Symbolic Powers of Edge Ideals

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Abstract. In this paper we discuss a connection between graph theory and ring theory. Given a graph G, there exists a corresponding edge ideal I generated by $x_i x_j$ where x_i and x_j are vertices in G connected by an edge. Simis, Vasconcelos, and Villarreal show that a graph G is bipartite (contains only even cycles) if and only if its corresponding edge ideal I satisfies $I^{(n)} = I^n$ for all $n \geq 1$. We explore what happens when G is not bipartite - in particular, when G is an odd sided polygon.

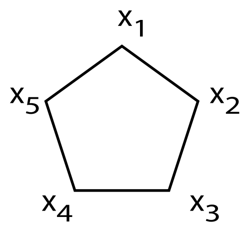
1 Introduction

In 1992, Aron Simis, Wolmer V. Vasconcelos, and Rafael H. Villarreal published the paper, *On the Ideal Theory of Graphs*. In this paper, they studied the correspondence between graph theory and ring theory via graphs and their associated edge ideals. Their main interest was relating properties of a graph G to properties of certain algebras defined by the edge ideal of G and vice versa. Our research stems from one of their results which characterizes the bipartiteness of a graph in terms of a special property of its edge ideal. In this paper we will address what properties of the edge ideal are expected if the graph is an odd sided polygon.

2 Our Problem

In [3], the authors explored properties between edge ideals in ring theory and their associated graphs in graph theory. The connection between the two theories is as follows: given a graph G consisting of vertices and edges, the corresponding *edge ideal* $I(G)$ is the ideal generated by the degree two (sum of exponents is 2), square-free (exponents ≤ 1) monomials representing edges. That is, x_ix_j is a generator of $I(G)$ if and only if x_i and x_j are vertices in G joined by an edge of G.

Example 1. The graph G of a pentagon labelled as



has corresponding edge ideal $I(G) = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5\rangle$.

A theorem from [3], which motivates the work in this paper, is the following.

Theorem 2. *Let G be a graph and let $I = I(G)$ be its edge ideal. The following conditions are equivalent:*
(i) G is bipartite
(ii) I is normally torsion free.

A graph G is *bipartite* if it contains only even cycles. The second statement from the theorem above is equivalent to $I^{(n)} = I^n$ for $n \geq 1$. That is, the n^{th} symbolic power of I is equal to the n^{th} power of I for all n; these terms will be defined shortly.

Theorem 2 prompts one to ask the following question: What happens if G is not bipartite? Using the theorem, there exists an $n \geq 1$ such that $I^{(n)} \neq I^n$. In particular, we want to know for what n does $I^{(n)} = I^n$? If equality does not hold, then what is the symbolic power? In this article, we explore graphs of odd sided polygons denoted by C_{2n+1} where n is a nonnegative integer.

A first step in answering our questions is to compute the regular powers of I. In general, if $I = \langle f_1, f_2, \ldots, f_n\rangle$, then

$$I^k = \langle f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} | m_1 + m_2 + \cdots + m_n = k\rangle.$$

For example, if $I = \langle x, y\rangle$, then $I^2 = \langle x^2, xy, y^2\rangle$ and $I^3 = \langle x^3, x^2y, xy^2, y^3\rangle$.

Next let us define the n^{th} *symbolic power* of an ideal I:

Definition 3. The n^{th} *symbolic power* of an ideal $I \subset R$ is the ideal $I^{(n)} = \{r \in R \mid sr \in I^n \text{ for some } 0 \neq s \in R, s \text{ is a nonzerodivisor of } R/I\}$.

Instead of using the definition above, we chose to use an alternate way to compute the symbolic power of an edge ideal. A simplified version of this method states [4]:

Proposition 4. *Let I be a radical ideal of a ring R and $p_1, \cdots, p_r$ the minimal primes of I. Then*

$$I^{(n)} = p_1^n \cap \ldots \cap p_r^n$$

for $n \geq 1$.

While the ideals described above can be computed by hand, we turned to the computer algebra system, Macaulay, to do the tedious task of computation for $n > 1$. For $n = 1$ observe that $I^{(1)} = I$ since we can use $s = 1$ in the definition of $I^{(1)}$.

