# A SPECIAL CASE OF SELBERG'S INTEGRAL

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Abstract. We evaluate the integral

$$\int_0^\infty \cdots \int_0^\infty \prod_{1\le i<j\le n} (x_j - x_i)^2 e^{-x_1} dx_1 \cdots e^{-x_n} dx_n$$

using orthogonal polynomials and techniques from linear algebra.

## 1. Introduction

The integral evaluated in this paper arose in the study of a class of invariant differential operators of a matrix argument. Briefly, these polynomials can be described by considering invariant polynomials on a cross section of diagonal matrices that is essentially $\mathbb{R}^n$ ([BHR1] [BHR2]). The way in which the relevant measure is pushed down to this cross section can be determined by computing the integral

$$\int_0^\infty \cdots \int_0^\infty \prod_{1\le i<j\le n} (x_j - x_i)^2 e^{-x_1} dx_1 \cdots e^{-x_n} dx_n \tag{1.1}$$

The author was supported by a Philip S. Zivnuska scholarship and the Summer Research Experience for Undergraduates program at the University of Wisconsin-Eau Claire under the supervision of R. Michael Howe.

Since the integrand is a polynomial, and using the fact that, for $k \in \mathbb{N}$ the Euler Gamma function

$$\Gamma(k+1) = \int_0^\infty x^k e^{-x} dx = k!$$

we see that the value of this integral must be an integer that depends on $n$. We denote the value of the integral (1.1) by $I(n)$.

Somewhat later we discovered that the above integral is a special case of Selberg's integral ([AAR], hence the title of this paper) and can be evaluated using published formulae, but the direct computation of this integral demonstrates the use of algebraic techniques to answer a question that at first appears to be strictly analytic.

## 2. Preliminaries

We first computed this integral for manageable values of $n$ using a computer algebra system and obtained the following results:

| n | $I(n)$ |
|---|---|
| 2 | 2 |
| 3 | 24 |
| 4 | 3465 |
| 5 | 9953280 |

After some consideration we realized that we could write these values as:

| n | I(n) |
|---|---|
| 2 | 2! |
| 3 | 3! (2!)$^2$ |
| 4 | 4! (3! 2!)$^2$ |
| 5 | 5! (4! 3! 2!)$^2$ |

This led us to conjecture that, for $n \in \mathbb{N}$

$$I(n) = n![(n-1)!(n-2)!\cdots(2)(1)]^2$$

It was then suggested that we consider Laguerre polynomials, which we now recall. The Laguerre polynomial in the variable $x$ of degree $k$ can be defined using Rodrigues' formula by

$$L_k(x) = e^x \frac{d^k}{dx^k}(x^k e^{-x}).$$

Note that the leading term will be $x^k$. Direct computation using integration by parts shows that

$$\int_0^\infty L_k(x)L_m(x)e^{-x}dx = \begin{cases} (k!)^2 & \text{k=m,} \\ 0 & \text{k} \neq \text{m.} \end{cases}$$

so that the Laguerre polynomials are orthogonal with respect to the inner product

$$\langle f\,,\,g\rangle = \int_0^\infty f(x)g(x)e^{-x}dx.$$

It is well known that the Laguerre polynomials are an orthogonal basis for the vector space of polynomials in $x$.

Note also that products of Laguerre polynomials in the variable $x$ and Laguerre polynomials in the variable $y$ will be a basis for the vector space of polynomials in the two variables $x$ and $y$, orthogonal with respect to the inner product given by

$$\langle f\,,\,g\rangle = \int_0^\infty \int_0^\infty f(x,y)\,g(x,y)\,e^{-x}\,e^{-y}\,dx\,dy.$$

In particular, by Fubini's Theorem we have

$$\langle L_n(x)L_m(y)\,,\,L_n(x)L_m(y)\rangle = \langle L_n(x)\,,\,L_n(x)\rangle\,\langle L_m(y)\,,\,L_m(y)\rangle = (n!)^2(m!)^2.$$

This idea extends to polynomials in $n$ variables. The products of Laguerre polynomials in the variables $x_1\cdots x_n$ are a basis for the vector space of polynomials in these variables, orthogonal with respect to the inner product

$$\langle f\,,\,g\rangle = \int_0^\infty \cdots \int_0^\infty f(x_1,\cdots,x_n)g(x_1,\cdots,x_n)e^{-x_1}dx_1\cdots e^{-x_n}dx_n$$

with

$$\langle L_{k_1}(x_1)\cdots L_{k_n}(x_n)\,,\,L_{k_1}(x_1)\cdots L_{k_n}(x_n)\rangle$$
$$=\langle L_{k_1}(x_1)\,,\,L_{k_1}(x_1)\rangle\cdots\langle L_{k_n}(x_n)\,,\,L_{k_n}(x_n)\rangle$$
$$=(k_1!)^2\cdots(k_n!)^2.$$

Our first step will be to expand the integrand of (1.1) using Laguerre polynomials.

