

# Another Proof of the Steiner Result for Three Equidistant Points in Euclidean Space and an Analogous Proof in Hyperbolic Space

Diana Dimond

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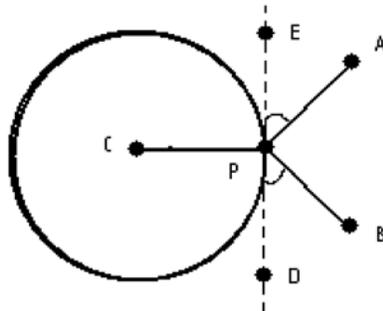
## 1 Introduction

The Steiner problem deals with the shortest distance to connect a given number of points. For three points, the Steiner problem is usually credited to Cavalieri, and Italian goemeter in the early 1600's. However, there is some evidence it was first posed by Fermat and solved by Torricelli around the same time. Despite the questions surrounding the origin of the problem, the general  $n$  case appeared to be due to Steiner [11]. Because of this, I will use Steiner's proof for three points. In the early 19th century, Jacob Steiner wanted to find the shortest path to connect three villages. He concluded that if we think of the three villages as vertices of a triangle, then the shortest path depended on the angles of the triangle. If all the angles were less than  $120^\circ$ , the shortest path involved a fourth interior point, a Steiner point, at which the segments from the vertices all meet at  $120^\circ$ . If any angle was larger than  $120^\circ$ , the shortest path consisted of the two sides of the triangle adjacent to the vertex of the greatest angle (see figure 1).



**Figure 1:** Steiner Result

There are different proofs for the case in which all the angles are less than  $120^\circ$ . One common proof uses circles. Let the points be represented by the vertices of the triangle  $ABC$  with the fourth interior point being  $P$ . Steiner showed that the minimum path is  $PA + PB + PC$  with the three segments intersecting at  $120^\circ$ . To prove this consider a circle  $K$ . Let  $K$  be a circle of radius  $PC$  centered at  $C$  (see figure 2). Then, in order for the path to be minimized,  $P$  must be such that  $PA + PB$  is minimum. This will occur when the angle between the tangent to the circle, at  $P$ , and  $PA$  is equal to the angle between the tangent to the circle, at  $P$ , and  $PB$ . That is,  $\angle APE \cong \angle BPD$  (see figure 2). This implies  $\angle APC \cong \angle BPC$ . When you repeat this same reasoning for circles centered at  $A$  and at  $B$ , you find that the angles between  $PA$ ,  $PB$ , and  $PC$  must all be equal; thus they all equal  $120^\circ$ . For more background on the Steiner problem, see [3],[5],[6].



**Figure 2:** Steiner's Proof

Various slicing methods have been used to solve minimization problems. Our research group developed a different “slicing” method that can be used to prove conjectured minimal paths are indeed minimal.

## 2 New Slicing Method

Slicing methods have been developed to prove minimization conjectures (see [8],[9]). The basis for this new slicing method is solving problems by transformation. First, transform the problem into a different setting. Next, solve the problem in the new space, where it is, ideally, an easier problem. Lastly, transform the solution back into the original setting, where it should, hopefully, solve the original problem. In the process, pay special attention to the properties preserved by the transformation since they are the keys for the method’s success [10]. This process can be separated into four main steps:

1. Partition the conjectured minimal path with equidistant lines in the given space. Each partition line corresponds to a single point in the new metric.
2. Define how distances will be measured in the new metric.
3. Once the conjectured minimal path is constructed point-wise in the new metric, partition the new metric in such a way that it shows that the conjectured minimal path is a minimal path.
4. Consider all the competitor paths and show that they will be longer than the path used in step 3.