3 Computations

To get an idea if the symbolic and ordinary powers of edge ideals are equal, we used Macaulay to compute several examples. Here we present the computations for C_3 and C_5. We were hoping to find a pattern from these computations, and we did, which will be announced shortly.

Example 5. The edge ideal corresponding to C_3 is $I = \langle xy, yz, xz \rangle$.

$$I^2 = \langle x^2y^2, y^2z^2, x^2z^2, xy^2z, x^2yz, xyz^2 \rangle$$
$$I^{(2)} = \langle x^2y^2, y^2z^2, x^2z^2, xyz \rangle$$

Note that every generator in I^2 is a generator of $I^{(2)}$ or is a multiple of the product of the vertices xyz. Also, notice that xyz cannot be in I^2 because of degree reasons; xyz is of degree 3 while all elements in I^2 are of degree 4 or higher.

$$I^3 = \langle x^3y^3, y^3z^3, x^3z^3, x^3y^2z, x^3yz^2, x^2y^3z, x^2y^2z^2, x^2yz^3, xy^3z^2, xy^2z^3 \rangle$$

$$I^{(3)} = \langle x^3y^3, y^3z^3, x^3z^3, x^2y^2z, x^2yz^2, xy^2z^2 \rangle$$

Observe again that the generators of I^3 are generators of $I^{(3)}$ or are multiples of the remaining generators in $I^{(3)}$. Also, the generators of $I^{(3)}$ that are not generators of I^3 cannot be in I^3 because of degree reasons.

$$I^4 = \langle x^4y^4, y^4z^4, x^4z^4, x^4y^3z, x^4y^2z^2, x^4yz^3, x^3y^4z, x^3y^3z^2,$$
$$x^3y^2z^3, x^3yz^4, x^2y^4z^2, x^2y^3z^3, x^2y^2z^4, xy^4z^3, xy^3z^4\rangle$$

$$I^{(4)} = \langle x^4y^4, y^4z^4, x^4z^4, x^3y^3z, x^3yz^3, x^2y^2z^2\rangle$$

Once again, all generators of I^4 are contained in $I^{(4)}$ but the generators of $I^{(4)}$ of degrees six and seven cannot be contained in I^4 due to degree reasons.

Thus for $G = C_3$, we have $I^{(1)} = I$, $I^{(2)} \neq I^2$, $I^{(3)} \neq I^3$, and $I^{(4)} \neq I^4$.

Example 6. The edge ideal corresponding to C_5 is $I = \langle xy, yz, zw, wv, xv\rangle$.

$$I^2 = \langle x^2y^2, y^2z^2, z^2w^2, w^2v^2, x^2v^2, x^2yv, xy^2z, xyzw,$$
$$xyzv, xywv, xzwv, xwv^2, yz^2w, yzwv, zw^2v\rangle$$

$$I^{(2)} = \langle x^2y^2, y^2z^2, z^2w^2, w^2v^2, x^2v^2, x^2yv, xy^2z, xyzw,$$
$$xyzv, xywv, xzwv, xwv^2, yz^2w, yzwv, zw^2v\rangle$$

Here, $I^2 = I^{(2)}$.

$$I^3 = \langle x^3y^3, y^3z^3, z^3w^3, w^3v^3, x^3v^3, x^3y^2v, x^3yv^2, x^2y^3z, x^2y^2zw, x^2y^2zv,$$
$$x^2y^2wv, x^2yzwv, x^2yzv^2, x^2ywv^2, x^2zwv^2, x^2wv^3, xy^3z^2, xy^2z^2w,$$
$$xy^2z^2v, xy^2zwv, xyz^2w^2, xyz^2wv, xyzw^2v, xyzwv^2, xyw^2v, xz^2w^2v,$$
$$xw^2v^3, y^2z^3w, y^2z^2wv, yz^3w^2, yz^2w^2v, yzw^2v^2, z^2w^3v, zw^3v^2\rangle$$

