Non-constant stable solutions to reaction-diffusion equations in star-shaped domains

Gregory Drugan[*][†]

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Abstract

In the following we will discuss some known results on the behavior of solutions to reaction-diffusion equations. We will be concerned with the stability of steady-state solutions in different classes of domains. A result in [1] states that for convex domains, every non-constant stable steady-state solution to the reaction-diffusion equation (2) is unstable. As an application of a theorem in [1], we show that this result for convex domains does not generalize to the larger class of star-shaped domains.

1 Introduction

Reaction-diffusion equations model a variety of scientific phenomena such as chemical reactions, heat conduction, electron flow, and population dynamics. For example, if we let $u(x,t)$ represent the concentration of a chemical substance in a spatial domain Ω, at a time $t > 0$, then we can model the effects of chemical reactions and diffusion on $u(x,t)$ with a partial differential equation. Often times it is of heuristic interest to understand the long-term behavior of the solutions to these models: do they approach a steady-state as time goes to infinity? do the steady-state solutions exhibit stability?

Considering the chemical model, we intoduce a function $f(x,t,u)$ to account for the change in $u(x,t)$ per unit time caused by chemical reactions. We also define the flux density $\mathbf{\Phi}(x,t)$ of the chemical as the rate of flow of $u(x,t)$ per unit time caused by diffusion. Supposing these are the only factors affecting $u(x,t)$, we find that the rate of change of the total quantity of chemical substance in Ω equals the total flux entering the region plus the total amount of substance generated by the reaction term:

$$\frac{\partial}{\partial t}\int_{\Omega} u(x,t)dx = -\int_{\partial\Omega} \mathbf{\Phi}(x,t)\cdot\nu(x)dS + \int_{\Omega} f(x,t,u)dx,$$

[*]University of Texas at Austin

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where $\nu(x)$ denotes the unit outward normal. Moving the differentiation inside and applying the divergence formula to $\mathbf{\Phi}(x,t)$, we get the following general reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = -div(\mathbf{\Phi}(x,t)) + f(x,t,u).$$

According to Fick's law of diffusion, the flux density points in the direction where the density decreases most rapidly:

$$\mathbf{\Phi}(x,t) = -k(x)\nabla u(x,t).$$

This means that regions of higher concentration flow to regions of lower concentration.

We will assume that the diffusion is homogeneous so that $k(x) \equiv k > 0$ is constant. If we replace t by kt, which is just a rescaling of time, then the constant k is absorbed into the reaction term f. So we can choose the constant k to be 1. We will also assume that the reaction term $f(x,t,u) = f(u)$ depends only on u. Then we have the following reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u + f(u). \tag{1}$$

In order to solve (1), initial and boundary conditions must be prescribed. We will assume that the system is closed, so that the flux across the boundary is zero (homogeneous Neumann boundary condition):

$$\frac{\partial u}{\partial \nu} \equiv \nabla u(x,t) \cdot \nu(x) = 0.$$

In Sections 3 and 4 we will discuss some results on the stability of steady-state solutions to (1) with the above boundary condition. In Section 5, as an application of a theorem in [1], we construct a class of star-shaped domains in $\mathbf{R}^2$ that admit non-constant stable steady-state solutions to the initial-boundary value problem (2). In Section 6, we generalize this result to $\mathbf{R}^n$.

2 Definition of stability

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem where ν denotes the unit outward normal:

$$\begin{aligned} \tfrac{\partial u}{\partial t} &= \Delta u + f(u) \qquad x \in \Omega,\, t > 0, \\ \tfrac{\partial u}{\partial \nu} &= 0 \qquad x \in \partial\Omega,\, t > 0, \\ u(x,0) &= u_0(x) \qquad x \in \Omega. \end{aligned} \tag{2}$$

It is known that, for any bounded continuous initial condition, a unique solution to (2) exists whenever t is small and $f(u)$ is of class C^1 (see [2] for a discussion on local existence and regularity for parabolic operators). For each $t \geq 0$, we can define an operator $Q(t)$ on $L_\infty(\Omega) \cap C(\Omega)$ by

$$[Q(t)\omega](x) = u(x,t)$$

where u is the solution of (2) with initial condition $u_0 = \omega$. A function ω belongs to the domain of $Q(t')$ if and only if $Q(\cdot)\omega$ can be extended as a solution of (2) for some $t > t'$.

We call a function $v = v(x)$ a steady-state solution of (2) if it is a solution of the following boundary value problem:

$$\begin{aligned} \Delta v + f(v) = 0 \quad & x \in \Omega, \\ \tfrac{\partial v}{\partial \nu} = 0 \quad & x \in \partial\Omega. \end{aligned} \tag{3}$$

When modeling scientific phenomena, we are often interested in the long-term behavior of the solutions, so we would like to know if a solution to (2) blows up in finite time. If a solution $u(x,t)$ of (2) approaches a steady-state as t goes to infinity, then $u(x,t)$ cannot blow up in finite time. However, a solution of (2) may be close (in the $L_\infty(\Omega)$ sense) to a steady-state solution at some time t_0 and still blow up in finite time.