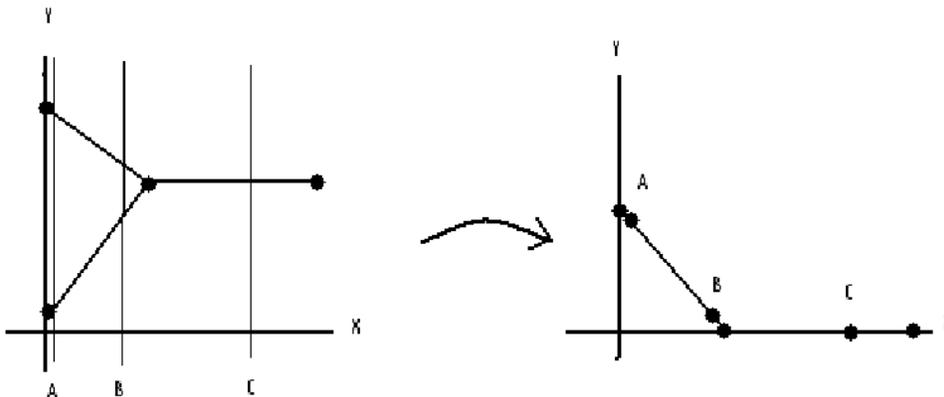
This new slicing method can be used to give new proofs of old results and potentially prove new conjectures. To demonstrate this new method and its effectiveness, we will show how it can be used to give new proof of Steiner’s result for three equidistant points. Then we will use it to prove an analogous result in hyperbolic space; that is, the shortest path between three equidistant point in hyperbolic space is formed by hyperbolic geodesics that meet at  $120^\circ$ .

### 3 A New Proof of Steiner's Problem

Three equidistant points can be represented as vertices of an equilateral triangle. In an equilateral triangle all the angles are less than  $120^\circ$ , and so by Steiner's results the conjectured minimal path is the path that has all the segments from the vertices intersecting at a fourth interior point with angles of  $120^\circ$  (see figure 1).

**Steiner Result:** The shortest path between three equidistant points is formed by line segments that meet at  $120^\circ$ .

*Proof. Step 1: Constructing the new metric.* First we partition the conjectured minimal path with equidistant vertical lines in Euclidean Space. According to the slicing method, each vertical line will map to one point in the new metric, where the  $x$ -value remains the same and the  $y$ -value is  $\frac{1}{2}$  the Euclidean distance between the two segments. We chose this  $y$ -value because we wanted to preserve the angle between the segments. For our conjectured minimal path, this mapping preserves the  $120^\circ$  between the segments. Therefore, in the new metric the end points are  $(0, \frac{1}{2}l)$  and  $(\frac{\sqrt{3}}{2}l, 0)$ , where  $l$  is the length of the side of the triangle (see figure 3).

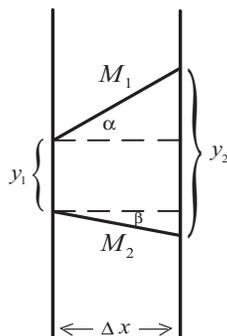


**Figure 3:** Euclidean Metric

**Step 2: Distances in the new metric.** In the new metric, distances will not be calculated in the same way as in Euclidean space. The distance in the new metric needs to be less than or equal to the corresponding segment in our original path so that once we have shown the minimum path in the

new metric we know it is also minimal in the original space. First consider a segment in the new metric off the  $x$ -axis. Then we need to show what the shortest preimage will be. The following lemma, or the compartment problem, will show the shortest preimage for a segment off the  $x$ -axis in the new metric.

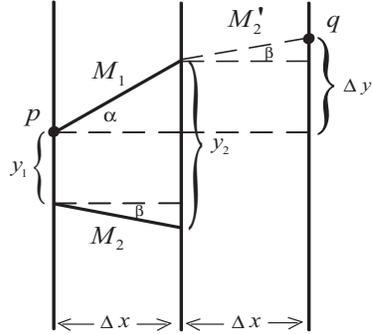
**Lemma 1.** *Given the following diagram*



**Figure 4:** *Finding a minimum crossing length.*

where  $y_1, y_2$ , and  $\Delta x$  are fixed,  $L(M)$  is minimized when  $\angle\alpha = \angle\beta$ .

