Symbolic Powers of Edge Ideals

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Abstract. In this paper we discuss a connection between graph theory and ring theory. Given a graph G, there exists a corresponding edge ideal I generated by $x_i x_j$ where x_i and x_j are vertices in G connected by an edge. Simis, Vasconcelos, and Villarreal show that a graph G is bipartite (contains only even cycles) if and only if its corresponding edge ideal I satisfies $I^{(n)} = I^n$ for all $n \geq 1$. We explore what happens when G is not bipartite - in particular, when G is an odd sided polygon.

1 Introduction

In 1992, Aron Simis, Wolmer V. Vasconcelos, and Rafael H. Villarreal published the paper, *On the Ideal Theory of Graphs.* In this paper, they studied the correspondence between graph theory and ring theory via graphs and their associated edge ideals. Their main interest was relating properties of a graph G to properties of certain algebras defined by the edge ideal of G and vice versa. Our research stems from one of their results which characterizes the bipartiteness of a graph in terms of a special property of its edge ideal. In this paper we will address what properties of the edge ideal are expected if the graph is an odd sided polygon.

Throughout the paper we will denote a graph by G, an ideal by I, and a ring by R. In particular, if G contains vertices $x_1, \ldots, x_n$, then I will be an edge ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$ where k is a field. Standard terminology used in this paper consists of *bipartite* graphs, *degree* two monomials, and *square-free* monomials. Bipartite graphs are graphs that include only even cycles. The degree of a monomial is the sum of the exponents of that monomial, so a degree two monomial is a monomial in which the sum of the exponents is two. A square-free monomial is a monomial in which all exponents are 0 or 1.

2 Our Problem

In [3], the authors explored properties between edge ideals in ring theory and their associated graphs in graph theory. The connection between the two theories is as follows: given a graph G consisting of vertices and edges, the corresponding *edge ideal* $I(G)$ is the ideal generated by the degree two, square-free monomials representing edges. That is, $x_i x_j$ is a generator of $I(G)$ if and only if x_i and x_j are vertices in G joined by an edge of G.

Example 1. The graph C_5 labelled as Figure 1 has corresponding edge ideal $I(C_5) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5 \rangle$.

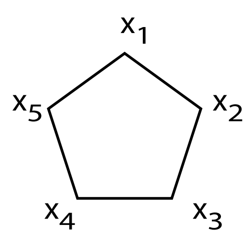

Figure 1: An illustration of C_5.

A slightly modified version of a theorem from [3] which motivates the work in this paper is the following:

Theorem 2. *Let G be a graph and let $I = I(G)$ be its edge ideal. The following conditions are equivalent:*
(i) G is bipartite
(ii) $I^{(n)} = I^n$ for $n \geq 1$.

The ideal $I^{(n)}$ is called the n^{th} *symbolic power* of the ideal I and I^n is called the n^{th} *power* of the ideal I.

Definition 3. The n^{th} *symbolic power* of an ideal $I \subset R$ is the ideal

$$I^{(n)} = \{r \in R \mid sr \in I^n \text{ because } s + I \text{ is a regular element of } R/I\}.$$

Definition 4. The n^{th} *power* of an ideal $I = \langle f_1, f_2, \ldots, f_r \rangle \subset R$ is the ideal

$$I^n = \langle f_1^{m_1} f_2^{m_2} \cdots f_r^{m_r} | m_1 + m_2 + \cdots + m_r = n \rangle.$$

Computing the power of an ideal is relatively easy. For example, if $I = \langle x, y\rangle$, then $I^2 = \langle x^2, xy, y^2\rangle$ and $I^3 = \langle x^3, x^2y, xy^2, y^3\rangle$.

Theorem 2 prompts one to ask the following question: What happens if G is not bipartite? Using the theorem, there exists an $n \geq 1$ such that $I^{(n)} \neq I^n$. In particular, we want to know for what n does $I^{(n)} = I^n$? If equality does not hold, then what is the symbolic power? In this article, we explore graphs of odd sided polygons denoted by C_{2n+1} where n is a nonnegative integer.

Instead of using the definition of the symbolic power, we choose to use an alternate way to compute the symbolic power of an edge ideal. A simplified version of this method is stated in [4]:

Proposition 5. *Let I be a radical ideal of a ring R and $p_1, \cdots, p_r$ the minimal primes of I. Then*

$$I^{(n)} = p_1^n \cap \ldots \cap p_r^n$$

for $n \geq 1$.

