

# A SEARCH FOR PATTERNS IN A SEQUENCE OF INTEGERS: PROVING TRENDS IN A RECURRENCE RELATION

QAISER SAIFY

ABSTRACT. Let  $\{f(n)\}$  be the sequence of integers determined by the recurrence relation  $f(n+2) \equiv 2f(n+1) - f(n) \pmod{(n+2)}$ , the inequality  $0 \leq f(n) < n$  for every integer  $n \geq m$ , where  $m$  is a certain positive integer, and initially assigned arbitrary integers  $f(m)$  and  $f(m+1)$ . We investigate how the sequence  $\{f(n)\}$  increases or decreases. We call a maximal increasing or decreasing subsequence of consecutive elements of the sequence a run. We show that each run is an arithmetic progression and that the common difference in an increasing run is one more than the common difference in the previous increasing run, and similiary for decreasing runs.

## 1. INTRODUCTION

The aim of this paper is to study the sequence  $\{f(n)\}$  of integers determined by the recurrence relation

$$(1) \quad f(n+2) \equiv 2f(n+1) - f(n) \pmod{(n+2)}$$

and the inequality  $0 \leq f(n) < n$  for every integer  $n \geq m$ ,  $m$  being a given positive integer, where  $f(m)$  and  $f(m+1)$  are, as initial values, arbitrarily given integers. If  $f(n+1) - f(n) = 0$  then  $f(k) = f(n)$  for all  $k \geq n$ , and if  $f(n+1) - f(n) = 1$  then  $f(k) = f(n) + k - n$  for all  $k \geq n$ . We exclude these trivial cases in our discussion. For such a sequence  $\{f(n)\}$  we define a run of the sequence to be a maximal increasing or decreasing subsequence of consecutive elements of the sequence of length at least 2.

We call a sequence  $\{f(n)\}$  a solution of the recurrence relation (1) or simply a solution if  $f(n)$  satisfies the relation (1) and the inequality  $0 \leq f(n) < n$  for every  $n \geq m$ , where  $m$  is a given positive integer.

In what follows,  $\{f(n)\}$  means a solution of the recurrence relation (1).

Here we give some examples of solutions:

**Example 1.**

$n$	7	8	9	10	11	12	13	14	15	16	17	18
$f(n)$	0	3	6	9	1	5	9	13	2	7	12	17

In this example the increasing runs are 0, 3, 6, 9; 1, 5, 9, 13 and 2, 7, 12, 17.

It can be immediately seen that for  $n = 7, \dots, 10$ , the difference in consecutive terms is 3, while for  $n = 11, \dots, 14$ , the difference in consecutive terms is 4, which is one more than the previous difference. We will show that this property holds in general.

**Example 2.**

$n$	5	6	7	8	9	10	11	12	13	14	15	16	17
$f(n)$	0	5	3	1	8	5	2	11	7	3	14	9	4

In this example the decreasing runs are 5, 3, 1; 8, 5, 2; 11, 7, 3 and 14, 9, 4.

In the first decreasing run the numbers decrease by 2, in the second by 3, and in the third by 4. We will also show that this property holds in general.

---

*Date:* October 17, 2004.

Research supported by Brandeis University Undergraduate Research Program, Summer 2003.

## 2. RESULTS AND PROOFS

The following theorem describes properties of the increasing and decreasing runs of a solution  $\{f(n)\}$ :

**Theorem.** *Let  $\{f(n)\}$  be a solution of the recurrence (1). Then*

(i) *Each run is an arithmetic progression.*

(ii) *If an increasing run is followed by another increasing run, then the common difference in the second run is one more than the common difference in the first run.*

(iii) *If a decreasing run is followed by another decreasing run, then the common difference in the second run is one more than the common difference in the first run.*

To prove the Theorem we need the following three lemmas:

**Lemma 1.** *Let  $\{f(n)\}$  be a solution. Let  $g(n) = n - 1 - f(n)$  for every  $n \geq m$ . Then  $\{g(n)\}$  is also a solution; that is, we have*

$$(2) \quad g(n+2) \equiv 2g(n+1) - g(n) \pmod{n+2}$$

and  $0 \leq g(n) < n$  for every  $n \geq m$ .

