

Extensions of Turán's Theorem for Multigraphs

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1. Introduction

A graph $G = \langle V(G), E(G) \rangle$ consists of a nonempty collection of nodes $V(G) = \{v_1, v_2, \dots, v_n\}$ together with a (possibly empty) set of unordered pairs of them called edges $E(G) = \{x_1, x_2, \dots, x_n\}$. If there is more than one edge between a pair of nodes, each is referred to as a multiple edge. If there is only one edge between a pair of nodes, it is a simple edge. The term *simple graph* refers to a graph that has neither loops nor multiple edges, while *multigraph* describes a graph with no loops but at least one multiple edge.

This article is concerned with exploring ideas in Extremal Graph Theory and extending them for multigraphs. The terminology and notation, except where noted, follows that of Harary [H69].

2. Extremal Graph Theory

Some results in graph theory state that if a graph G is of fixed order n (number of nodes) and has at least size $f(n)$ (number of edges), then G contains a particular subgraph, or G has some kind of property. If the bound $f(n)$ on the number of edges is *sharp*, this means there exists a graph of order n with the same number of nodes and one fewer edge, $f(n)-1$, that does not contain the subgraph or does not possess the property involved.

Determining the sharp bounds and the resulting extremal graphs is an area of graph theory known as *Extremal Graph Theory*.

Extremal Graph Theory has given rise to many interesting problems, and these problems have relevance in a wide range of graph-theoretic applications. This field of study is a relatively recent one.

In 1941 Paul Turán initiated the study of Extremal graph theory with his result on the maximum number of edges in a triangle-free graph. In 2001, Boesch, Gross, Kazmierczak, Stiles, and Suffel restated this theorem in a more straight-forward form. Before stating their version of the theorem, we recall that graph G is *bipartite* if it is possible to partition the nodes into two sets, U and V , such that all edges of G join nodes of U to nodes of V , but none within U and V . Further, a graph G is *complete bipartite* if it is bipartite, and every node in U is adjacent to every node in V . It is denoted $K_{|U|,|V|}$. In general, graphs that partition nodes into more than two parts with no edges within the r individual partitions are called *multipartite*, or *r-partite*, and those in which nodes in one part are adjacent to all nodes in the other parts are *complete multipartite*. [See figure 1.]

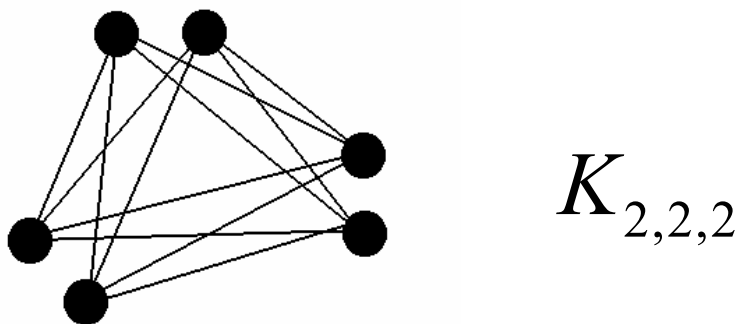


Figure 1: An example of a complete multipartite graph

Theorem 2.1 (Boesch, Gross, Kazmierczak, Stiles, and Suffel) [B2001]

Let G be a graph of order n and size $e(G)$: Then

1. If G is triangle-free, then $e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$
2. If G is triangle-free and $\left\lfloor \frac{n^2}{4} \right\rfloor - \left(\left\lfloor \frac{n}{2} \right\rfloor - 2 \right) \leq e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$ then G is bipartite
3. If G is triangle-free and $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor$, then $G = K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$
4. The lower bound given in (2) is best possible, i.e., there exists a triangle-free graph having $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right)$ edges which is not bipartite.

After Theorem 2.1 was presented at the June, 2000: Quadrennial Conference on Discrete Math, Graph Theory and Combinatorics at Western Michigan University, Balister and Bollobas generalized the result for graphs that were bipartite and triangle-free with $\left\lfloor \frac{n^2}{4} \right\rfloor - \left(\left\lfloor \frac{n}{2} \right\rfloor - 2 \right) \leq e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Note that an almost regular graph is one in which all degrees in the degree sequence differ by at most one.

