An analytic result in a star-shaped domain

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Abstract

A star-shaped domain is constructed that admits non-constant stable steady-state solutions to a semilinear diffusion equation.

1 Introduction

In the following we will discuss some known results on the behavior of solutions to semilinear diffusion equations. We will be concerned with the stability of solutions in different classes of domains. As an application of a known theorem, we will construct a star-shaped domain that admits non-constant stable solutions to a steady-state problem. Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem where ν denotes the outward normal derivative:

$$\begin{aligned} \tfrac{\partial u}{\partial t} &= \Delta u + f(u) \qquad x \in \Omega,\ t > 0, \\ \tfrac{\partial u}{\partial \nu} &= 0 \qquad x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x) \qquad x \in \Omega. \end{aligned} \tag{1}$$

It is known that, for any bounded continuous initial condition, a unique solution to (1) exists whenever t is small.

For each $t \geq 0$, we define an operator $Q(t)$ on $L_\infty(\Omega) \cap C(\Omega)$ by

$$[Q(t)\omega](x) = u(x,t)$$

where u is the solution of (1) with initial condition $u_0 = \omega$. A function ω belongs to the domain of $Q(t')$ if and only if $Q(\cdot)\omega$ can be extended as a solution of (1) for some $t > t'$.

When studying problems of the form (1), we are interested in the long-term behavior of solutions. A function $v = v(x)$ is said to be a steady-state solution of (1) if it satisfies the following boundary value problem:

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$$\Delta v + f(v) = 0 \quad x \in \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad x \in \partial\Omega, \tag{2}$$

Suppose v is a solution to (2), then v is also a solution to (1), where the initial condition u_0 equals v. We are interested in the behavior of solutions to (1) when the initial condition u_0 is close (in the $L_\infty(\Omega)$ sense) to v. If $\|u_0 - v\|_{L_\infty(\Omega)}$ is small, we would like to know whether or not $Q(t)u_0$ remains close to v for all $t > 0$. This motivates the following definition of stability.

Definition 1 A solution v of (2) is said to be *stable* if given any $\epsilon > 0$, there exists a $\delta > 0$ so that

$$\|Q(t)\psi - v\|_{L_\infty(\Omega)} < \epsilon, \quad 0 < t < \infty \tag{3}$$

for any $\psi \in L_\infty(\Omega) \cap C(\Omega)$ satisfying $\|\psi - v\|_{L_\infty(\Omega)} < \delta$.

In the next section we will present some background on the stability of solutions to (2).

2 Background

Consider the following conditions on the initial-boundary value problem:

(i) Ω is bounded and its boundary is sufficiently smooth, say of class C^3.

(ii) $f = f(u)$ is of class C^2.

We are interested in determining which classes of domains admit non-constant stable solutions to (2). The following theorem from [1] answers this question in the case where Ω is convex:

Theorem 1 *Let the above conditions hold. If Ω is convex and v is a non-constant solution of* (2)*, then v is unstable.*

In [1] sufficient conditions are given on Ω and f which guarantee the existence of non-constant stable solutions to (2). Let the nonlinear term f be of the form $f(u) = k\, g(u)$ where g is a function of class C^2 satisfying:

$$\begin{aligned} g(a) = g(0) = g(b) &\quad \text{for some } a < 0 < b, \\ u \le g(u) < 0 &\quad \text{for } a < u < 0, \\ 0 < g(u) \le u &\quad \text{for } 0 < u < b. \end{aligned} \tag{4}$$

Let $G(u) = \int_0^u g(w)dw$. Then G is non-negative on $a \le u \le b$ and achieves a maximum at $u = a$ or $u = b$. We make the additional assumption that $G(a) = G(b)$. Let Ω be a bounded domain in $\mathbf{R}^n$ with a sufficiently smooth boundary. Assume Ω satisfies the following conditions (P):

(1) D_1 and D_2 are subdomains of Ω with smooth boundaries in which Poincaré's second inequality holds.

(2) There is a component of $\Omega \cap \{x : -\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}\}$, denoted by D_3, such that $D \setminus D_3$ is divided into disjoint open sets O_1 and O_2 containing D_1 and D_2 respectively and satisfying $\partial D_3 \cap \partial O_1 \subset \{x : x_1 = -\frac{\ell}{2}\}$ and $\partial D_3 \cap \partial O_2 \subset \{x : x_1 = \frac{\ell}{2}\}$.

(3) The $(n-1)$-dimensional measure of the intersection of D_3 and the hyperplane $\{x : x_1 = \xi\}$ does not exceed S for $-\frac{l}{2} \leq \xi \leq \frac{l}{2}$.