$$I^{(3)} = \langle x^3y^3, y^3z^3, z^3w^3, w^3v^3, x^3v^3, x^3y^2v, x^3yv^2, x^2y^3z, x^2y^2zw, x^2y^2zv,$$
$$x^2y^2wv, x^2yzwv, x^2yzv^2, x^2ywv^2, x^2zwv^2, x^2wv^3, xy^3z^2, xy^2z^2w,$$
$$xy^2z^2v, xy^2zwv, xyz^2w^2, xyz^2wv, xyzw^2v, xyzwv^2, xyw^2v, xz^2w^2v,$$
$$xw^2v^3, y^2z^3w, y^2z^2wv, yz^3w^2, yz^2w^2v, yzw^2v^2, z^2w^3v, zw^3v^2, xyzwv\rangle$$

Notice that the remaining generator in $I^{(3)}$ that is not in I^3 is the product of the vertices, $xyzwv$.

Thus for $G = C_5$, we have $I^{(1)} = I$, $I^{(2)} = I^2$, and $I^{(3)} \neq I^3$.

We continued this process for C_7 and C_9, and after further analysis, we developed our conjecture. At first, it seemed like the symbolic powers were equal to the ordinary powers, but we quickly realized otherwise as one can see from the computations above.

Conjecture 7. *Let C_{2n+1} be an odd cycle of length $2n+1$ and I be its edge ideal. Then*

1. $I^{(t)} = I^t$ *for* $1 \leq t \leq n$
2. $I^{(n+r)} \supseteq I^{n+r} + I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$ *for all* $r \geq 1$*, where equality holds for* $r = 1$.

In order to prove this conjecture, we develop a more appropriate theoretical framework in the following sections.

4 Preliminaries

In Section 2, Proposition 4 involves minimal primes of an ideal and radical ideals. In this section we provide the reader with the necessary definitions and background for these ideas to be fully understood.

4.1 Prime Ideals

A *prime ideal* p in a commutative ring R is a proper ideal of R such that $a, b \in R$ and $ab \in p$ implies $a \in p$ or $b \in p$. A nice class of prime ideals are monomial prime ideals in $k[x_1, x_2, \ldots, x_n]$ where k is a field. These ideals are generated by a subset of the variables $x_1, \ldots, x_n$.

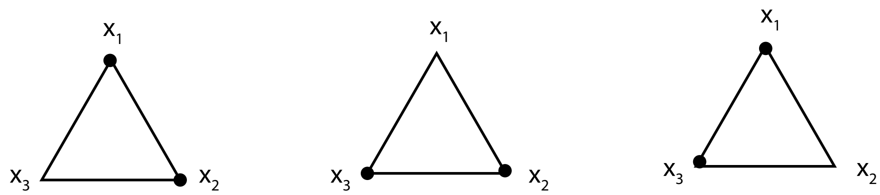
Given an ideal I, the *minimal primes of* I are the collection of "smallest" primes containing I. That is, p is a minimal prime of I if $p \supseteq I$ and there does not exist another prime q such that $p \supset q \supseteq I$.

Example 8. Let $I = \langle x_1x_2, x_2x_3, x_1x_3 \rangle$. The minimal primes of I are:

$$\langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \langle x_1, x_3 \rangle$$

Note that $\langle x_1, x_2, x_3 \rangle$ is a prime of I. However, it is not a minimal prime of I because it properly contains another prime ideal containing I.

Notice that the previous example of an edge ideal I corresponds to the triangle. Also, every minimal prime of I is composed of a minimum number of vertices needed to make each edge of the triangle incident with one vertex as seen below. This is not a coincidence.



Definition 9. Let G be a graph with vertex set V. A subset $A \subset V$ is a minimal vertex cover for G if: (i) every edge of G is incident with one vertex in A, and (ii) there is no proper subset of A with the first property.

One can deduct from the definitions that minimal primes of $I(G)$ correspond to minimal vertex covers of G. In fact, we found minimal vertex covers rather than calculating the minimal primes because, in our opinion, it was easier to analyze pictures.

4.2 Radical Ideals

An ideal I is *radical* if $I = \sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \geq 1\}$. It is a standard exercise using the previous definition that in a polynomial ring over a field a monomial ideal I is radical if and only if it has square-free generators. Therefore since the generators of edge ideals are square-free monomials of degree two, edge ideals are radical.

