

Symbolic Powers of Edge Ideals

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Referee's Report

Relationships between graph theory and ring theory have been discussed rather frequently during the past ten years. In this article the authors specifically discuss a connection between regular powers and symbolic powers of the edge ideal of an odd-sided polygon. This work came about by asking some very natural questions that arise from the work done by Simis, Vasconcelos, and Villareal that appeared in the *Journal of Algebra* in 1992. Studying properties of rings using graph theoretic properties is a nontrivial exercise, and contributions to this subject are definitely worthy of publication.

Mathematically, this article is very sound. The new results are proven with an appropriate amount of rigor, and the exposition is nicely and clearly written. In particular, the authors should be commended for the development of Section 5 - Results. (This being said, the proof of Lemma 12 does need a little more explanation.) As such, many of the suggestions that follow are stylistic in nature to help the authors create an article that allows the reader to follow along more easily and that is self-contained.

1. The introduction to your paper should include basic terminology and notation that will be used throughout. For example, you should state in your introduction that G will always represent a graph, I will always represent an ideal (in your case, you could probably even state that I will specifically refer to the edge ideal of G), and R will always represent a ring. You could also define such terms as *bipartite*, *degree two monomials*, and *square-free monomials* here. In general, if a definition is more of a background definition, it is better to lead with it in the introduction so that the reader can build on these definitions that may already be familiar.
2. It is important that you explicitly state where your edge ideals live. This can also be done in the introduction. Since Simis, et. al., discussed k -algebras that include polynomials in n indeterminates over a field k , a reader familiar with that work may assume the ideals are in $k[x_1, x_2, \dots, x_n]$, but someone reading about edge ideals for the first time may be left wondering.
3. In line 7, page 2, the commas before “which” and after “paper” can be deleted.
4. In Theorem 2 on page 2, *normally torsion free* should be defined either before the theorem or in the statement of the theorem.
5. In general you should order your article as linearly as possible. For example, saying, “these terms will be defined shortly” in lines 15 and 16 of page 2 requires the reader to jump around to find the appropriate definitions and results to gain a full understanding of the material that was just presented. In this instance, immediately following

Theorem 2 the definitions of the regular and symbolic powers should be given and then a restatement of (ii) of Theorem 2 using the regular and symbolic powers should be provided.

6. Be consistent in what punctuation, if any, you are using before something is set off from the rest of the text. Before Theorem 2 you end a sentence with a period. Before the definition of I^k , there is no punctuation after “then.” Before Definition 3 the sentence is ended with a colon. No one way is correct; consistency is the key issue. Be sure to check for other instances throughout the article.
7. In Definition 3, it is easier to say “ $s + I$ is a regular element of R/I .” Then, you do not have to mention $s \neq 0$ nor $s \in R$. Also, most of the literature either uses “zero divisor” or “zero-divisor” as opposed to running the two words together.
8. In line 3 of page 3, “chose” should be “choose.”
9. In line 5 of page 3, “states” should be “is stated in.”
10. Proposition 4 uses minimal primes of an ideal, but the definition for minimal primes of an ideal is not provided until page 5.
11. In line 11 of page 3, “Macaulay” should probably be italicized. Also, the commas around the word “Macaulay” can be deleted.
12. In line 12 of page 3, your argument only shows $I^{(1)} \supseteq I$. You could just eliminate the end of the sentence starting with the word “since,” or possibly even say, “We leave it to the reader to show $I^{(1)} = I$.”
13. In line 16 of page 3, “Macaulay” should probably be italicized.
14. In line 17 of page 3, eliminate the sentence beginning with, “We were hoping to ...” You may also want to say you performed several computations for the regular and symbolic powers of the edge ideal of C_{2n+1} and are including the computations for C_3 and C_5 .
15. It might be nice to see how one would compute the regular and symbolic powers I^2 and $I^{(2)}$ for C_3 to get a feel for why using the computer is so advantageous for this type of an inquiry.
16. Since the regular and symbolic powers for C_3 were computed up to the fourth power, the results for the regular and symbolic powers for C_5 should as well. This is not to say the generators for I^4 and $I^{(4)}$ should be provided; clearly, this would take up an undue amount of space in the article and time to type. It should simply be noted that $I^{(4)} \neq I$ in this case.
17. The paragraph on page 5 beginning with, “We continued this process ...” should be replaced. In its place the results for the regular and symbolic powers should be listed for C_7 and C_9 much as they are on line 9 of page 4 and line 3 of page 5. You can then follow how you formulated Conjecture 7.