*Proof.* This follows from a geometric argument. Derive  $M'_2$  by vertically reflecting and translating  $M_2$  as shown in figure 6.



**Figure 5:** Geometric argument for Lemma.

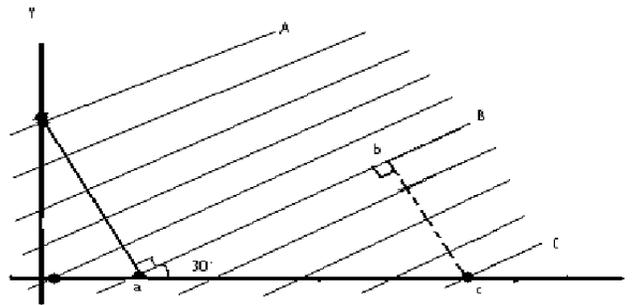
When  $y_1$  and  $y_2$  are fixed, so is  $\Delta y = y_2 - y_1$ . Observe that the shortest path from the point  $p$  to the point  $q$  is a straight line segment and that any other path is of longer length. Thus  $L(M)$  is minimized when  $\angle\alpha = \angle\beta$ .  $\square$

Therefore, since the shortest path in the preimage is when the segments are symmetric between two partition lines, then we define distance of off the  $x$ -axis in the new metric as twice Euclidean length. When dealing with a segment on the  $x$ -axis in the new metric, it is easy to see that the shortest preimage will be the path that is perpendicular to the partition lines. Therefore, we define length on the  $x$ -axis in the new metric as Euclidean length.

**Step 3: Proof of the conjectured minimal path.** Now that we have constructed our conjectured minimal path in the new metric, it is necessary to show that it is a minimal path. If we partition the new metric space with parallel lines having a reference angle of  $30^\circ$ , then we can do a piecewise proof of a minimal path. It is important to note that a minimal path will be a path that crosses all the partition lines with the shortest distance. Therefore, we

can split a minimal path into two parts: the part of the line off the  $x$ -axis and the part of the line on the  $x$ -axis.

First consider the part of the line off the  $x$ -axis (i.e., the part between the partition lines A and B) (see figure 4). Our conjectured minimal path has an angle of  $120^\circ$  between the segments. This angle will be preserved in our new metric. Therefore, since the partition lines have reference angle of  $30^\circ$  and our path has reference angle of  $120^\circ$ , then the partition lines and our path will intersect at  $90^\circ$ . We know that the shortest path between parallel lines is the line perpendicular. Thus, we know that the conjectured minimal path off the  $x$ -axis is minimal.



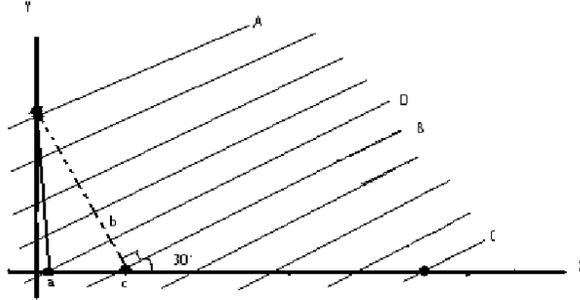
**Figure 6:** A minimal path in Euclidean

Next, consider the segment of our path on the  $x$ -axis (i.e., the path between partition lines B and C). In order to prove this path is minimized we can construct a  $30^\circ, 60^\circ, 90^\circ$  triangle and use the length properties of it. The hypotenuse of the triangle is  $\overline{ac}$  with legs  $\overline{ab}$  and  $\overline{bc}$ .