*Proof.* Since  $0 \leq f(n+2) \leq n+1$ , we have  $0 \leq g(n+2) \leq n+1$ . Thus it is sufficient to show that  $g(n+2) \equiv 2g(n+1) - g(n) \pmod{n+2}$ . We have

$$\begin{aligned} g(n+2) &= n+1 - f(n+2) \\ &\equiv n+1 - (2f(n+1) - f(n)) \pmod{n+2} \\ &= n+1 - 2(n - g(n+1)) + (n-1 - g(n)) \\ &= 2g(n+1) - g(n). \end{aligned}$$

Thus  $g(n+2) \equiv 2g(n+1) - g(n) \pmod{n+2}$ . □

Note that if  $f$  and  $g$  are as in Lemma 1 then  $f$  is a trivial solution if and only if  $g$ . If  $f$  and  $g$  are not trivial solutions then  $f(n) < f(n+1)$  if and only if  $g(n) > g(n+1)$ , so increasing runs of  $f$  correspond to decreasing runs of  $g$  and vice-versa.

**Lemma 2.** *For a solution  $\{f(n)\}$ , we have*

$$(3) \quad f(n+2) = 2f(n+1) - f(n) + K(n+2),$$

where

$$(4) \quad K = \begin{cases} 0 & \text{if } f(n) < f(n+1) \text{ and } f(n+1) < f(n+2) \\ -1 & \text{if } f(n) < f(n+1) \text{ and } f(n+1) > f(n+2) \\ 1 & \text{if } f(n) > f(n+1) \text{ and } f(n+1) < f(n+2) \\ 0 & \text{if } f(n) > f(n+1) \text{ and } f(n+1) > f(n+2) \end{cases}$$

*Proof.* We know that  $f(n+2) = 2f(n+1) - f(n) + K(n+2)$  for some integer  $K$  and  $0 \leq f(n+2) \leq n+1$ . We can rewrite this as

$$(5) \quad f(n+2) = f(n+1) + (f(n+1) - f(n)) + K(n+2)$$

or

$$(6) \quad f(n+2) - f(n+1) = f(n+1) - f(n) + K(n+2)$$

We now consider the first two cases separately.

Case 1: If  $K \geq 1$ , then since  $f(n+1) \geq 0$  and  $f(n+1) - f(n) \geq 0$ , it follows from (5) that  $f(n+2) \geq n+2$ , which is a contradiction since  $f(n+2) \leq n+1$ . If  $K \leq -1$ , then from (6),

$$f(n+2) - f(n+1) \leq f(n+1) - f(n) - (n+2)$$

But  $f(n+1) - f(n) < n$ , since  $f(n+1) \leq n$  and  $f(n) \geq 0$ , so  $f(n+2) - f(n+1)$  is negative, which is a contradiction since  $f(n+2) > f(n+1)$ . Since  $K \not\geq 1$  and  $K \not\leq -1$ ,  $K = 0$ .

Case 2: We have  $f(n+1) - f(n) \geq 0$ , so if  $K \geq 0$  then by (5),  $f(n+2) > f(n+1)$ , which is a contradiction. If  $K \leq -2$  then  $f(n+1) - f(n) + K(n+2) \leq n - 0 - 2(n+2) = -n - 4$ . But by (6),  $f(n+1) - f(n) + K(n+2) = f(n+2) - f(n+1) \geq 0 - n = -n$ , a contradiction. Since  $K \not\geq 0$  and  $K \not\leq -2$ ,  $K = -1$ .

Cases 3 and 4 follow from Lemma 1 and cases 1 and 2 .