Definition 1 A solution v of (3) is said to be *stable* if given any $\epsilon > 0$, there exists a $\delta > 0$ so that

$$\|Q(t)\psi - v\|_{L_\infty(\Omega)} < \epsilon, \quad 0 < t < \infty \tag{4}$$

for any $\psi \in L_\infty(\Omega) \cap C(\Omega)$ satisfying $\|\psi - v\|_{L_\infty(\Omega)} < \delta$.

This says that a solution v of (3) is stable if whenever u is a solution of (2) and $u(x,0)$ is close to v, then $u(x,t)$ will remain close to v for all $t > 0$.

3 Stability in convex domains

Since stable steady-state solutions to (3) provide insight into the behavior of solutions to the initial-boundary value problem (2), we are interested in finding stable solutions to (3). We observe that a solution to (3) is constant only if $f(u) \equiv 0$. Since we are interested in finding stable solutions to (3) when $f(u)$ is not identically zero, we would like to know (Q): when does a domain Ω admit non-constant stable solutions to (3)?

Definition 2 A domain Ω is *convex* if for all $p, q \in \Omega$, the line segement connecting p and q is contained entirely in Ω.

Consider the following conditions on the initial-boundary value problem (2):

(i) Ω is bounded and its boundary is sufficiently smooth, say of class C^3.

(ii) $f = f(u)$ is of class C^2.

The following theorem from [1] provides an answer to (Q) in the case where Ω is convex:

Theorem 1 *Let the conditions* (i) *and* (ii) *hold. If* Ω *is convex and* v *is a non-constant solution of* (3)*, then* v *is unstable.*

Equivalently, the theorem says that if Ω is convex and v is a stable solution of (3), then v is constant.

A convex domain Ω has the property that for every point $p \in \Omega$: the line segment connecting p and q is contained entirely in Ω whenever $q \in \Omega$. If we relax this condition and require only that there exists one such point in Ω, then we say that Ω is a star-shaped domain.

Definition 3 A domain Ω is *star-shaped* with respect to the point $p \in \Omega$ if whenever $q \in \Omega$, then the line segement connecting p and q is contained entirely in Ω.

It is clear that every convex domain is star-shaped (in fact, a convex domain is star-shaped with respect to each of its points). This brings about the question as to whether or not Theorem 1 holds for star-shaped domains. In Sections 5 and 6, we show that Theorem 1 does not generalize to the larger class of star-shaped domains.

4 Existence of non-constant stable solutions

In this section we state a simplified version of a theorem from [1] which guarantees the existence of non-constant stable solutions to (3).

Let Ω be a bounded domain in $\mathbf{R}^n$ with a sufficiently smooth boundary. Assume the nonlinear term f is of the form $f(u) = k\,g(u)$ where g is a function of class C^2 satisfying:

$$\begin{aligned} g(a) = g(0) = g(b) = 0 \quad & \text{for some } a < 0 < b, \\ u \leq g(u) < 0 \quad & \text{for } a < u < 0, \\ 0 < g(u) \leq u \quad & \text{for } 0 < u < b. \end{aligned} \tag{5}$$

Define $G(u) = \int_0^u g(w)dw$. Then G is non-negative on $a \leq u \leq b$ and achieves a maximum at $u = a$ or $u = b$. We make the additional assumption that $G(a) = G(b)$.

In [3], it is shown for any bounded convex domain D in $\mathbf{R}^n$ that:

$$\lambda_2(D) \geq \frac{\pi^2}{\text{dia}(D)^2}, \tag{6}$$

where $\text{dia}(D)$ is the diameter of D and $\lambda_2(D)$ is the second eigenvalue of $-\Delta$ on D with Neumann boundary conditions. We know that the eigenvalue $\lambda_2(D)$

satisfies:

$$\lambda_2(D) = \inf \frac{\int_D |\nabla w|^2\, dx}{\int_D w^2\, dx}, \tag{7}$$

where the inf is taken over all functions w in the Sobolev space $W^{1,2}(D)$ with $\int_D w\, dx = 0$.

Lemma 1 *If D is a bounded convex domain in $\boldsymbol{R}^n$, then*

$$\frac{1}{\lambda_2(D)} \int_D |\nabla w|^2\, dx + \frac{1}{\mu(D)} \left(\int_D w\, dx \right)^2 \geq \int_D w^2\, dx, \tag{8}$$

holds for all w in the Sobolev space $W^{1,2}(D)$.