Now, according to the properties of a  $30^\circ, 60^\circ, 90^\circ$  triangle, the side across from the  $30^\circ$  is half the length of the side across from the  $90^\circ$ . This means that  $\overline{bc}$  is half the distance of  $\overline{ac}$ . But in this metric any line off the  $x$ -axis is twice Euclidean length, while any line on the  $x$ -axis is Euclidean length. Therefore, the two paths are equal. The leg that is perpendicular to the partition is a minimal path that crosses this set of partition lines. Since the length of this line in the new metric is equal to the length of the segment on the  $x$ -axis, the segment on the  $x$ -axis is also a minimal path. Hence, our conjectured path is a minimal path, and we will now call it path  $P_E$ .

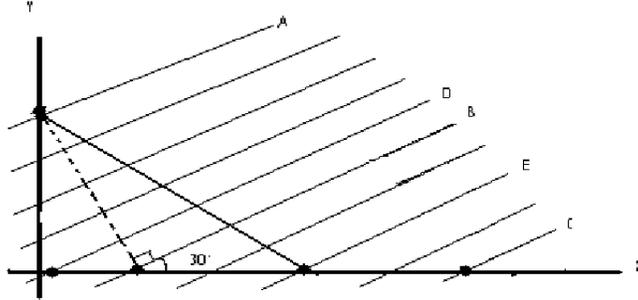
**Step 4: Competitor paths.** The final step in the proof is to show why any competitor path is longer than path  $P_E$  used in step 3. In Euclidean Space, the shortest distance between any two points is the straight line connecting them. Therefore, in considering all the competitor paths we only need to consider those with straight line components. In looking at the new metric we can see that the competitors will fall into two main categories: those that reach the  $x$ -axis before path  $P_E$ , and those that reach the  $x$ -axis after path  $P_E$ . In both cases we will do a piecewise comparison with path  $P_E$ .

First, consider the case where the competitor reaches the  $x$ -axis before path  $P_E$  (see figure 5). Since we are doing a piecewise comparison test, we need to prove that the competitor path is longer than path  $P_E$  when they cross the same number of partition lines. The first section is between the partition lines A and D. Since path  $P_E$  is crossing perpendicularly to these partition lines, it is the shortest path. Therefore, since the competitor is not crossing perpendicularly, its path will be longer. Next consider the path between the partition lines D and B. As in step 3, we can construct a  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$  triangle whose hypotenuse is  $\overline{ac}$  with legs  $\overline{ab}$  and  $\overline{bc}$ . The competitor path is  $\overline{ac}$ , and path  $P_E$  is  $\overline{bc}$ . Therefore, given the properties of a  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$  triangle, and the fact that distance off the  $x$ -axis is twice Euclidean distance, these paths are equal. The last segment is the path between the partition lines B and C. In this section the competitor and path  $P_E$  are the same, so they will have the same length. Thus, since the competitor path is longer than path  $P_E$  in the first segment and equal in the other two, then overall the competitor's path is longer.



**Figure 7:** Competitor that reaches the  $x$ -axis before path  $P_E$

Next, consider the case where the competitor reaches the  $x$ -axis after path  $P_E$  (see figure 6). First, consider the section between the partition lines A and B. As in the previous case, the competitor path will be longer. Next, consider the section of the path between the partition lines B and E. In the previous case we saw that if the path off the  $x$ -axis were perpendicular to the partition line, then it would have the same length in the new metric as the path on the  $x$ -axis. However, in this case it is not perpendicular and hence longer. The last section between partition lines E and C, the paths are the same, so the lengths are the same. Hence, since the competitor path is longer than path  $P_E$  in the first and second sections and equal in the last, overall the competitor path is longer.



**Figure 8:** Competitor that reaches the  $x$ -axis after path  $P_E$

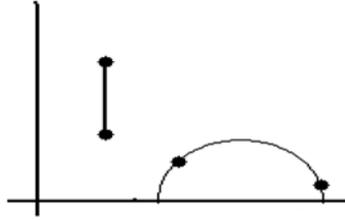
Therefore, the conjectured minimal path, path  $P_E$ , is indeed the shortest path because any possible other path has been shown to be longer. This means that the minimum path to connect 3-equidistant points in Euclidean space is with a fourth interior point at which the three segments meet at  $120^\circ$ .

