

A SEARCH FOR PATTERNS IN A SEQUENCE OF NUMBERS: PROVING TRENDS IN A RECURRENCE RELATION

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ABSTRACT. In this paper we study the sequence of numbers generated by the recurrence relation $f(n+2) = 2f(n+1) - f(n) \pmod{n+2}$, where $n \geq m$, and $f(m)$ and $f(m+1)$ are initially assigned certain integer values. More precisely we want to look at how the sequence of numbers generated increases, calling a maximal increasing or decreasing subsequence of consecutive elements of the sequence a run. We show that each run is an arithmetic progression. We prove that the common difference in an increasing run is one more than the common difference in the previous increasing run, and similarly for decreasing runs.

1. INTRODUCTION

The aim is to study the sequence of numbers generated by the recurrence relation

$$(1) \quad f(n+2) = 2f(n+1) - f(n) \pmod{n+2}$$

for $n \geq m+2$ where $f(m)$ and $f(m+1)$ are initial values which are arbitrary integers satisfying $0 \leq f(m) < m$ and $0 \leq f(m+1) < m+1$. Here $a \pmod b$ means the non-negative remainder when a is divided by b , so (1) means $f(n+2) \equiv 2f(n+1) - f(n) \pmod{n+2}$ and $0 \leq f(n+2) \leq n+1$. Define a run to be a maximal increasing or decreasing subsequence of consecutive elements of the sequence of length at least 2. Look at the solution to the recurrence below. In this example, the increasing runs are 0 3 6 9, 1 5 9 13, and 2 7 12 17.

n	7	8	9	10	11	12	13	14	15	16	17	18
$f(n)$	0	3	6	9	1	5	9	13	2	7	12	17

Here $f(n)$ is the solution to the recurrence, and it can be seen at a glance that for $n = 7, \dots, 10$, the increase in consecutive terms is 3, while for $n = 11, \dots, 14$, the increase in consecutive terms is 4, which is one more than the previous increase. We will show that this trend holds in general.

In the following example, the decreasing runs are 5 3 1, 8 5 2, 11 7 3, and 14 9 4. In the first decreasing run the numbers decrease by 2, in the second by 3, and in the third by 4.

n	5	6	7	8	9	10	11	12	13	14	15	16	17
$f(n)$	0	5	3	1	8	5	2	11	7	3	14	9	4

2. DEFINITIONS, THEOREMS, AND PROOFS

The following theorem describes the increasing and decreasing runs of a sequence f satisfying 1:

Theorem.

- (i) *Each run is an arithmetic progression.*
- (ii) *If an increasing run is followed by another increasing run then the common difference in the second run is one more than the common difference in the first run.*

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(iii) If a decreasing run is followed by another decreasing run then the common difference in the second run is one more than the common difference in the first run.

In order to prove the theorem, we first prove several lemmas:

Lemma 1. If $f(n) < f(n+1)$ and $f(n+1) < f(n+2)$, then

$$f(n+2) = 2f(n+1) - f(n).$$

Proof. We know that $f(n+2) = 2f(n+1) - f(n) + K(n+2)$ for some integer K and $0 \leq f(n+2) \leq n+1$. Rewrite

$$f(n+2) = 2f(n+1) - f(n) + K(n+2)$$

as

$$f(n+2) = f(n+1) + (f(n+1) - f(n)) + K(n+2).$$

□

Lemma 2. If $f(n) < f(n+1)$ and $f(n+1) > f(n+2)$, then

$$f(n+2) = 2f(n+1) - f(n) - (n+2).$$

Proof. We know that $f(n+2) = 2f(n+1) - f(n) + K(n+2)$ for some integer K and $0 \leq f(n+2) \leq n+1$. Rewrite

$$f(n+2) = 2f(n+1) - f(n) + K(n+2)$$

as

$$f(n+2) = f(n+1) + (f(n+1) - f(n)) + K(n+2).$$

Now $f(n+1) \geq 0$ and $f(n+1) - f(n) \geq 0$. If K is non-negative, then $f(n+2) \geq f(n+1)$, which is a contradiction since $f(n+1) > f(n+2)$.

If $K \leq -2$, then we want to show that

$$2f(n+1) - f(n) + K(n+2) < 0.$$

Now $n < f(n)/2 + (n+2)$, since $f(n) \geq 0$. Also we know $f(n+1) \leq n$. Then

$$f(n+1) < f(n)/2 + (n+2).$$

Therefore

$$2f(n+1) < f(n) + 2(n+2).$$

Finally

$$2f(n+1) - f(n) - K(n+2) \leq 2f(n+1) - f(n) - 2(n+2) < 0.$$

Since $K \not\geq 0$ and $K \not\leq -2$, $K = -1$.

