmapsto \text{Log}_n(K, L) = \ln(|K|) + inL$$

(  $\ln$  denotes the real Napierian logarithm ).

*Proof.* The proof is a calculation. Set

$$\exp_{\mathbb{C}}(z) \equiv K, \quad \frac{1}{n}\text{Im}(z) \equiv L$$

Then,

$$z = \log_{\mathbb{C}}(K) = \ln(|K|) + i \arg(K)$$

Comparing the last relation with the formula  $\log_{\mathbb{C}}(K) = \ln(|\text{Re}(z)|) + inL$  we conclude to the requested relation.  $\square$

We are now in position to introduce a well-defined complex logarithm on the limit covering space produced by the exponential.

**Notational convention:** In the sequel we adopt the following convention:  $\Sigma_{log}^{C^\infty}$  denotes  $\Sigma_{log}$  equipped with the induced smooth sub-atlas of the holomorphic one coming from  $\mathbb{R}^4$ :

$$\Sigma_{log}^{C^\infty} \equiv (\Sigma_{log}, \mathcal{A}_{\Sigma_{log}}^{C^\infty}) \hookrightarrow (\Sigma_{log}, \mathcal{A}_{\Sigma_{log}}^O) \subseteq (\mathbb{R}^4, Id)$$

**2.7 Corollary. (Convergence to a well-defined  $\log_{\mathbb{C}}$  on its Riemann Surface)**

The sequence of maps  $(\text{Log}_n)_{n \in \mathbb{N}}$  of 2.6 converges point-wise to a well-defined complex logarithm and uniformly on the subsets of  $\text{Exp}_n(\mathbb{C}) \subseteq \mathbb{C} \setminus \{0\} \times \mathbb{R}$  of the form  $\text{Exp}_n(\mathbb{C}) \cap \mathbb{C} \setminus \{0\} \times (a, b)$ :

$$\text{Log}_n \xrightarrow{n \rightarrow \infty} (\exp_{\mathbb{C}} \times \{0\})^{-1} \equiv \log_{\mathbb{C}}|_{\exp_{\mathbb{C}}(\mathbb{C})}$$

and the limit surface is diffeomorphic to the (immersed into  $\mathbb{C} \setminus \{0\}$ ) Riemann Surface of the logarithm:

$$\pi_{\mathbb{C}}(\Sigma_{log}^{C^\infty}) \cong \varinjlim (\text{Exp}_n(\mathbb{C})) = \exp_{\mathbb{C}}(\mathbb{C}) \equiv \mathbb{C} \setminus \{0\}$$

*Proof.* Every set contained into  $\text{Exp}_n(\mathbb{C})$  (for any  $n \in \mathbb{N}$ ) of this form with bounded height (that is,  $\mathbb{R}$ -component) is contained into a sub-helicoid  $\text{Exp}_n(B) \subseteq \text{Exp}_n(\mathbb{C})$  of finite height, thus under its inversed image via  $\text{Exp}_n$  into a bounded strip  $B$  of  $\mathbb{C}$ , for some  $M > 0$ .

The diffeomorphism comes from the previous *Th.* 2.2 and the construction of injective complex exponential on  $\exp_{\mathbb{C}}(\mathbb{C})$ , as well as the remarks in the beginning of the present *Section* concerning the covering space approach. We note that the covering space map coincides with the immersion map into  $\mathbb{C} \setminus \{0\}$ .  $\square$

**(Part II) The Realization Theorem.**

We have not so far shown the explicit relation between the surfaces  $\text{Exp}_a(\mathbb{C})$  and  $\Sigma_{log}$ .

We recall that we manage to construct an injective  $\exp_{\mathbb{C}}$  and thus a well-defined  $\log_{\mathbb{C}}$  on  $\text{Exp}_a(\mathbb{C}) \subseteq \mathbb{R}^3$  via convergence of helicoids. Taking into consideration the initial "defining characterization" of B. Riemann himself on the surfaces of multi-valued holomorphic functions (as the one of  $\log_{\mathbb{C}}$ ), "surface onto which the function considered becomes single-valued" (see also [18]), we conclude that  $\text{Exp}_a(\mathbb{C})$  and  $\Sigma_{\log}$  must necessarily be different representations of the "same" object, but the former living into  $\mathbb{R}^3$  instead of the latter which lives into  $\mathbb{R}^4$ .