□

## 4 A Proof of Steiner's Result in Hyperbolic Space

Now we will use this new slicing method to prove the conjectured minimal path to connect 3-equidistant points in hyperbolic space. In hyperbolic space we will use the upper-half plane model (for background material, see [1], [2]). The geodesics (i.e., shortest distance curves between two points) of this model are vertical lines and arcs of circles that intersect the  $x$ -axis at  $90^\circ$  (see figure

7).



**Figure 9:** Hyperbolic Geodesics

When calculating distances in Euclidean space the formula is

$$ds = \sqrt{dx^2 + dy^2}.$$

In the upper-half plane model the formula is

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

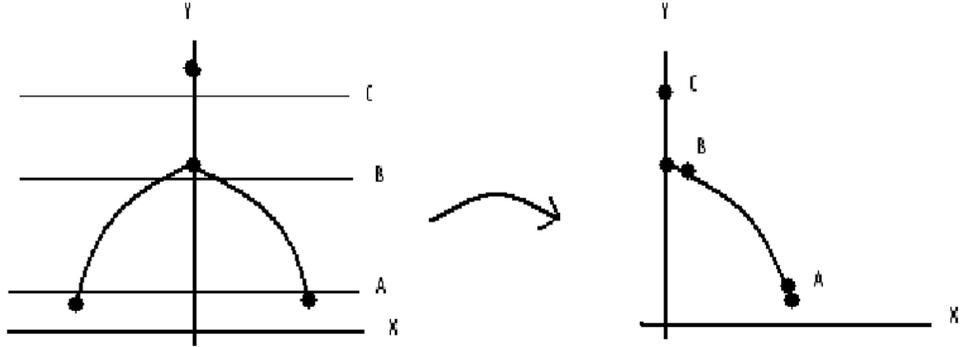
Therefore, the distance between two points in Euclidean space is constant as you move along the  $y$ -axis, but in hyperbolic space as you move towards the  $x$ -axis the distance between two points increases to infinity. Also, in the upper-half plane model corresponding angles in Euclidean and Hyperbolic space are equal.

This last property suggests that the conjectured minimal path for connecting 3-equidistant points in hyperbolic space is with a fourth interior point at which the segments intersect at  $120^\circ$ . Now we will apply the slicing method to prove that this indeed is the minimal path.

**Steiner Result II:** The shortest path between three equidistant points in hyperbolic space is formed by hyperbolic geodesics that meet at  $120^\circ$ .

*Proof. Step 1: Constructing the new metric.* First, we partition the conjectured minimal path with horizontal lines, which are equidistant in hyperbolic space. As described in the slicing method, each horizontal line will map to one point in the new metric, where the  $y$ -value remains the same and

the  $x$ -value is  $\frac{1}{2}$  the Euclidean distance between the two segments. Therefore, any part with one segment will be mapped to the  $y$ -axis, and any part with two segments will map to an arc of a circle off the  $y$ -axis (see figure 8).



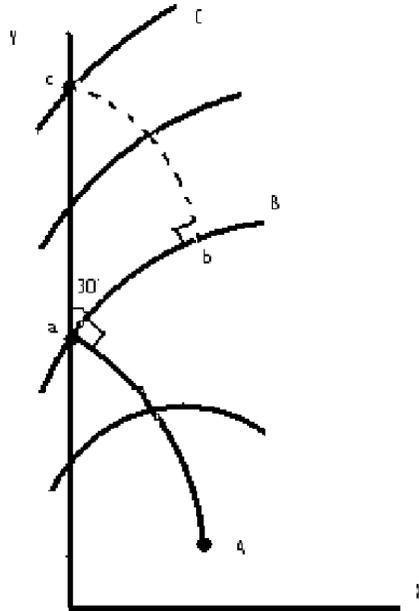
**Figure 10:** Hyperbolic Metric

**Step 2: Distances in the new metric.** Distances in the new metric will not be calculated the same way as in hyperbolic space. Instead, any line that is on the  $y$ -axis will have hyperbolic length, but any line off the  $y$ -axis will have twice hyperbolic length. It is necessary that any line off the  $y$ -axis have twice hyperbolic length because it is representing two segments of the path, and both of these paths need to be accounted for in the new metric.