□

Lemma 3. If $f(n) > f(n+1)$ and $f(n+1) < f(n+2)$ then

$$f(n+2) = 2f(n+1) - f(n) + (n+2).$$

Proof. We know that $f(n+2) = 2f(n+1) - f(n) + K(n+2)$ for some integer K and $0 \leq f(n+2) \leq n+1$.

Thus it is enough to show that

$$0 \leq 2f(n+1) - f(n) + n + 2 \leq n + 1$$

Suppose that $K = 2$. We know that $f(n) \leq (n-1)$. Therefore $-f(n) \geq -(n-1)$.

Suppose that $K \leq 0$. We can rewrite

$$f(n+2) = 2f(n+1) - f(n) - K(n+2)$$

as

$$f(n+1) = \frac{f(n+2)}{2} + \frac{f(n)}{2} + \frac{K(n+2)}{2}$$

This is a contradiction, since for $f(n+2) > f(n+1)$, $f(n) > f(n+1)$, and $K \geq 0$,

$$\frac{f(n+2)}{2} + \frac{f(n)}{2} + \frac{K(n+2)}{2} > f(n+1).$$

□

Lemma 4. *If $f(n) < f(n+1)$ and $f(n+1) > f(n+2)$ and $f(n+2) < f(n+3)$ then $f(n+3) - f(n+2) = 1 + f(n+1) - f(n)$.*

$$\begin{aligned} f(n+3) - f(n+2) &\equiv 2f(n+1) - f(n) - (n+2) - f(n+1) \pmod{n+3} \\ f(n+3) - f(n+2) &= f(n+1) - f(n) - (n+2) \pmod{n+3}. \end{aligned}$$

By Lemma 3,

$$f(n+3) - f(n+2) = f(n+1) - f(n) - (n+2) + (n+3).$$

Finally,

$$f(n+3) - f(n+2) = f(n+1) - f(n) + 1.$$

Lemma 5. *Suppose that $f(n)$ satisfies the recurrence (1). Let $g(n) = n - 1 - f(n)$. Then g satisfies the recurrence $g(n+2) = 2g(n+1) - g(n) \pmod{n+2}$.*

Proof. Since $0 \leq f(n+2) \leq n+1$, we have $0 \leq g(n+2) \leq n+1$. Thus it is sufficient to show that $g(n+2) \equiv 2g(n+1) - g(n) \pmod{n+2}$. We have

$$\begin{aligned} g(n+2) &= n+1 - f(n+2) \\ &\equiv n+1 - (2f(n+1) - f(n)) \pmod{n+2} \\ &= n+1 - 2(n - g(n+1)) + (n-1 - g(n)) \\ &= 2g(n+1) - g(n). \end{aligned}$$

Thus $g(n+2) \equiv 2g(n+1) - g(n) \pmod{n+2}$. □

Proof of the theorem. Part (i) of the theorem for increasing runs follows from Lemma 1. Part (ii) of the theorem follows from Lemma 4. The results for decreasing runs follow easily from the corresponding results for increasing runs by Lemma 5. □

We observe empirically that except for the trivial cases in which $f(n)$ is constant, every solution of (1) has the property that either it consists entirely of increasing runs or it consists entirely of decreasing runs. The following result is a step in this direction.

Proposition. *If f satisfies (1) then it is impossible that $f(n) < f(n+1)$, $f(n+1) < f(n+2)$, $f(n+2) > f(n+3)$, and $f(n+3) > f(n+4)$.*

Proof. To show that the set of inequalities above is impossible, we use contradiction.

Assume that the inequalities hold. Then by Lemma 1, $f(n+2) - f(n+1) = f(n+1) - f(n) = A$. Since $f(n) \geq 0$, and $f(n+1) \leq n$, we have $A \leq n$. Also $2A \leq f(n) + 2A = f(n+2) \leq n+1$, so $2A \leq n+1$.

We now want to express $f(n+2) - f(n+3) = f(n+3) - f(n+4) = S$ in terms of A . By Lemma 2, $f(n+3) = 2f(n+2) - f(n+1) - (n+3)$. Then

$$\begin{aligned} S &= f(n+2) - f(n+3) = f(n+2) - [2f(n+2) - f(n+1) - (n+3)] \\ &= -f(n+2) + f(n+1) + (n+3) \\ &= (n+3) - [f(n+2) - f(n+1)] \\ &= (n+3) - A. \end{aligned}$$

Now $2S \leq f(n+4) + 2S = f(n+2) \leq n+1$, so $2S \leq n+1$ and $f(n+2) \leq n+1$. Then $2S \leq n+1$. So $2(n+3-A) \leq n+1$. Then $2n+6-2A \leq n+1$. Therefore $n+5 \leq 2A$, which is a contradiction since $2A \leq n+1$.

□

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