Note that a radical ideal is an ideal J in which $f^m \in J$ implies $f \in J$. In a polynomial ring over a field, radical monomial ideals are exactly the monomial ideals with square-free generators. Examples of radical ideals include edge ideals. Also, minimal primes of I are the "smallest" prime ideals containing I. The details of minimal primes are discussed in Section 4.

In the next example we use Proposition 5 to compute a symbolic power by hand.

Example 6. The edge ideal corresponding to C_3 is $I = \langle xy, yz, xz\rangle$. Let's compute $I^{(2)}$.

$$\begin{aligned}
I^{(2)} &= \langle x, y\rangle^2 \cap \langle x, z\rangle^2 \cap \langle y, z\rangle^2 \\
&= \langle x^2, xy, y^2\rangle \cap \langle x^2, xz, z^2\rangle \cap \langle y^2, yz, z^2\rangle \\
&= \langle x^2, x^2z, x^2z^2, x^2y, xyz, xyz^2, x^2y^2, xy^2z, y^2z^2\rangle \cap \langle y^2, yz, z^2\rangle \\
&= \langle x^2, xyz, y^2z^2\rangle \cap \langle y^2, yz, z^2\rangle \\
&= \langle x^2y^2, x^2yz, x^2z^2, xy^2z, xyz, xyz^2, y^2z^2, y^2z^2, y^2z^2\rangle \\
&= \langle x^2y^2, x^2z^2, xyz, y^2z^2\rangle
\end{aligned}$$

As can be seen from the computation above, symbolic powers can be rather tedious to compute by hand. Thus, we turned to the computer algebra system *Macaulay* to do the task of computation for $n > 1$. We leave it to the reader to show $I^{(1)} = I$.

3 Computations

To get an idea if the symbolic and ordinary powers of edge ideals are equal, we used *Macaulay* to compute several examples. We performed several computations for the regular and symbolic powers of the edge ideal of C_{2n+1} and are including the computations for C_3 and C_5.

Example 7. The edge ideal corresponding to C_3 is $I = \langle xy, yz, xz \rangle$.

$$I^2 = \langle x^2y^2, y^2z^2, x^2z^2, xy^2z, x^2yz, xyz^2 \rangle$$
$$I^{(2)} = \langle x^2y^2, y^2z^2, x^2z^2, xyz \rangle$$

Note that every generator in I^2 is a generator of $I^{(2)}$ or is a multiple of the product of the vertices xyz. Also, notice that xyz cannot be in I^2 because of degree reasons; xyz is of degree 3 while all elements in I^2 are of degree 4 or higher.

$$I^3 = \langle x^3y^3, y^3z^3, x^3z^3, x^3y^2z, x^3yz^2, x^2y^3z, x^2y^2z^2, x^2yz^3, xy^3z^2, xy^2z^3 \rangle$$
$$I^{(3)} = \langle x^3y^3, y^3z^3, x^3z^3, x^2y^2z, x^2yz^2, xy^2z^2 \rangle$$

Observe again that the generators of I^3 are generators of $I^{(3)}$ or are multiples of the remaining generators in $I^{(3)}$. Also, the generators of $I^{(3)}$ that are not generators of I^3 cannot be in I^3 because of degree reasons.

$$I^4 = \langle x^4y^4, y^4z^4, x^4z^4, x^4y^3z, x^4y^2z^2, x^4yz^3, x^3y^4z, x^3y^3z^2, \\ x^3y^2z^3, x^3yz^4, x^2y^4z^2, x^2y^3z^3, x^2y^2z^4, xy^4z^3, xy^3z^4 \rangle$$

$$I^{(4)} = \langle x^4y^4, y^4z^4, x^4z^4, x^3y^3z, x^3yz^3, x^2y^2z^2 \rangle$$

Once again, all generators of I^4 are contained in $I^{(4)}$ but the generators of $I^{(4)}$ of degrees six and seven cannot be contained in I^4 due to degree reasons.

Thus for $G = C_3$, we have $I^{(1)} = I$, $I^{(2)} \neq I^2$, $I^{(3)} \neq I^3$, and $I^{(4)} \neq I^4$.