**Step 3: Proof of the conjectured minimal path.** Now that we have the conjectured minimal path constructed in the new metric, we need to show that it is a minimal path. If we partition the new metric with hyperbolic circles centered at the endpoint, we can do a piecewise proof that this is a minimal path. It is important to note that a shortest path is a path that crosses all the partition lines with the shortest length. Therefore, we will split a minimal path into two sections: the part of the line that is off the  $y$ -axis and the part of the line that is on the  $y$ -axis.

First, consider the part that is off the  $y$ -axis (i.e., the part between partition lines A and B) (see figure 9). Since the partition lines are hyperbolic circles centered at the end point, our minimizer is the radii of these circles. The shortest path to cross circles is a path that intersects each circle perpendicularly. Since the radii of circles intersect the circle perpendicularly, this

path is a minimal path. Thus, we know that the conjectured minimal path off the  $y$ -axis is minimal.



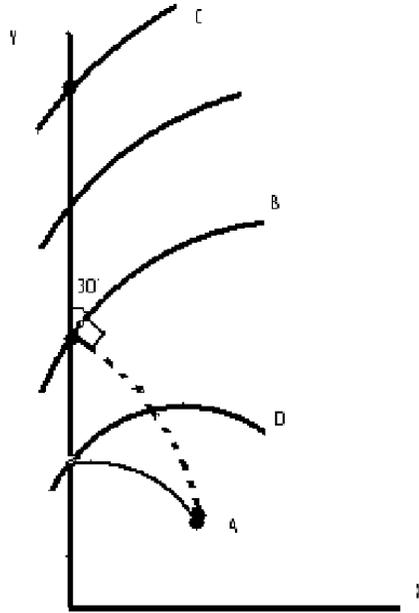
**Figure 11:** A minimal path in Hyperbolic

Next, consider the part of our path that is on the  $y$ -axis (i.e., the part between partition lines B and C). Since the angle between the segments of our path is  $120^\circ$  and our path intersects the partition circles perpendicularly, the line tangent to circle where our path intersects the  $y$ -axis is  $30^\circ$  off the  $y$ -axis. Therefore, we can construct  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$  triangles with the hypotenuse  $\overline{ac}$  and legs  $\overline{ab}$  and  $\overline{bc}$ . At the point where the path intersects the  $y$ -axis there is a  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$  triangle with our path on the  $y$ -axis. Thus, given the properties of a  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$  triangle, and the fact that any path off the  $y$ -axis is twice hyperbolic length, our path on the  $y$ -axis is equal to the geodesic between the partition lines. However, in hyperbolic space the distance between points decreases as you move up the  $y$ -axis so the  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$  relationship will only hold at one point. But looking at the partition circles, the angle between the  $y$ -axis and the tangent line to these circles increases as you go up the

$y$ -axis. This means that the sine of that angle increases so the edge opposite also increases. So, our path on the  $y$ -axis will be less than twice the path off the  $y$ -axis. Hence, a shortest path to cross the partition lines after the path intersects the  $y$ -axis is our path on the  $y$ -axis. Therefore, our conjectured path is minimal. Call this path  $P_H$ .

**Step 4: Competitor paths.** The final step in the proof is to show why any competitor path is longer than path  $P_H$  used step 3. As in the Euclidean case, the number of paths that need to be considered can be reduced because only those composed of the hyperbolic geodesics could possibly be shorter. Again, all the competitors will fall into two cases: those that reach the  $y$ -axis before path  $P_H$ ; and those that reach the  $y$ -axis after path  $P_H$ . In both cases we will do a piecewise comparison proof with path  $P_H$ .