This indeed turns to be the case, where "same" is expounded due to the dimensional reduction to a realizable level (of 3-dimensions) as follows:

**2.8 Theorem. (The Realization of  $\Sigma_{\log}$  into  $\mathbb{R}^3$ .)** *The following diffeomorphism holds true*

$$\Xi : \text{Exp}_a(\mathbb{C}) \xrightarrow{\cong} \Sigma_{\log}^{C^\infty}$$

Thus, the (exponential) helicoid is  $C^\infty$ -diffeomorphic to the Logarithmic Riemann surface, when the latter is equipped with the induced smooth sub-atlas  $\mathcal{A}_{\Sigma_{\log}}^{C^\infty}$  of the holomorphic  $\mathcal{A}_{\Sigma_{\log}}^{\mathcal{O}}$  coming from  $\mathbb{R}^4$ .

The diffeomorphism  $\Xi$  will be constructed in the proof. As a consequence, 2.8 implies that we can equip the helicoid surface  $\text{Exp}_a(\mathbb{C})$  with a holomorphic structure via the bijective correspondence of it with  $\Sigma_{\log}$ , that is, if  $(U, \phi)$  is a holomorphic chart of  $\mathcal{A}_{\Sigma_{\log}}^{\mathcal{O}}$ , then define an atlas of  $\text{Exp}_a(\mathbb{C})$  as follows

$$\mathcal{A}_{\text{Exp}_a(\mathbb{C})}^{\mathcal{O}} := \{(\Xi^{-1}(U), \Xi^{-1} \circ \phi) \mid (U, \phi) \in \mathcal{A}_{\Sigma_{\log}}^{\mathcal{O}}\}$$

Of course, this holomorphic atlas of the helicoid does not coincide with its induced smooth atlas coming from the ambient 3- space.

*Proof.*  $\text{Exp}_a(\mathbb{C})$  is given as the following subset of  $\mathbb{R}^3$ :

$$\{(e^u \cos v, e^u \sin v, av) \mid u, v \in \mathbb{R}\}$$

and  $\Sigma_{\log}$  as the subset of  $\mathbb{R}^4$

$$\{(u, v, e^u \cos v, e^u \sin v) \mid u, v \in \mathbb{R}\}$$

Define a map  $\Xi$  as

$$\Xi : (e^u \cos v, e^u \sin v, av) \longmapsto (u, v, e^u \cos v, e^u \sin v)$$

Simple algebraic manipulations show that this map is a bijection with inverse the projection onto  $\mathbb{R}^3$  composed with a rigid motion

$$\Omega \equiv \Xi^{-1} : (u, v, e^u \cos v, e^u \sin v) \longmapsto (e^u \cos v, e^u \sin v, av)$$

both maps can be easily seen to have smooth components and this completes the proof. □

### 3. Multi-dimensional Generalizations.

In this section we present the reasonable high-dimensional analogues of the results presented in the previous *Sections*. Notwithstanding, the new context suggests that the results will no longer stand in the region of Riemann Surfaces, but will be of a more general differential-geometric nature.

The following definitions arise naturally from the respective 1.1, 1.3 of the *Section 1*.

**3.1 Definition. (Multi-Exponential field)** The (*exponential*) *multi-helicoid* is (in complex coordinates) the map ( $a_1, \dots, a_m > 0$  parameters):

$$\text{Exp}_{a_1, \dots, a_m} : \mathbb{C}^m \longrightarrow (\mathbb{C} \setminus \{0\})^m \times \mathbb{R}^m :$$

defined as

$$(z_1, \dots, z_m) \longmapsto (\exp_{\mathbb{C}}(z_1), \dots, \exp_{\mathbb{C}}(z_m); a_1 \text{Im}(z_1), \dots, a_m \text{Im}(z_m))$$

that is, (up to a natural identification  $\mathbb{C}^m \times \mathbb{R}^m \cong (\mathbb{C} \times \mathbb{R})^m$ )

$$\text{Exp}_{a_1, \dots, a_m} := \prod_{1 \leq k \leq m} \text{Exp}_{a_k}$$

This is a vector field by 1.4 (used component-wise), defined on the (flat) trivially embedded submanifold  $\mathbb{C}^m$  of  $\mathbb{R}^{3m}$  and valued in (the tangent bundle of) the ambient  $\mathbb{R}^{3m}$  by 1.2.

Using 1.5,  $\text{Exp}_{a_1, \dots, a_m}(\mathbb{C}^m)$  is a globally coordinated smooth submanifold of (real) dimension  $\dim_{\mathbb{R}}(\text{Exp}_{a_1, \dots, a_m}(\mathbb{C}^m)) = 2m$ . In fact, a **multi-helicoid**, in complete analogy with 1.1.



The proofs of the following results go without saying using the respective results of the previous sections, so we will not bother showing anything explicitly. A component-wise argument suffices in all cases, due to the product form of the definition 3.1.