Example 8. The edge ideal corresponding to C_5 is $I = \langle xy, yz, zw, wv, xv \rangle$.

$$I^2 = \langle x^2y^2, y^2z^2, z^2w^2, w^2v^2, x^2v^2, x^2yv, xy^2z, xyzw, \\ xyzv, xywv, xzwv, xwv^2, yz^2w, yzwv, zw^2v \rangle$$

$$I^{(2)} = \langle x^2y^2, y^2z^2, z^2w^2, w^2v^2, x^2v^2, x^2yv, xy^2z, xyzw,$$
$$xyzv, xywv, xzwv, xwv^2, yz^2w, yzwv, zw^2v\rangle$$

Here, $I^2 = I^{(2)}$.

$$I^3 = \langle x^3y^3, y^3z^3, z^3w^3, w^3v^3, x^3v^3, x^3y^2v, x^3yv^2, x^2y^3z, x^2y^2zw, x^2y^2zv,$$
$$x^2y^2wv, x^2yzwv, x^2yzv^2, x^2ywv^2, x^2zwv^2, x^2wv^3, xy^3z^2, xy^2z^2w,$$
$$xy^2z^2v, xy^2zwv, xyz^2w^2, xyz^2wv, xyzw^2v, xyzwv^2, xyw^2v, xz^2w^2v,$$
$$xw^2v^3, y^2z^3w, y^2z^2wv, yz^3w^2, yz^2w^2v, yzw^2v^2, z^2w^3v, zw^3v^2\rangle$$

$$I^{(3)} = \langle x^3y^3, y^3z^3, z^3w^3, w^3v^3, x^3v^3, x^3y^2v, x^3yv^2, x^2y^3z, x^2y^2zw, x^2y^2zv,$$
$$x^2y^2wv, x^2yzwv, x^2yzv^2, x^2ywv^2, x^2zwv^2, x^2wv^3, xy^3z^2, xy^2z^2w,$$
$$xy^2z^2v, xy^2zwv, xyz^2w^2, xyz^2wv, xyzw^2v, xyzwv^2, xyw^2v, xz^2w^2v,$$
$$xw^2v^3, y^2z^3w, y^2z^2wv, yz^3w^2, yz^2w^2v, yzw^2v^2, z^2w^3v, zw^3v^2, xyzwv\rangle$$

Notice that the remaining generator in $I^{(3)}$ that is not in I^3 is the product of the vertices, $xyzwv$.

We also have computations to show that $I^{(4)} \neq I^4$ but we do not provide them here due to space issues.

Thus for $G = C_5$, we have $I^{(1)} = I$, $I^{(2)} = I^2$, $I^{(3)} \neq I^3$, and $I^{(4)} \neq I^4$.

Additional Macaulay computations show that $I^{(1)} = I$, $I^{(2)} = I^2$, $I^{(3)} = I^3$, and $I^{(4)} \neq I^4$ for $G = C_7$ and $I^{(1)} = I$, $I^{(2)} = I^2$, $I^{(3)} = I^3$, and $I^{(4)} = I^4$ for $G = C_9$. From these computations and the ones above, we developed our conjecture.

Conjecture 9. *Let C_{2n+1} be an odd cycle of length $2n+1$ and I be its edge ideal. Then*

1. $I^{(t)} = I^t$ *for* $1 \leq t \leq n$
2. $I^{(n+r)} \supseteq I^{n+r} + I^{r-1} \cdot \langle x_1x_2 \cdots x_{2n+1}\rangle$ *for all* $r \geq 1$, *where equality holds for* $r = 1$.

In order to prove this conjecture, we develop a more appropriate theoretical framework in the following sections.

4 Preliminaries

In Section 2, Proposition 5 involves minimal primes of an ideal and radical ideals. In this section we provide the reader with the necessary definitions and background for these ideas to be fully understood.

4.1 Prime Ideals

A *prime ideal* p in a commutative ring R is a proper ideal of R such that $a, b \in R$ and $ab \in p$ implies $a \in p$ or $b \in p$. A nice class of prime ideals are monomial prime ideals in $k[x_1, x_2, \ldots, x_n]$ where k is a field. These ideals are generated by a subset of the variables $x_1, \ldots, x_n$.

Given an ideal I, the *minimal primes of* I are the collection of "smallest" primes containing I. That is, p is a minimal prime of I if $I \subset p$ and there does not exist another prime q such that $I \subset q \subset p$.

Example 10. Let $I = \langle x_1x_2, x_2x_3, x_1x_3\rangle$. The minimal primes of I are:

$$\langle x_1, x_2\rangle, \langle x_2, x_3\rangle, \langle x_1, x_3\rangle$$

Note that $\langle x_1, x_2, x_3\rangle$ is a prime of I. However, it is not a minimal prime of I because it properly contains another prime ideal containing I.

Notice that the previous example of an edge ideal I corresponds to the triangle. Also, every minimal prime of I is composed of a minimum number of vertices needed to make each edge of the triangle incident with one vertex as seen below. This is not a coincidence.

Definition 11. Let G be a graph with vertex set V. A subset $A \subset V$ is a *minimal vertex cover* for G if: (i) every edge of G is incident with at least one vertex in A, and (ii) there is no proper subset of A with the first property.