## 3. Expansion of the integrand using Laguerre polynomials

Making further investigations using computer algebra computations, we expanded $\prod_{1\le i<j\le n}(x_j-x_i)$ for $n=3$ and then subtracted off successive Laguerre polynomials in $x_1$, $x_2$ and $x_3$ by matching the highest order terms. For example, we have

$$(x_2-x_1)(x_3-x_1)(x_3-x_2)=x_2*x_3^2-x_2^2*x_3+x_2^2*x_1-x_1*x_3^2+x_1^2*x_3-x_1^2*x_2.$$

Since the first term in the left hand side is $x_2*x_3^2$, we first subtract off $L_1(x_2)L_2(x_3)$. We then continue in this manner until the result was zero (this is guaranteed since the products of Laguerre polynomials form a basis) and obtain

$$(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) =$$

$$L_1(x_2)L_2(x_3) - L_1(x_3)L_2(x_2)$$

$$+L_1(x_1)L_2(x_2) - L_1(x_1)L_2(x_3)$$

$$+L_1(x_3)L_2(x_1) - L_1(x_2)L_2(x_1)$$

This led us to conjecture the following formula:

$$\prod_{1 \le i < j \le n} (x_j - x_i) = \sum_{\sigma \in S_n} sgn(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)}) \tag{3.1}$$

where $S_n$ is the symmetric group of degree n and $sgn(\sigma) = \pm 1$, depending on whether the permutation $\sigma$ is even or odd. We remark that the left hand side is the well known Vandermonde determinant.

To prove (3.1) we first show that, for the right hand side of (3.1), the polynomial defined by

$$P(x_1, x_2, \cdots, x_n) = \sum_{\sigma \in S_n} sgn(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) L_2(x_{\sigma(3)}) \cdots L_{n-1}(x_{\sigma(n)})$$

is alternating. That is, that the polynomial changes sign whenever two variables are interchanged. Let (i,j).P(x) denote the transposition in $S_n$ that interchanges the $i^{th}$ and $j^{th}$ variables. Then we have

$$(i, j).P(x_1, x_2, \cdots, x_n) = \sum_{\sigma \in S_n} sgn(\sigma) L_0(x_{(i,j)\sigma(1)}) L_1(x_{(i,j)\sigma(2)}) \cdots L_{n-1}(x_{(i,j)\sigma(n)})$$

Now let $\tau = (i,j)\sigma$, so that $sgn(\sigma) = -sgn(\tau)$. Then we have

$$
\begin{aligned}
&(i,j).P(x_1, x_2, \cdots, x_n) \\
&\qquad = \sum_{\sigma \in S_n} sgn(\sigma) L_0(x_{\tau(1)}) L_1(x_{\tau(2)}) \cdots L_{n-1}(x_{\tau(n)}) \\
&\qquad = \sum_{\tau \in S_n} -sgn(\tau) L_0(x_{\tau(1)}) L_1(x_{\tau(2)}) \cdots L_{n-1}(x_{\tau(n)}) \\
&\qquad = -P(x_1, x_2, \cdots, x_n)
\end{aligned}
$$

Therefore, $P$ is alternating.

Now set $P_n(x_n) = P(x_1, x_2, \cdots, x_n)$. That is, consider $P_n$ as a polynomial in $x_n$, with the coefficients being polynomials in $x_1, x_2, \cdots, x_{n-1}$. Since $P$ is alternating, we have $P_n(x_i) = 0$ for $i \neq n$. Thus, by the "factor theorem" from elementary algebra, each $x_1, x_2, \cdots, x_{n-1}$ is a root of $P_n(x_n)$ and so for each $i \neq n$, $(x_n - x_i)$ is a factor of $P_n(x_n)$. Therefore

$$
\begin{aligned}
P_n(x_n) &= (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_1) A_{n-1}(x_1, \cdots, x_{n-1}) \\
&= \prod_{i=1}^{n-1} (x_n - x_i) A_{n-1}(x_1, \cdots, x_{n-1}) \qquad (3.2)
\end{aligned}
$$

for some polynomial $A_{n-1}(x_1, \cdots, x_{n-1})$. Note that $A_{n-1}(x_1, \cdots, x_{n-1})$ is the coefficient of $x_n^{n-1}$ in $P_n(x_n)$.

