Sequentially Decreasing Subsets of Metric Spaces *

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Abstract

We introduce and discuss various properties of sequences of subsets $\{O_n : n \in \mathbb{Z}^+\}$ of metric spaces with the property that $\lim_{n\to\infty} \delta(O_n) = 0$ where δ denotes the diameter of a set, which we call sequentially decreasing subsets. As applications of the theory developed, we give a short proof of a well known necessary condition for a metric space to be connected, give sufficient conditions for subsets of a connected metric space to be totally disconnected, and discuss a specific outer measure on metric spaces.

1 Introduction and Definitions

By placing an open sphere around each point of a Lindelof metric space (a metric space with the property that every open cover has a countable subcover), we can construct a sequence of spheres that cover the space where each member of the sequence has diameter smaller than any preassigned positive number. However, if we have a sequence of spheres where the diameters of the spheres in the sequence of spheres converge to zero, then no matter where the spheres are centered, they may not end up covering the space. This observation naturally leads to an interesting question that has not been largely discussed in the literature: when can a metric space be covered by a sequence of subsets whose diameters converge to zero? Therefore, we investigate this question and some of its related applications.

Before we begin, however, it will be useful to define some terms and introduce standard notation that will be used throughout the paper.

Definition 1 *A metric space X is* ***connected*** *if it can not be written as the disjoint union of two non-empty open subsets, or equivalently if it can not be written as the disjoint union of two non-empty closed subsets. A subset $S \subseteq X$ is connected if it is connected as a subspace of X.*

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Definition 2 *A metric space X or a subset $S \subseteq X$ is* ***totally disconnected*** *if all subsets with more than one element are disconnected.*

Definition 3 *A metric space X or a subset $S \subseteq X$ is* ***locally compact*** *if every element of X or S is contained in a neighborhood whose closure is compact.*

Definition 4 *The* ***diameter*** *of a subset $S \subseteq X$ of a metric space X with metric d, denoted by $\delta(S)$, is the non-negative number sup $\{d(x, y) : x, y \in S\}$, assuming that this supremum exists. If the supremum does not exist, then we say S has infinite diameter, or that $\delta(S) = \infty$.*

Definition 5 *Let A and B be two subsets of a metric space X. The* ***distance*** *between the two subsets, denoted by $d(A, B)$, is the non-negative number inf $\{d(x, y) : x \in A, y \in B\}$. If x is an element of X instead of a subset, then we define the distance between x and B by treating x as a one-point subset of X and using the same definition and the notation $d(x, B)$.*

Definition 6 *A* ***closed sphere*** *with center x_0 (or around x_0) in a metric space X with radius r is the set $\{x \in X : d(x, x_0) \leq r\}$. If $\leq$ is changed to $<$, then the resulting set is an* ***open sphere*** *with center x_0 (or around x_0) and radius r. If the center of a sphere is not important, then often we will just say "a sphere" or "a sphere of radius r" or something to that nature.*

Definition 7 *Let $\{O_\alpha\}$ be a cover of a metric space X or a cover of a subset $S \subseteq X$, and let x and y be any points in S or X respectively. A countable subcover containing x and y such that for any $O_j \in O_1, O_2, ...$, there exists an $O_k \in \{O_1, O_2, ...\}$ such that $O_j \cap O_k \neq \emptyset$ is called a* ***chain*** *connecting x and y. If this subcover also has the property that for each $O_j \in \{O_1, O_2, ...\}, O_j$ intersects only O_{j-1} and O_{j+1} with the exception that O_1 only intersects O_2 , then the chain is called a* ***simple chain*** *connecting x and y. If $\{O_1, O_2, ...\}$ is a finite subcover with the above properties, then $\{O_1, O_2, ...\}$ is called a* ***finite chain*** *and a* ***finite simple chain*** *respectively (where in the latter case, we have the obvious condition that O_n intersects only O_{n-1}, where O_n is the last set in $\{O_1, O_2, ...\}$.)*

Definition 8 *A real function $f : \mathbb{R} \to \mathbb{R}$ is* ***countably subadditive*** *if for any sequence of real numbers $\{x_n : n \in \mathbb{Z}^+\}$ where $\sum_{i=11}^{\infty} f(x_i) < \infty$, we have $f(\sum_{i=1}^{n} x_i) \leq \sum_{i=1}^{n} f(x_i)$. If O is a family of sets that is closed under unions, then a function $f : O \to \mathbb{R}$ is said to be* ***countably subadditive*** *if for any $\{O_n : n \in \mathbb{Z}^+\} \subseteq O$, we have $f(\bigcup_{i=1}^{\infty} O_i) \leq \sum_{i=1}^{\infty} f(O_i)$.*

In dealing with a sequence of subsets $\{O_n : n \in \mathbb{Z}^+\}$ of some metric space X where $\lim_{n\to\infty} \delta(O_n) = 0$, it is convenient to separate the case when the sum of the diameters converges or diverges, and we will first treat the case when the sum converges. In the former case, we call the sequence a **convergent sequentially decreasing** sequence of subsets and in the latter case we will call the sequence of subsets a **divergent sequentially decreasing** sequence of subsets, or a c.s.d. and d.s.d. sequence respectfully. If the subsets cover X, then we call the cover a **convergent sequentially decreasing cover** and **divergent sequentially decreasing cover** respectively, or a c.s.d. cover and d.s.d. cover respectfully.

2 Convergent Sequentially Decreasing Subsets

The difficulties concerning the existence and properties of c.s.d. covers are illustrated by the following example. Let $\{r_n : n \in \mathbb{Z}^+\}$ be any enumeration of the rationals in $\mathbb{R}$ and let $\{O_n : n \in \mathbb{Z}^+\}$ be any sequence of subsets of the reals such that for each $n \in \mathbb{Z}^+, r_n \in O_n$. If $\sum_{n=1}^{\infty} \delta(O_n)$ converges, then it is not immediately clear that $\{O_n : n \in \mathbb{Z}^+\}$ will not cover $\mathbb{R}$ (however, those who are familiar with Lebesgue measure will see that this fact is easy to prove, and this idea will be discussed in Section 3.)