Proof: From (6) we see that $\lambda_2(D)$ is positive. Let $w \in W^{1,2}(D)$. Let $\bar{w} = w - \frac{1}{\mu(D)} \int_D w\, dx$, then $\int_D w\, dx = 0$. It follows from (7) that

$$\int_D \bar{w}^2\, dx \leq \frac{1}{\lambda_2(D)} \int_D |\nabla \bar{w}|^2\, dx.$$

Substituting in $\bar{w}$, we see that

$$\int_D w^2\, dx \leq \frac{1}{\lambda_2(D)} \int_D |\nabla w|^2\, dx + \frac{1}{\mu(D)} \left(\int_D w\, dx \right)^2.$$

QED

Finally, we assume that Ω satisfies the following conditions (P):

(1) D_1 and D_2 are subdomains of Ω with smooth boundaries in which the Poincaré inequality (8) holds.

(2) There is a component of $\Omega \cap \{x : -\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}\}$, denoted by D_3, such that $D \setminus D_3$ is divided into disjoint open sets O_1 and O_2 containing D_1 and D_2 respectively and satisfying $\partial D_3 \cap \partial O_1 \subset \{x : x_1 = -\frac{\ell}{2}\}$ and $\partial D_3 \cap \partial O_2 \subset \{x : x_1 = \frac{\ell}{2}\}$.

(3) The $(n-1)$-dimensional measure of the intersection of D_3 and the hyperplane $\{x : x_1 = \xi\}$ does not exceed S for $-\frac{l}{2} \leq \xi \leq \frac{l}{2}$.

We define $R[-,+] \subset C^1(\overline{\Omega}) \cap C^2(\Omega)$ so that:

$$R[-,+] = \{w :\ a \leq w \leq b \text{ on } \overline{\Omega},\ \int_{D_1} w(x)dx < 0,\ \int_{D_2} w(x)dx > 0\}.$$

We define $\epsilon_0 > 0$ so that

$$\epsilon_0 = G(b) \cdot min\,\{\, \mu(D_1) \cdot min\{k, \lambda_2(D_1)\}\,,\ \mu(D_2) \cdot min\{k, \lambda_2(D_2)\,\}\}\,.$$

We have the following result from [1]:

Theorem 2 *Suppose the above conditions hold. If*

$$\left\{ \frac{(b-a)^2}{2\,\ell} + k\, G(b)\, \ell \right\} S \leq \epsilon_0, \tag{9}$$

then (3) *has at least one stable solution belonging to* $R[-,+]$.

It is clear that any function belonging to $R[-,+]$ is non-constant.

5 An application to a star-shaped domain in two-dimensions

Let $f(u) = k\, g(u)$ where g is a C^2 function satisfying (5) and $G(b) = G(a)$. For example, take $g(u) = u - u^3$ where $b = 1 = -a$. We will fix a length $\ell > 0$ and a radius $R > 0$, and then we will find a $\delta > 0$ so that the domain Ω depicted below admits a non-constant stable solution to (3). We construct Ω so that its boundary is smooth.

Let D_1 and D_2 be open disks of radius R, then Ω satisfies (P.1). From (6) we have $\lambda_2(D_1) = \lambda_2(D_2) \geq \frac{\pi^2}{4R^2}$. We let $D_3 = \Omega \cap \{x : -\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}\}$. Then Ω satisfies (P.2) where $O_1 = \{x \in \Omega : x_1 < -\frac{\ell}{2}\}$ and $O_2 = \{x \in \Omega : x_1 > \frac{\ell}{2}\}$. For each $\xi \in \mathbf{R}$, the length of $D_3 \cap \{x : x_1 = \xi\}$ is either 0 or δ. Letting $S = \delta$, we see that Ω satisfies (P.3). Therefore, Ω satisfies the conditions (P). By construction, we see that Ω is star-shaped with respect to the origin.

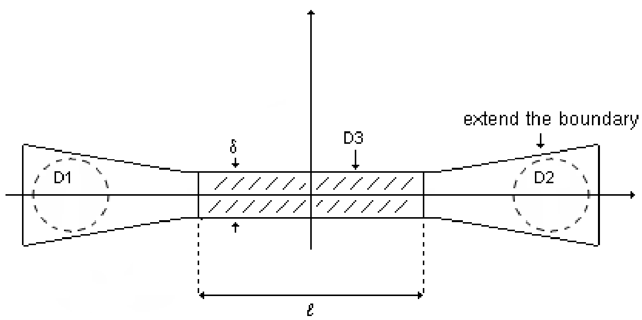

Figure 1: $\Omega \subset \mathbf{R}^2$

Fix $\ell > 0$ and $R > 0$. For simplicity, we choose k so that $0 < k < \frac{\pi^2}{4R^2}$. Then $\epsilon_0 = k\, G(b)\pi R^2$. Choose $\delta > 0$ so that:

$$\delta < \frac{kG(b)\pi R^2}{\frac{(b-a)^2}{2\,\ell} + k\, G(b)\,\ell}.$$

Then,

$$\begin{aligned} \left\{\frac{(b-a)^2}{2\ell} + kG(b)\ell\right\} S &= \left\{\frac{(b-a)^2}{2\ell} + kG(b)\ell\right\} \delta \\ &< \left\{\frac{(b-a)^2}{2\ell} + kG(b)\ell\right\} \frac{kG(b)\pi R^2}{\frac{(b-a)^2}{2\,\ell} + k\, G(b)\,\ell} \\ &= k\, G(b)\pi R^2 \\ &= \epsilon_0 \end{aligned} \tag{10}$$

Under the above assumptions, we see that the hypotheses of Theorem 2 are satisfied. Therefore, Ω is a star-shaped domain that admits a non-constant stable solution to (3).

6 A generalization to n-dimensions

Fix $n \geq 3$. Let $x_1, x_2, ..., x_n$ be rectangular coordinates for $\mathbf{R}^n$. Consider the domain Ω constructed in Section 5 as a subset of the $x_1 x_2$ plane. We require that R be chosen so that

$$\pi R^2 \leq \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} R^n,$$

since $n > 2$, we can choose R sufficiently large. We also require that $\delta < 1$. These are additional assumptions on R and δ which do not affect the analysis done in Section 5. Since δ depends on R, we must fix R before we choose δ.

Let $A = \{a_1 \,:\, (a_1, a_2, 0, ..., 0) \in \Omega \text{ for some } a_2\}$. Define $\phi : A \to (0, \infty)$ by

$$\phi(a) = \sup\{|x_2| \,:\, (a, x_2, 0, ..., 0) \in \Omega\}.$$

Define

$$\Omega' = \{y = (y_1, ..., y_n) \,:\, y_1 \in A \text{ and } |y - (y_1, 0, ..., 0)| < \phi(y_1)\}.$$

Then Ω' is a bounded star-shaped domain in $\mathbf{R}^n$ with smooth boundary. Similarly, define D_1', D_2', D_3', O_1', and O_2'. We see that D_1' and D_2' are n-dimensional balls of radius R. It then follows that (6), (7), and (8) hold for D_1' and D_2'.

It is clear that Ω' satisfies (P.1) and (P.2). To see that Ω' satisfies (P.3), we need to calculate the $(n-1)$-dimensional measure of the intersection of D_3 and the hyperplane $\{x : x_1 = \xi\}$ for $-\frac{l}{2} \leq \xi \leq \frac{l}{2}$. Let's denote this value by $V(\xi)$. By construction, we see that $V(\xi)$ equals the volume of the $(n-1)$-dimensional ball of radius $\frac{\delta}{2}$:

$$V(\xi) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} \left(\frac{\delta}{2}\right)^{n-1} < \left(\frac{\pi}{4}\right)^{\frac{n-1}{2}} \delta^{n-1} < \delta^{n-1}.$$

We note that δ was chosen so that $\delta < 1$. It follows that $V(\xi) < \delta$, so Ω' satisfies (P.3) with $S = \delta$.

Recall from (6) and our previous choice of k that $0 < k < \frac{\pi^2}{4R^2} \leq \lambda_2(D_1) = \lambda_2(D_2)$. Hence

$$\epsilon_0 = k\, G(b) \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} R^n.$$

It follows that

$$\begin{aligned}
\left\{\frac{(b-a)^2}{2\ell}+kG(b)\ell\right\}S &= \left\{\frac{(b-a)^2}{2\ell}+kG(b)\ell\right\}\delta \\
&< \left\{\frac{(b-a)^2}{2\ell}+kG(b)\ell\right\}\frac{kG(b)\pi R^2}{\frac{(b-a)^2}{2\,\ell}+k\,G(b)\,\ell} \\
&= k\,G(b)\pi R^2 \\
&\leq k\,G(b)\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}R^n \\
&= \epsilon_0
\end{aligned} \tag{11}$$

Under the above assumptions, we see that the hypotheses of Theorem 2 are satisfied. Therefore, Ω' is a star-shaped domain that admits a non-constant stable solution to (3).

References

[1] H. Matano, Asymptotic Behavior and Stability of Solutions of Semilinear Diffusion Equations, *Publ. RIMS, Kyoto Univ.* **15** (1979), 401-454.

[2] R.C. McOwen, *Partial Differential Equations: Methods and Applications*, 2nd ed., Prentice Hall, New Jersey, 2003.

[3] L.E. Payne and H.F. Weinberger, An Optimal Poincaré Inequality for Convex Domains, *Arch. Rat. Mech. Anal.* **5** (1960), 286-292.