Next we split the sum (3.1) over those elements of $S_n$ which leave $n$ fixed (a subgroup of $S_n$ isomorphic to $S_{n-1}$), and those which do not.

$$\begin{aligned}
P_n(x_n) = \sum_{\sigma(n)=n} & sgn(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)})\cdots L_{n-1}(x_{\sigma(n)}) \\
& + \sum_{\sigma(n)\neq n} sgn(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)})\cdots L_{n-1}(x_{\sigma(n)}) \\
= & \sum_{\sigma\in S_{n-1}} sgn(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)})\cdots L_{n-2}(x_{\sigma(n-1)})L_{n-1}(x_n) \\
& +(\text{lower degree terms}) \\
= & L_{n-1}(x_n)\left[\sum_{\sigma\in S_{n-1}} sgn(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)})\cdots L_{n-2}(x_{\sigma(n-1)})\right] \\
& +(\text{lower degree terms})
\end{aligned}$$

Equating the leading coefficient of $x_n^{n-1}$ above with that in (3.2) we see that

$$\begin{aligned}
A_{n-1}(x_1, x_2, \cdots, x_{n-1}) &= \sum_{\sigma\in S_{n-1}} sgn(\sigma)L_0(x_{\sigma(1)})L_1(x_{\sigma(2)})\cdots L_{n-2}(x_{n-1}) \\
&= P_{n-1}(x_{n-1})
\end{aligned}$$

Combining these results we have

$$\begin{aligned} P_n(x_n) &= (x_n - x_{n-1})(x_n - x_{n-2}) \cdots (x_n - x_1) A_{n-1}(x_1, \cdots, x_{n-1}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) A_{n-1}(x_1, \cdots, x_{n-1}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) P_{n-1}(x_{n-1}) \end{aligned}$$

Now by the same argument used earlier, we have $x_1, x_2, \cdots, x_{n-2}$ are roots of $P_{n-1}(x_{n-1})$, so

$$\begin{aligned} P_n(x_n) &= \prod_{i=1}^{n-1} (x_n - x_i) P_{n-1}(x_{n-1}) A_{n-2}(x_1, x_2, \cdots, x_{n-2}) \\ &= \prod_{i=1}^{n-1} (x_n - x_i) \prod_{j=1}^{n-2} (x_{n-1} - x_j) A_{n-2}(x_1, x_2, \cdots, x_{n-2}) \end{aligned}$$

where $A_{n-2}$ is the coefficient of $x_{n-1}^{n-2}$.

Now, for $n = 2$ we can experimentally verify that

$$\sum_{\sigma \in S_2} sgn(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) = L_0(x_1) L_1(x_2) - L_0(x_2) L_1(x_1) = x_2 - x_1$$

and so the result follows by induction. Namely

$$\prod_{0 \le i < j \le n} (x_j - x_i) = \sum_{\sigma \in S_n} sgn(\sigma) L_0(x_{\sigma(1)}) L_1(x_{\sigma(2)}) \cdots L_{n-1}(x_{\sigma(n)})$$

as required.

## 4. EVALUATING THE INTEGRAL AS AN INNER PRODUCT

Recall that we have an inner product on the vector space of polynomials on $\mathbb{R}^n$ given by

$$\langle f\,,\,g\rangle = \int_0^\infty \cdots \int_0^\infty f(x_1,\cdots,x_n)g(x_1,\cdots,x_n)e^{-x_1}dx_1\cdots e^{-x_n}dx_n.$$

Therefore we can treat our integral

$$I(n) = \int_0^\infty \cdots \int_0^\infty \prod_{1\le i<j\le n}(x_j-x_i)^2 e^{-x_1}dx_1\cdots e^{-x_n}dx_n \tag{4.1}$$

as the inner product

$$I(n) = \langle \prod_{1\le i<j\le n}(x_j-x_i)\,,\,\prod_{1\le i<j\le n}(x_j-x_i)\rangle.$$

By (3.1), we have

$$I(n) =$$

$$\langle \sum_{\sigma\in S_n} sgn(\sigma)L_0(x_{\sigma(1)})\cdots L_{n-1}(x_{\sigma(n)})\,,\,\sum_{\sigma\in S_n} sgn(\sigma)L_0(x_{\sigma(1)})\cdots L_{n-1}(x_{\sigma(n)})\rangle$$

But each of the terms in the above sums are mutually orthogonal, i.e.

$$\langle L_0(x_{\sigma(1)})\cdots L_{n-1}(x_{\sigma(n)})\,,\,L_0(x_{\tau(1)})\cdots L_{n-1}(x_{\tau(n)})\rangle$$

$$= \begin{cases} (0!)^2(1!)^2(2!)^2\cdots(n-1!)^2 & \sigma=\tau, \\ 0 & \sigma\neq\tau \end{cases}$$

Evaluating our integral in this context yields the required result:

$$I(n) =$$

$$\langle \sum_{\sigma \in S_n} sgn(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}) \, , \sum_{\sigma \in S_n} sgn(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}) \rangle$$

$$= \sum_{\sigma \in S_n} \langle \; sgn(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}) \, , \;\; sgn(\sigma) L_0(x_{\sigma(1)}) \cdots L_{n-1}(x_{\sigma(n)}) \rangle$$

$$= \sum_{\sigma \in S_n} (0!)^2 (1!)^2 (2!)^2 \cdots (n-1!)^2$$

$$= n! [(n-1)! \cdots (2)(1)]^2$$

## References

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