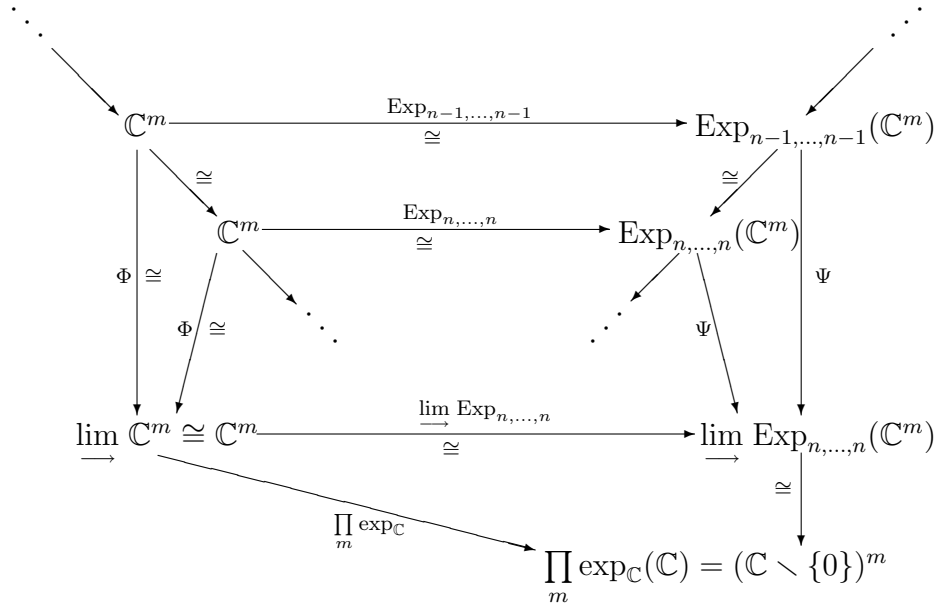
If we set  $a_1 = \dots = a_m = 1/n$ , we obtain a multi-sequence of vector fields and in turn of multi-helicoid submanifolds of  $\mathbb{R}^{3m}$ :

$$\text{Exp}_{n,\dots,n}(z_1, \dots, z_m) := (\exp_{\mathbb{C}}(z_1), \dots, \exp_{\mathbb{C}}(z_m); \frac{1}{n} \text{Im}(z_1), \dots, \frac{1}{n} \text{Im}(z_1))$$

We are now in position to present the analogous result of Theorem 2.2.

**3.2 Theorem. (Convergence to Injective multi-Exponential)**

The sequence of exponential vector fields  $(\text{Exp}_{n,\dots,n})_{n \in \mathbb{N}}$  converges point-wise to a  $C^\infty$ - diffeomorphism  $\prod_m \exp_{\mathbb{C}} : \mathbb{C}^m \longrightarrow (\mathbb{C} \setminus \{0\})^m :$



The convergence is uniform on the bounded multi-strips  $B^m \equiv \{z_k \in \mathbb{C} : |\text{Im}(z_k)| < M_k / M_k > 0, 1 \leq k \leq m\} \subseteq \mathbb{C}^m$ .

$$\text{Exp}_{n,\dots,n}|_{B^m} \xrightarrow[U]{n \rightarrow \infty} \prod_m \exp_{\mathbb{C}} \times \{0\}|_B \equiv \prod_m \exp_{\mathbb{C}}|_B$$

**3.3 Remark.** The methods we expounded for these specific construction raise the question if and how they may be extended and applied to other surfaces, in the framework of a general approach that would, at least, be in position to give back the already known facts concerning the classical Riemann surfaces, e.g. those of  $\log_{\mathbb{C}}(z)$  and  $\sqrt[n]{z}$  which, the latter, more or less constituted the beginnings of the subject of the study of holomorphic 1-manifolds.

Our problem is focused in the quest of an appropriate sequence of surfaces in  $\mathbb{R}^3$  (if any), whereon the complex function in question (properly extended as a vector field) will become single valued, and the covering surface will be obtained in a uniform limit.

The above attractive concept might be of significant importance in the finite-sheeted Riemann surfaces of algebraic functions satisfying the general formula  $a_n(z)[f(z)]^n + a_{n-1}(z)[f(z)]^{n-1} + \dots + a_0(z) = 0$ .

*In a nutshell, recovering via differential geometry pure analytic information concerning complex functions through their Riemann surfaces would give a measure of the amount that (the latter) in fact depend on the holomorphic structure of the plane and, more generally, of every complex manifold modeled upon it.*

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