5 Results

The following section consists of a few lemmas that will be used to prove our results later in the section. But first, a definition:

Definition 10. A walk of length n in a graph G is an alternating sequence of vertices and edges

$$w = v_0, z_1, v_1, \cdots v_{n-1}, z_n, v_n,$$

where $z_i = v_{i-1}, v_i$ is the edge joining v_{i-1} and v_i.

Lemma 11. *Let C_{2n+1} be an odd cycle of length $2n+1$. There are at least $2n+1$ minimal vertex covers.*

Proof. We will show that

$$\{x_{imod(2n+1)}, x_{(i+1)mod(2n+1)}, x_{(i+3)mod(2n+1)}, \cdots, x_{(i+2n-1)mod(2n+1)}\}$$

for $i = 1, 2, \cdots, 2n+1$ are minimal vertex covers. After renaming variables, it is enough to show $\{x_1, x_2, x_4, \cdots, x_{2n}\}$ is a minimal vertex cover.

In order for us to show that $\{x_1, x_2, x_4, \cdots, x_{2n}\}$ is a vertex cover, we must show that at least one endpoint of every edge is in $\{x_1, x_2, x_4, \cdots, x_{2n}\}$. First, we will pick an edge whose endpoints are x_j and x_{j+1} where $1 \leq j \leq 2n$. If j is even, then, $x_j \in \{x_1, x_2, x_4, \cdots, x_{2n}\}$ because all even vertices are in $\{x_1, x_2, x_4, \cdots, x_{2n}\}$. If j is odd, then we know $j+1$ is even, and $2 \leq j+1 \leq 2n$, which gives $x_{j+1} \in \{x_1, x_2, x_4, \cdots, x_{2n}\}$. Therefore, $\{x_1, x_2, x_4, \cdots, x_{2n}\}$ is a vertex cover.

Now we must show $\{x_1, x_2, x_4, \cdots, x_{2n}\}$ is a *minimal* vertex cover. This will be shown by eliminating a vertex from the vertex cover and noting it is not still a vertex cover.

Case 1: First, let x_1 be eliminated. This leaves $\{x_2, x_4, \cdots, x_{2n}\}$. This is not a vertex cover because the edge whose endpoints are x_1 and x_{2n+1} is not in $\{x_2, x_4, \cdots, x_{2n}\}$, and by definition of vertex cover, every edge must have a vertex incidental to it listed in the vertex cover.

Case 2: Now, let x_{2i} where $1 \leq i \leq n$ be eliminated. This will leave $\{x_1, x_2, \cdots, x_{2(i-1)}, x_{2(i+1)}, \cdots, x_{2n}\}$, which is not a vertex cover because it leaves the edge whose endpoints are x_{2i} and x_{2i+1} without an incidental vertex listed in $\{x_1, x_2, \cdots, x_{2(i-1)}, x_{2(i+1)}, \cdots, x_{2n}\}$.

Therefore, there are at least $2n+1$ minimal vertex covers in C_{2n+1}. $\square$

Note that there may be more than $2n+1$ minimal vertex covers in some C_{2n+1} where $n \geq 4$. This was observed in C_9 as seen in Figure 1.

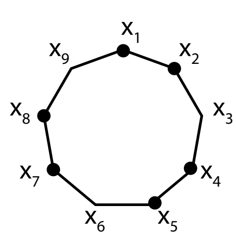


Figure 1: A “different” minimal vertex cover for C_9.

Lemma 12. *Let $G = C_{2n+1}$ be an odd cycle of length $2n+1$. Any minimal vertex cover of the edge ideal $I(G)$ contains at least $n+1$ vertices.*

Proof. For sake of contradiction assume there are at least $n+1$ vertices not in some minimal vertex cover p of C_{2n+1}. Since there cannot be two adjacent vertices not in p, there are at least $n+1$ walks. Because C_{2n+1} has only $2n+1$ vertices and at least $n+1$ have been accounted for, there are at most n vertices left in the minimal vertex cover for the $n+1$ walks, which is impossible. Therefore, there must be at most n vertices not in p, which implies there are at least $n+1$ vertices in each minimal vertex cover p. $\square$

The previous lemmas can be restated in terms of minimal primes of edge ideals. That is, there are at least $2n+1$ minimal primes containing at least $n+1$ vertices for the edge ideal of an odd sided polygon, C_{2n+1}.