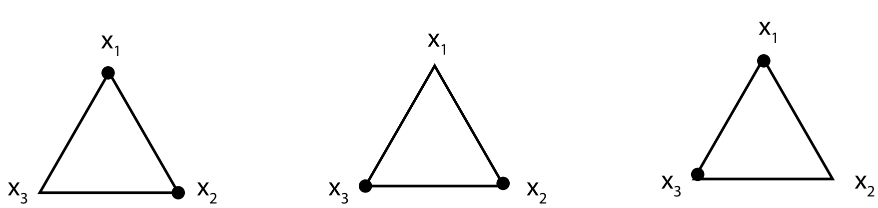

Figure 2: From left to right, the darkened vertices on each triangle represent the only minimal vertex covers $\{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\}$ of C_3.

One can deduce from the definitions that minimal primes of $I(G)$ correspond to minimal vertex covers of G. In fact, we found minimal vertex covers rather than calculating the minimal primes because, in our opinion, it was easier to analyze pictures.

5 Results

The following section consists of a few lemmas that will be used to prove our results later in the section.

Lemma 12. *Let C_{2n+1} be an odd cycle of length $2n+1$. There are at least $2n+1$ minimal vertex covers.*

Proof. We will show that

$$\{x_{imod(2n+1)}, x_{(i+1)mod(2n+1)}, x_{(i+3)mod(2n+1)}, \cdots, x_{(i+2n-1)mod(2n+1)}\}$$

for $i = 1, 2, \cdots, 2n+1$ are minimal vertex covers. Without loss of generality, it is enough to show $\{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$ is a minimal vertex cover.

In order for us to show that $\{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$ is a vertex cover, we must show that at least one endpoint of every edge is in $\{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$. First, note that the edge whose endpoints are x_{2n+1} and x_1 has the vertex $x_1 \in \{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$. Any other edge of C_{2n+1} has endpoints x_j and x_{j+1} for some $1 \leq j \leq 2n$. If j is even, then, $x_j \in \{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$ because all even vertices are in $\{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$. If j is odd, then we know $j+1$ is even, and $2 \leq j+1 \leq 2n$, which gives $x_{j+1} \in \{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$. Therefore, $\{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$ is a vertex cover.

Now we must show $\{x_1, x_2, x_4, x_6, \cdots, x_{2n}\}$ is a *minimal* vertex cover. This will be shown by eliminating a vertex from the vertex cover and noting it is not still a vertex cover.

Case 1: First, let x_1 be eliminated. This leaves $\{x_2, x_4, x_6, \cdots, x_{2n}\}$. This is not a vertex cover because the edge whose endpoints are x_1 and x_{2n+1} is not in $\{x_2, x_4, x_6, \cdots, x_{2n}\}$, and by definition of vertex cover, every edge must have a vertex incidental to it listed in the vertex cover.

Case 2: Now, let x_{2i} where $1 \leq i \leq n$ be eliminated. This will leave $\{x_1, x_2, \cdots, x_{2(i-1)}, x_{2(i+1)}, \cdots, x_{2n}\}$, which is not a vertex cover because it leaves the edge whose endpoints are x_{2i} and x_{2i+1} without an incidental vertex listed in $\{x_1, x_2, \cdots, x_{2(i-1)}, x_{2(i+1)}, \cdots, x_{2n}\}$. □

For some $n \geq 4$, there are more than $2n+1$ minimal vertex covers in C_{2n+1}. Notice that the $2n+1$ minimal vertex covers from the previous lemma started with 2 adjacent vertices in the minimal vertex cover and then began alternating so that every other vertex would be in a minimal vertex cover. To see a different vertex cover for C_9 consider $x_1, x_2, x_4, x_5, x_7, x_8$. We can see that this is a minimal vertex cover and it has a different pattern than the previous minimal vertex covers mentioned in the lemma. This new pattern starts with 2 adjacent vertices in the minimal vertex cover, then one not in the cover, then another pair in the cover, then one not in cover, etc as seen in Figure 3.

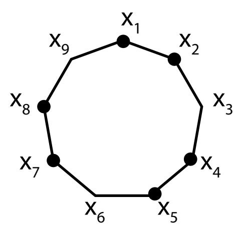

Figure 3: A "different" minimal vertex cover for C_9.

Definition 13. A walk of length n in a graph G is an alternating sequence of vertices and edges

$$w = v_0, z_1, v_1, \cdots v_{n-1}, z_n, v_n,$$

where $z_i = v_{i-1}, v_i$ is the edge joining v_{i-1} and v_i.