Define $R[-,+] \subset C^1(\overline{\Omega}) \cap C^2(\Omega)$ as follows:

$$R[-,+] = \{w : \, a \leq w \leq b \text{ on } \overline{\Omega}, \, \int_{D_1} w(x)dx < 0, \, \int_{D_2} w(x)dx > 0\}.$$

It is clear that any function belonging to $R[-,+]$ is non-constant. Let μ denote the Lebesgue measure in $\mathbf{R}^n$. Let $\lambda_2(D_i)$ be the second eigenvalue of $-\Delta$ on D_i under the Neumann boundary conditions. Finally, define $\epsilon_0 > 0$ so that

$$\epsilon_0 = G(b) \cdot min \{\mu(D_1) \cdot min\{k, \lambda_2(D_1)\}, \, \mu(D_1) \cdot min\{k, \lambda_2(D_1)\}\}.$$

We have the following result from [1]:

Theorem 2 *Suppose the above conditions hold. If*

$$\left\{ \frac{(b-a)^2}{2\,\ell} + k\, G(b)\, \ell \right\} S \leq \epsilon_0, \tag{5}$$

then (2) *has at least one stable solution belonging to* $R[-,+]$.

3 An application to a star-shaped domain

In this section we construct a star-shaped domain in $\mathbf{R}^2$ that allows us to apply theorem 2.

Let $f(u) = k\, g(u)$ where g is a C^2 function satisfying (4) and $G(b) = G(a)$. We will fix a length $\ell > 0$ and a radius $R > 0$, and then we will find a $\delta > 0$ so that the domain Ω depicted below admits a non-constant stable solution to (2). We construct Ω so that its boundary is smooth.

Let D_1 and D_2 be open disks of radius R, then Ω satisfies (P.1). Since the second eigenvalue of the Neumann problem in a disk is positive, it follows that $\lambda_2(D_1) = \lambda_2(D_2) > 0$. We let $D_3 = \Omega \cap \left\{x : -\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}\right\}$. Then Ω satisfies (P.2) where $O_1 = \{x \in \Omega : x_1 < -\frac{\ell}{2}\}$ and $O_2 = \{x \in \Omega : x_1 > \frac{\ell}{2}\}$. For each $\xi \in \mathbf{R}$, the length of $D_3 \cap \{x : \, x_1 = \xi\}$ is either 0 or δ. Letting $S = \delta$, we see that Ω satisfies (P.3). Therefore, Ω satisfies the conditions (P). By construction, we see that Ω is star-shaped with respect to the origin.

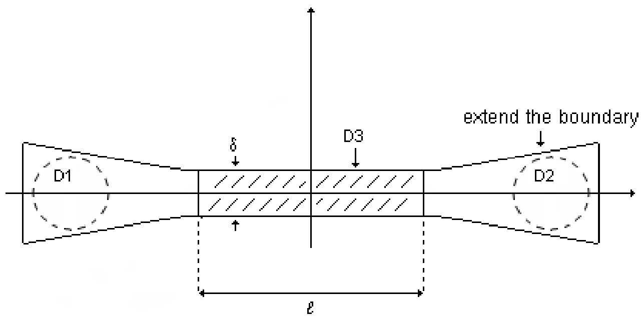

Figure 1: $\Omega \subset \mathbf{R}^2$

Fix $\ell > 0$ and $R > 0$. Since $\lambda_2(D_1) = \lambda_2(D_2) > 0$, we can choose k so that $0 < k < \lambda_2(D_1)$. Then $\epsilon_0 = k\,G(b)\pi R^2$. Choose $\delta > 0$ so that:

$$\delta < \frac{kG(b)\pi R^2}{\frac{(b-a)^2}{2\,\ell} + k\,G(b)\,\ell}.$$

Then,

$$\begin{aligned} \left\{\tfrac{(b-a)^2}{2\ell} + kG(b)\ell\right\} S &= \left\{\tfrac{(b-a)^2}{2\ell} + kG(b)\ell\right\} \delta \\ &< \left\{\tfrac{(b-a)^2}{2\ell} + kG(b)\ell\right\} \frac{kG(b)\pi R^2}{\frac{(b-a)^2}{2\,\ell} + k\,G(b)\,\ell} \\ &= k\,G(b)\pi R^2 \\ &= \epsilon \end{aligned} \tag{6}$$

Under the above assumptions, we see that the hypotheses of theorem 2 are satisfied. Therefore, Ω is a star-shaped domain that admits a non-constant stable solution to (2).

References

[1] Matano, H., Asymptotic Behavior and Stability of Solutions of Semilinear Diffusion Equations, *Publ. RIMS, Kyoto Univ.* **15** (1979), 401-454.