Their result is as follows:

Let $r \geq 2$ and let $t_r(n)$ denote the number of edges in almost regular complete r -partite graph on n vertices. If G is a K_{r+1} -free graph with n vertices and e edges, where $t_r(n) - \left(\left\lfloor \frac{n}{r} \right\rfloor - 2 \right) \leq e \leq t_r(n)$, then G is r -partite. Moreover the lower bound is still best possible.

3. Multigraph extension to Turán

As stated previously, Turán's Theorem holds for simple graphs. Our extension to multigraphs introduces the maximum multiplicity of the number of edges of a multigraph to observe whether or not the inequality remains true.

Theorem 3.1: Given a multigraph M having n nodes and whose multiple edges have maximum multiplicity μ , if $e \geq \mu \left\lfloor \frac{n^2}{4} \right\rfloor + 1$, then M must contain at least one triangle as a subgraph.

Proof: Observe that a multigraph M has a triangle if and only if the underlying simple graph G contains a triangle. This can be obtained by replacing each bank of multiple edges by a single edge.

If M has no triangles, then G has no triangles. It follows that $e(G) \leq \left\lfloor \frac{n(G)^2}{4} \right\rfloor$ which is the theorem by Turán.

Let $e(M) = \sum_{x \in e(G)} m(x)$ where $m(x)$ is the multiplicity of the multiple edge corresponding to x , i.e. the edges in the multigraph are represented as the summation of the multiplicity of the edges in the underlying graph G .

Since $m(x)$ is the multiplicity of one of the edges in G and μ is the maximum multiplicity of the edges in M , it follows that $m(x) \leq \mu$ for all x . Because $e(M) = \sum_{x \in e(G)} m(x)$, the total number of edges in the multigraph can be at most $\mu(e(G))$ so

$e(M) \leq \mu(e(G)) \leq \mu \left\lfloor \frac{n(G)^2}{4} \right\rfloor = \mu \left\lfloor \frac{n(M)^2}{4} \right\rfloor$. Therefore, the addition of one more edge creates a triangle. ■

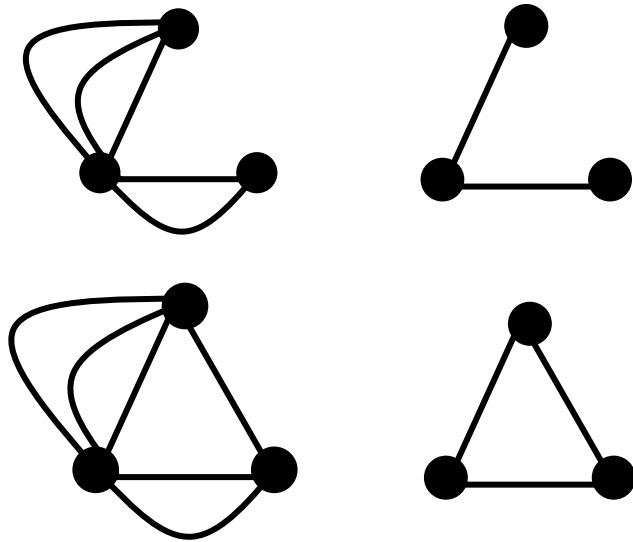


Figure 2: Examples of multigraphs and their underlying simple graphs, also illustrating maximum multiplicity 3

Theorem 3.1 can be extended in much the same manner as Turán's in Theorem 2.1.