□

**Lemma 3.** For a solution  $\{f(n)\}$ , we have:

(i) If  $f(n) < f(n+1)$ ,  $f(n+1) > f(n+2)$  and  $f(n+2) < f(n+3)$ , then  $f(n+3) - f(n+2) = f(n+1) - f(n) + 1$ .

(ii) If  $f(n) > f(n+1)$ ,  $f(n+1) < f(n+2)$  and  $f(n+2) > f(n+3)$ , then  $f(n+3) - f(n+2) = f(n+1) - f(n) - 1$ .

*Proof.* By case 3 of Lemma 2,

$$f(n+3) = 2f(n+2) - f(n+1) + (n+3)$$

and by case 2 of Lemma 2,

$$f(n+2) = 2f(n+1) - f(n) - (n+2).$$

So

$$\begin{aligned} f(n+3) - f(n+2) &= f(n+2) - f(n+1) + (n+3) \\ &= f(n+1) - f(n) - (n+2) + (n+3) \\ &= f(n+1) - f(n) + 1. \end{aligned}$$

This proves (i). The proof of (ii) is similar, using cases 1 and 4 of Lemma 2. □

*Proof of the Theorem.* Part (i) of the Theorem for increasing runs (resp. decreasing runs) follows from the first case (resp. the final case) of Lemma 2. Parts (ii) and (iii) follow from Lemma 3. □

We empirically found the following fact:

**Conjecture.** A solution  $\{f(n)\}$  of the recurrence (1) consists entirely of increasing runs or entirely of decreasing runs, except possible for the first run.

Note that a solution consisting entirely of increasing runs of length 2 can also be described as a solution consisting entirely of decreasing runs of length 2.

As a step towards the conjecture, we show that an increasing run of length at least 3 cannot be followed by a decreasing run of length at least 3.

**Lemma 4.** For a solution  $\{f(n)\}$  it is impossible that  $f(n) < f(n+1)$ ,  $f(n+1) < f(n+2)$ ,  $f(n+2) > f(n+3)$ , and  $f(n+3) > f(n+4)$ .

*Proof.* To show that the set of inequalities above is impossible, we use contradiction.

Assume that the inequalities hold. Then by case 1 of Lemma 2,  $f(n+2) - f(n+1) = f(n+1) - f(n) = A$ . Since  $f(n) \geq 0$ , and  $f(n+1) \leq n$ , we have  $A \leq n$ . Also  $2A \leq f(n) + 2A = f(n+2) \leq n+1$ , so  $2A \leq n+1$ .

We now want to express  $f(n+2) - f(n+3) = f(n+3) - f(n+4) = S$  in terms of  $A$ . By case 2 of Lemma 2,  $f(n+3) = 2f(n+2) - f(n+1) - (n+3)$ . Then

$$\begin{aligned} S &= f(n+2) - f(n+3) = f(n+2) - [2f(n+2) - f(n+1) - (n+3)] \\ &= -f(n+2) + f(n+1) + (n+3) \\ &= (n+3) - [f(n+2) - f(n+1)] \\ &= (n+3) - A. \end{aligned}$$

Now  $2S \leq f(n+4) + 2S = f(n+2) \leq n+1$ , so  $2S \leq n+1$ . So  $2(n+3-A) \leq n+1$ . Then  $2n+6-2A \leq n+1$ . Therefore  $n+5 \leq 2A$ , which is a contradiction since  $2A \leq n+1$ . □

In addition, from Lemma 1 and Lemma 4 it follows that a decreasing run of length at least 3 cannot be followed by an increasing run of length at least 3—again this is a step towards proving our conjecture, but does not prove it fully.

Finally, we can say something about whether all runs have the same length. Empirically we know that they either have the same length (except the first, which might be truncated since since we're not starting at the beginning), or the lengths consist of two consecutive numbers.

DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MA 02454  
*E-mail address:* [qsaify@brandeis.edu](mailto:qsaify@brandeis.edu)