Lemma 14. *Let C_{2n+1} be an odd cycle of length $2n+1$. Any minimal vertex cover of the edge ideal $I(G)$ contains at least $n+1$ vertices.*

Proof. Assume there are at least $n+1$ vertices not in some minimal vertex cover p of C_{2n+1}. Let's say $v_1, \ldots, v_{n+1}$ are among the vertices not in the minimal vertex cover p, and without loss of generality, we may assume that these vertices are listed in order as we move clockwise around C_{2n+1}. Since there cannot be two adjacent vertices not in p, we must have a vertex w_i between vertices v_i and v_{i+1} for every i. Thus $v_1, \ldots, v_{n+1}, w_1, \ldots w_{n+1}$ are contained in C_{2n+1} which gives us a contradiction. Therefore, there are at most n vertices not in p, which implies there are at least $n+1$ vertices in each minimal vertex cover p. □

The previous lemmas can be restated in terms of minimal primes of edge ideals. That is, there are at least $2n + 1$ minimal primes containing at least $n + 1$ vertices for the edge ideal of an odd sided polygon, C_{2n+1}.

Recall the conjecture from a previous section:

Conjecture 15. *Let C_{2n+1} be an odd cycle of length $2n+1$ and I be its edge ideal. Then*

1. $I^{(t)} = I^t$ *for* $1 \leq t \leq n$
2. $I^{(n+r)} \supseteq I^{n+r} + I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$ *for all* $r \geq 1$*, where equality holds for* $r = 1$.

We only have a partial proof for the conjecture above. We should note that the ordinary power is always contained in the symbolic power (the argument is given in the proof for completeness). Therefore, for the reverse containment on (ii), we focus only on the second term, $I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$. The lemma and its proof are as follows:

Lemma 16. *Let C_{2n+1} be an odd cycle of length $2n + 1$ and I be its corresponding edge ideal. Then $I^{(n+r)} \supseteq I^{n+r} + I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$ for all $r \geq 1$.*

Proof. It is known that $I^{(k)} \supseteq I^k$ for all k. To see this, we first recall that

$$I^{(k)} = p_1^k \cap \cdots \cap p_m^k$$

from Proposition 5 where each p_i is a minimal prime. To show this containment, it is enough to show that all the generators of I^k are contained in $I^{(k)}$. Following the discussion in Section 2 about generators of I^k, each generator of I^k is of the form $f_1 f_2 \cdots f_k$, where each f_i is a generator in I. Since p_j is a minimal prime of I, we have $f_i \in p_j$ for every i and j. Hence $f_1 f_2 \cdots f_k \in p_j^k$ for any j. Thus $f_1 f_2 \cdots f_k \in p_1^k \cap \cdots \cap p_m^k$. Therefore, $I^{(k)} \supseteq I^k$.

To prove the containment $I^{(n+r)} \supseteq I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$, we induct on r. Suppose $r = 1$ and let $\{p_i\}$ be a minimal primes of I. By Lemma 14, every p_i contains at least $n + 1$ variables from the set $\{x_1, x_2, \cdots, x_{2n+1}\}$. Thus a product of $n + 1$ distinct variables belongs to p_i^{n+1} for every i, which implies that $x_1 x_2 \cdots x_{2n+1}$ belongs to p_i^{n+1}. Hence $x_1 \cdots x_{2n+1} \in p_1^{n+1} \cap \cdots \cap p_m^{n+1} = I^{(n+1)}$.

Assume $\langle x_1 \cdots x_{2n+1} \rangle \cdot I^{r-2} \subseteq I^{(n+r-1)}$. We must show that a generator of $\langle x_1 \cdots x_{2n+1} \rangle \cdot I^{r-1}$ is an element of $I^{(n+r)}$. A generator of $\langle x_1 \cdots x_{2n+1} \rangle \cdot I^{r-1}$

has the form $(x_1 \cdots x_{2n+1})(m_1 m_2 \cdots m_{r-1})$, where the $m_i' s$ are generators of I . Observe that

$$\begin{aligned}
(x_1 \cdots x_{2n+1})(m_1 m_2 \cdots m_{r-1}) &= [(x_1 \cdots x_{2n+1})(m_1 m_2 \cdots m_{r-2})] m_{r-1} \\
&\in I^{(n+r-1)} \cdot I \\
&= (p_1^{n+r-1} \cap \cdots \cap p_m^{n+r-1}) \cdot (p_1 \cap \cdots \cap p_m) \\
&\subseteq p_1^{n+r} \cap \cdots \cap p_m^{n+r} \\
&= I^{(n+r)}
\end{aligned}$$

□

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