18. In lines 25 and 26 on page 5, it is probably better to use " \subseteq " rather than " \supseteq ." The reason is because p is to be a minimal ideal, and since we are accustomed to order following a left-to-right pattern, "minimal" is usually associated with "left." (What you have is perfectly correct and can be left as is; this is only included as an observation.)
19. The figure on page 6 needs some explanation. It is possible it can take the reader some time to discern that the dots included on the graph represent the generators of the prime ideals.
20. In Definition 9, the words "minimal vertex cover" should be italicized.
21. In Definition 9, the words "at least" need to be placed before "one vertex in A ."
22. In line 8 of page 6, the word "deduct" should be "deduce."
23. Though true, the reasoning behind including the fact that edge ideals are radical ideals is unclear. This fact is not used anywhere in the article. Either this fact should be moved near the beginning of the article (as an observation about edge ideals) or left out entirely.
24. In line 20 of page 6, the phrase, "But first, a definition:" can be deleted.
25. Since Lemma 11 makes no references to walks, Definition 10 can be moved so that it precedes Lemma 12.
26. In line 5 of page 7, instead of saying, "After renaming variables," say, "Without loss of generality."
27. In line 6 of page 7, put in x_6 to be sure the reader knows that all even numbered vertices are in the set. Also, naming the set might be helpful; every reference to $\{x_1, x_2, x_4, x_6, \dots, x_{2n}\}$ could then be replaced by the name of the set.
28. In the proof of Lemma 11, the case of the edge between x_{2n+1} and x_1 is not included when showing the set does indeed form a vertex cover.
29. The last sentence in the proof of Lemma 11 can be deleted.
30. If you are going to label the figure on the bottom of page 7, then the figure on the top of page 6 needs a label as well.
31. It is worthwhile to explain the figure on page 7. For example, what is the vertex cover you are illustrating with the picture? Why is it different from the vertex covers of Lemma 11?
32. In line 27 of page 7, replace "This was observed" with "For C_9 we see that $\{x_1, x_2, x_4, x_5, x_7, x_8\}$ is a minimal vertex cover." Another interesting observation is that two different minimal vertex covers do not have to have the same cardinality. (This is hinted at in the statement of Lemma 12.)
33. In line 1 of page 8, write " C_{2n+1} " instead of " $G = C_{2n+1}$."

34. In line 3 of page 8, delete “For sake of contradiction.”
35. In Lemma 12 you refer to $n + 1$ walks. It is not clear how long these walks are, nor is it clear which vertices and edges are in the walks. Are they walks of maximal length? Are the vertices in the walks alternating between a vertex that is in the cover and one that is not? These issues need to be clarified.
36. In line 24 of page 8, write “ C_{2n+1} ” instead of “ $G = C_{2n+1}$.”
37. In the last line of page 8, the word “section” should be capitalized.
38. In line 1 of page 9, replace “looks like” with “is of the form.”
39. In line 4 of page 9, the comma after “containment” should be moved to just before the word “we.”
40. In line 7 of page 9, a comma is needed before the word “which.”
41. In line 10 of page 9, delete “Now for the inductive step.”
42. In lines 11-13 of page 9, move “is an element of $I^{(n+r)}$ ” up to line 11 immediately before the word “which” and end the sentence. Then state, “A generator of $\langle x_1 \cdots x_{2n+1} \rangle \cdot I^{r-1}$ has the form $(x_1 \cdots x_{2n+1})(m_1 m_2 \cdots m_{r-1})$, where the m_i ’s are generators of I .”
43. In the bibliography, the titles of the works and the journal titles need to be set off by making one or the other italicized. Since you are starting with the work of Simis, et. al., you might want to follow the form of their bibliography.