Recall the conjecture from a previous section:

Conjecture 13. *Let C_{2n+1} be an odd cycle of length $2n+1$ and I be its edge ideal. Then*

1. $I^{(t)} = I^t$ *for* $1 \leq t \leq n$
2. $I^{(n+r)} \supseteq I^{n+r} + I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$ *for all* $r \geq 1$, *where equality holds for* $r = 1$.

We only have a partial proof for the conjecture above. We should note that the ordinary power is always contained in the symbolic power (the argument is given in the proof for completeness). Therefore, for the reverse containment on (ii), we focus only on the second term, $I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$. The lemma and its proof are as follows:

Lemma 14. *Let $G = C_{2n+1}$ be an odd cycle of length $2n+1$ and I be its corresponding edge ideal. Then $I^{(n+r)} \supseteq I^{n+r} + I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$ for all $r \geq 1$.*

Proof. It is known that $I^{(k)} \supseteq I^k$ for all k. To see this, we first recall that

$$I^{(k)} = p_1^k \cap \cdots \cap p_m^k$$

from Proposition 4 where each p_i is a minimal prime. To show this containment, it is enough to show that all the generators of I^k are contained in $I^{(k)}$. Following the discussion in section 2 about generators of I^k, each generator of I^k looks like $f_1 f_2 \cdots f_k$, where each f_i is a generator in I. Since p_j is a minimal prime of I, we have $f_i \in p_j$ for every i and j. Hence $f_1 f_2 \cdots f_k \in p_j^k$ for any j. Thus $f_1 f_2 \cdots f_k \in p_1^k \cap \cdots \cap p_m^k$. Therefore, $I^{(k)} \supseteq I^k$.

To prove the containment, $I^{(n+r)} \supseteq I^{r-1} \cdot \langle x_1 x_2 \cdots x_{2n+1} \rangle$ we induct on r. Suppose $r = 1$ and let $\{p_i\}$ be a minimal primes of I. By Lemma 12, every p_i contains at least $n+1$ variables from the set $\{x_1, x_2, \cdots, x_{2n+1}\}$. Thus a product of $n+1$ distinct variables belongs to p_i^{n+1} for every i which implies that $x_1 x_2 \cdots x_{2n+1}$ belongs to p_i^{n+1}. Hence $x_1 \cdots x_{2n+1} \in p_1^{n+1} \cap \cdots \cap p_m^{n+1} = I^{(n+1)}$.

Now for the inductive step, assume $\langle x_1 \cdots x_{2n+1} \rangle \cdot I^{r-2} \subseteq I^{(n+r-1)}$. We must show that a generator of $\langle x_1 \cdots x_{2n+1} \rangle \cdot I^{r-1}$ which has the form $(x_1 \cdots x_{2n+1})(m_1 m_2 \cdots m_{r-1})$ where m_i are generators of I is an element of $I^{(n+r)}$. Observe that

$$\begin{aligned} (x_1 \cdots x_{2n+1})(m_1 m_2 \cdots m_{r-1}) &= [(x_1 \cdots x_{2n+1})(m_1 m_2 \cdots m_{r-2})] m_{r-1} \\ &\in I^{(n+r-1)} \cdot I \\ &= (p_1^{n+r-1} \cap \cdots \cap p_m^{n+r-1}) \cdot (p_1 \cap \cdots \cap p_m) \\ &\subseteq p_1^{n+r} \cap \cdots \cap p_m^{n+r} \\ &= I^{(n+r)} \end{aligned}$$

□

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References

[1] D. Bayer, M. Stillman, Macaulay, 1989.

[2] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Springer-Verlag, New York, 1995.

[3] A. Simis, W. Vasconcelos, R. Villarreal, On the Ideal Theory of Graphs, J. Alegbra 167, 389-416 (1994).

[4] R. Villarreal, Monomial Algebras, Marcel-Dekker, New York, 2001.