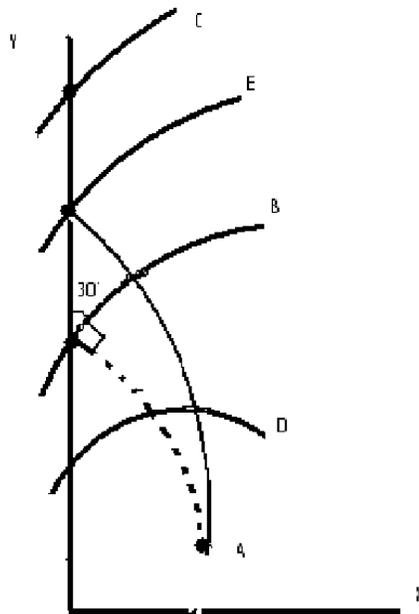
First, consider the competitors that reach the  $y$ -axis before path  $P_H$  (see figure 10). As in the Euclidean case, we need to prove that the competitor path is longer than path  $P_H$  when they cross the same partition lines. The first section is the path between partition lines A and D. Between these partition lines, path  $P_H$  is crossing the circles perpendicularly, so it is the shortest path. Since the competitor path does not intersect the circles perpendicularly, it will be longer. The second section is between partition lines D and B. In this section the competitor path is along the  $y$ -axis and path  $P_H$  is reaching the  $y$ -axis. As discussed in step 3, the angle of the tangent to the circles changes as you go up the  $y$ -axis. Below the point where path  $P_H$  intersects the  $y$ -axis, the angle between the  $y$ -axis and the tangent line is less than  $30^\circ$ . Hence, the sine of the angle decreases. Thus, the path on the  $y$ -axis is more than twice the path off the  $y$ -axis given the law of sines. Therefore, path  $P_H$  is shorter than the competitor. The last section is between the partition lines of B and C. In this section the paths are the same. Therefore, since the competitor path is longer than path  $P_H$  in two of the three sections, the overall length of the competitor is longer than path  $P_H$ .



**Figure 12:** Competitor that reaches the  $y$ -axis before path  $P_H$

Next, consider the competitor that reaches the  $y$ -axis after path  $P_H$  (see figure 11). We will once again do a piecewise comparison with path  $P_H$  in three sections. The first section is between the partition lines A and B. As in the previous case, the competitor path will be longer. The second section is between the partition lines B and E. As in the other competitor case, the lengths in this section depend on the angle between the  $y$ -axis and the tangent line of the partition circle. Since the tangents are above the point where path  $P_H$  intersects the  $y$ -axis, the angle between the  $y$ -axis and the tangent line is greater than  $30^\circ$ . Hence, the sine of the angle increases. Therefore, the path on the  $y$ -axis, path  $P_H$ , is less than twice the distance of the path off the  $y$ -axis, the competitor path. Thus, in this section path  $P_H$  is shorter than the competitor. The last section is between the partition

lines of E and C. In this section the paths are the same. Therefore, since the competitor is longer than path  $P_H$  in two of the sections, the overall length of the competitor is longer than path  $P_H$ .



**Figure 13:** Competitor that reaches the  $y$ -axis after path  $P_H$

Therefore, path  $P_H$  is indeed the minimal path since any competitor is longer. Thus, the shortest path to connect 3-equidistant points in hyperbolic space is the path that has the three segments from the vertices intersecting at  $120^\circ$ .

□

## 5 Future Applications

This new slicing method may be helpful in proving new results in geometric analysis. For example, it may be applied to regular  $n$ -gons in both Euclidean

and Hyperbolic space to show the minimal paths. Another potential application of this slicing method is with the multiple bubble problem, which deals with how to contain a given volume with the least amount of surface area. This problem has been solved for 2-bubbles in the plane and in space [7] and for 3-bubbles in the plane. [4] The slicing method could help to prove the conjectures for the multiple bubble problem in both the plane and in space.

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