

A SEARCH FOR PATTERNS IN A SEQUENCE OF NUMBERS: PROVING TRENDS IN A RECURRENCE RELATION*

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ABSTRACT. Let $\{f(n)\}$ be the sequence of integers determined by the recurrence relation $f(n+2) \equiv 2f(n+1) - f(n) \pmod{(n+2)}$, the inequality $0 \leq f(n) < n$ for every integer $n \geq m$, m is a certain positive integer, and initially assigned arbitrary integers $f(m)$ and $f(m+1)$. We investigate how the sequence $\{f(n)\}$ increases or decreases. We call a maximal increasing or decreasing subsequence of consecutive elements of the sequence a run. We show that each run is an arithmetic progression and that the common difference in an increasing run is one more than the common difference in the previous increasing run, and similiary for decreasing runs.

1. INTRODUCTION

The aim of this paper is to study the sequence $\{f(n)\}$ of integers determined by the recurrence relation

$$(1) \quad f(n+2) \equiv 2f(n+1) - f(n) \pmod{(n+2)}$$

and the inequality $0 \leq f(n) < n$ for every integer $n \geq m$, m being a given positive integer, where $f(m)$ and $f(m+1)$ are, as initial values, arbitrarily given integers. We exclude the trivial case in which $f(n)$ is constant. For such a sequence $\{f(n)\}$ we define a run of the sequence to be a maximal increasing or decreasing subsequence of consecutive elements of the sequence of length at least 2.

We call a sequence $\{f(n)\}$ a solution of the recurrence relation (1) or simply a solution if $f(n)$ satisfies the relation (1) and the inequality $0 \leq f(n) < n$ for every $n \geq m$, m being a given positive integer, where $f(m)$ and $f(m+1)$ are arbitrarily given integers.

In the sequel $\{f(n)\}$ means a solution of the recurrence relation (1).

Here we give some examples of solutions:

Example 1.

n	7	8	9	10	11	12	13	14	15	16	17	18
$f(n)$	0	3	6	9	1	5	9	13	2	7	12	17

In this example the increasing runs are 0, 3, 6, 9; 1, 5, 9, 13 and 2, 7, 12, 17.

It can be immediately seen that for $n = 7, \dots, 10$, the increase (difference ?) in consecutive terms is 3, while for $n = 11, \dots, 14$, the increase (difference ?) in consecutive terms is 4, which is one more than the previous increase (difference ?). We will show that this trend (property ?) holds in general.

Example 2.

n	5	6	7	8	9	10	11	12	13	14	15	16	17
$f(n)$	0	5	3	1	8	5	2	11	7	3	14	9	4

In this example the decreasing runs are 5, 3, 1; 6, 5, 2; 11, 7, 3 and 14, 9, 4.

In the first decreasing run the numbers decrease by 2, in the second by 3, and in the third by 4. We will also show that this trend (property ?) holds in general.

Example 3. Let $f(n) = n - a$ for $n \geq m$, where a is an integer with $1 \leq a \leq m$. Then $\{f(n)\}$ is a solution of infinite length.

This is a sample.

2. RESULTS AND PROOFS

The following theorem describes (properties of) the increasing and decreasing runs of a solution $\{f(n)\}$:

Theorem. *Let $\{f(n)\}$ be a solution of the recurrence (1). Then we have:*

(i) *Each run is an arithmetic progression.*

(ii) *If an increasing run is followed by another increasing run, then the common difference in the second run is one more than the common difference in the first run.*

(iii) *If an decreasing run is followed by another decreasing run, then the common difference in the second run is one more than the common difference in the first run.*

To prove Theorem we need the following Lemmas 1 and 2.

Lemma 1. *For a solution $\{f(n)\}$, we have*

$$(2) \quad f(n+2) = 2f(n+1) - f(n) + K(n+2),$$

where

$$(3) \quad K = \begin{cases} 0 & \text{if } f(n) < f(n+1) \text{ and } f(n+1) < f(n+2), \\ -1 & \text{if } f(n) < f(n+1) \text{ and } f(n+1) > f(n+2), \\ 1 & \text{if } f(n) > f(n+1) \text{ and } f(n+1) < f(n+2), \\ 0 & \text{if } f(n) > f(n+1) \text{ and } f(n+1) > f(n+2). \end{cases}$$

Proof. ...

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Lemma 2. *For a solution $\{f(n)\}$, we have:*

(i) *If $f(n) < f(n+1)$, $f(n+1) > f(n+2)$ and $f(n+2) < f(n+3)$, then $f(n+3) - f(n+2) = f(n+1) - f(n) + 1$.*

(ii) *If $f(n) > f(n+1)$, $f(n+1) < f(n+2)$ and $f(n+2) > f(n+3)$, then $f(n+3) - f(n+2) = f(n+1) - f(n) - 1$.*

Proof. ...

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Proof of Theorem. Part (i) of Theorem for increasing runs (resp. decreasing runs) follows from the first case (resp. the final case) of (3) in Lemma 1. Part (ii) of Theorem for increasing runs follows from Lemma 2 (i). Part (iii) of Theorem for decreasing runs follows from Lemma 2 (ii). \square

The results for decreasing runs follow from the corresponding ones for increasing runs, and vice versa, by

Proposition 1. *Let $\{f(n)\}$ be a solution. Let $g(n) = n - 1 - f(n)$ for every $n \geq m$. Then $\{g(n)\}$ is also a solution, that is, we have*

$$(4) \quad g(n+2) \equiv 2g(n+1) - g(n) \pmod{(n+2)}$$

and $0 \leq g(n) < n$ for every $n \geq m$.

Proof. ...

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We empirically found the following fact:

Proposition 2. *A solution $\{f(n)\}$ of the recurrence (1) consists entirely of increasing runs or entirely of decreasing runs.*

For the proof of Proposition 2 it suffices to show

Lemma 3. *For a solution $\{f(n)\}$ it is impossible that $f(n) < f(n+1)$, $f(n+1) < f(n+2)$, $f(n+2) > f(n+3)$, and $f(n+3) > f(n+4)$.*

Proof. ...

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