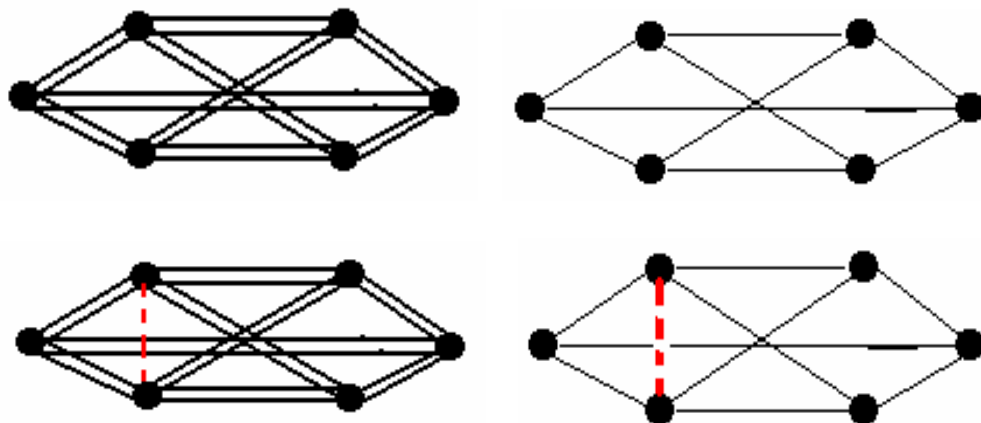


Figure 3: If a simple graph contains a triangle, then increasing the multiplicity of any edge or edges will not effect the existence of that original triangle

Figure 3 illustrates in general a property that applies to complete bipartite graphs specifically. Since complete bipartite graphs are the extremal simple graphs in Turán's Theorem, increasing the multiplicity of any or all existing edges will not create any triangles.

It can be observed that a multigraph M with maximum multiplicity μ is bipartite if and only if the underlying simple graph G is bipartite. Therefore:

Proposition 3.2: *If M is triangle-free and $\mu \left(\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) < e(M) \leq \mu \left\lfloor \frac{n^2}{4} \right\rfloor$, then M is bipartite.*

Proof: Assume M is not bipartite. Then G is not bipartite. If G is not bipartite, then

$e(G) > \left\lfloor \frac{n^2}{4} \right\rfloor$ or $e(G) < \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2$ by contrapositive of Theorem 2.1 part 2.

For $e(G) > \left\lfloor \frac{n^2}{4} \right\rfloor$: G has a triangle as a subgraph. This is a contradiction. Therefore, G is

bipartite. It follows that M is bipartite.

For $e(G) < \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2$: by contrapositive of Theorem 2.1 part 2, G has a triangle as a subgraph. This is a contradiction. Therefore, G is bipartite. It follows that M is bipartite. ■

Proposition 3.3 extends part (3) of Theorem 2.1 for multigraphs:

Proposition 3.3: *If M is triangle-free, $e(M) = \mu \left\lfloor \frac{n^2}{4} \right\rfloor$, then M is the multigraph obtained*

by increasing the multiplicity of edges in any complete bipartite graph.

Proof: M is bipartite and triangle free, as in Proposition 3.2, so the same is true for G because G is a subset of M . Hence, $e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$, but then $e(M) = \mu \left\lfloor \frac{n^2}{4} \right\rfloor \leq \mu e(G) \leq \mu \left\lfloor \frac{n^2}{4} \right\rfloor$ and so $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor$. Hence $G = K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor}$ by Theorem 2.1, part 3. The fact that $e(M) = \mu \left\lfloor \frac{n^2}{4} \right\rfloor$ forces each edge of $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor}$ to be repeated μ times. ■

The bounds that were given in proposition 3.2,

$$\mu \left(\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) < e(M) \leq \mu \left\lfloor \frac{n^2}{4} \right\rfloor,$$

point out the existence of a triangle if the number of edges exceeds the bound to the right, and shows the existence of a non-bipartite, triangle-free graph if the number of edges is less than or equal to the bound to

the left. Therefore, there exists a multigraph M , having edges $e(M) = \mu \left(\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 1 \right)$,

which is non-bipartite and triangle-free. So for a non-bipartite, triangle-free graph G with

$e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 1$, the multigraph M obtained by replacing each edge of G μ times

is also non-bipartite and triangle-free.

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References

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