We now show that this is in fact the case by proving a more general Theorem, (Theorem 2) regarding general metric spaces and show that the above problem is intimately related to connectivity. First however, we need an important lemma that will make our proof easier and will allow us to generalize Theorem 2.

Lemma 1 *Let X be a metric space and I be some countable indexing set. Then if $\bigcup_{i\in I} O_i \subseteq X$ is connected where $O_i \neq \emptyset$ for any $i \in I$, then $\delta(\bigcup_{i\in I} O_i) \leq \sum_{i\in I} \delta(O_i)$.*

In the proof of this lemma, we will be constructing a chain connecting any two elements x and y in $\bigcup_{i\in I} O_i$ that consists of sets of $\bigcup_{i\in I} O_i$ and a sequence of spheres $\{S_n : n \in \mathbb{Z}^+\}$ where the radius of each sphere can be made arbitrarily small. Therefore, to clarify the details of the proof, we give an example of such a chain.

Let $I = \mathbb{Z}^+$ and let $O_i = (\frac{1}{i+1}, \frac{1}{i}]$ for $i \in I = \mathbb{Z}^+$, so that

$$(0, 1] = \bigcup_{i\in I} O_i.$$

Now if S_n is a sphere of arbitrary radius around $\frac{1}{n+1}$ for each $n \in \mathbb{Z}^+$, then for any two elements $x, y \in (0, 1]$, the sets in $\{O_i : i \in I\}$ and $\{S_n : n \in \mathbb{Z}^+\}$ clearly form a chain connecting x and y.

It should be noted that this example shows that the spheres we constructed were essential in forming our chain, in the sense that the no subclass of $\{O_i : i \in I\}$ containing more than two sets form a chain on its own.

Proof: Let $Y = \bigcup_{i\in I} O_i$. Without loss of generality, assume that for any $\epsilon > 0$ and for any $x, y \in Y$, we can find a chain connecting x and y that consists of subsets in $\{O_i : i \in I\}$ and (open or closed) spheres $\{S_k : k \in \mathbb{Z}^+\}$ where each S_k has radius $\frac{\epsilon}{2^k}$ and and S_k intersects both O_k and O_{k+1}. Then for any

$x_k \in S_k : k \in \mathbb{Z}^+$ where $x_k \in O_k \cap S_k$ and for any $y_k \in S_k : k \in \mathbb{Z}^+$ where $y_k \in S_k \cap O_{k+1}$, we have

$$d(x, y) \leq (d(x, x_1) + d(x_1, y_1) + d(y_1, x_2) + d(x_2, y_3) + ...+$$

$$\leq \sum_{i \in I} \delta(O_i) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \sum_{i \in I} \delta(O_i) + \epsilon$$

and so the result immediately follows by the definition of diameter. Therefore, we prove that such a chain always exists.

Given any $x, y \in Y$ and any $\epsilon > 0$, let $x \in O_n$ and $y \in O_m$ where O_n and $O_m \in \{O_i : i \in I\}$ and assume O_n and O_m are disjoint, for otherwise we are done. Now let $\{O_n\}^1$ and $\{O_m\}^1$ be the class of sets in $\{O_i : i \in I\}$ where each set in $\{O_n\}^1$ has zero distance from O_n and each set in $\{O_m\}^1$ has zero distance from O_m. If any set in $\{O_n\}^1$ or $\{O_m\}^1$ has zero distance from both O_m and O_n then we are done, so assume the contrary. Now let $\{O_n\}^2$ and $\{O_m\}^2$ be the class of sets in $\{O_i : i \in I\}$ that have zero distance from every set in $\{O_n\}^1$ and $\{O_m\}^1$ respectively, where again if any set in $\{O_n\}^2$ or $\{O_m\}^2$ has zero distance from any set in both $\{O_n\}^1$ and $\{O_m\}^1$ then we are done, so assume the contrary, and in general, for $k = 3, 4, ...$ let $\{O_n\}^k$ and $\{O_m\}^k$ be the class of sets in $\{O_i : i \in I\}$ that have zero distance from every set in $\{O_n\}^{k-1}$ and $\{O_m\}^{k-1}$ respectively.

Let $O_n^{(k)}$ and $O_m^{(k)}$ denote the union of all sets in $\{O_n\}^k$ and $\{O_m\}^k$ respectively. If $\bigcup_{k=1}^{\infty}(O_n^{(k)} \cup O_m^{(k)}) = Y$, then the distance between $\bigcup_{k=1}^{\infty} O_n^{(k)}$ and $\bigcup_{k=1}^{\infty} O_m^{(k)}$ must be zero, for otherwise we have Y as the disjoint union of two non-empty and relatively open subsets of Y. Also, by the definition of our subsets $\{O_m\}^k$ and $\{O_n\}^k$ and since the distance between $\bigcup_{k=1}^{\infty} O_n^{(k)}$ and $\bigcup_{k=1}^{\infty} O_m^{(k)}$ is zero, it easily seen that we can find a chain connecting x and y that consists of members of $\bigcup_{k=1}^{\infty}(\{O_n\}^k \cup \{O_m\}^k)$ and spheres $\{S_k : k \in \mathbb{Z}^+\}$ where each S_k has radius $\frac{\epsilon}{2^k}$, so our results immediately follow by the first paragraph. However, if $\bigcup_{k=1}^{\infty}(O_n^{(k)} \cup O_m^{(k)}) \neq Y$ then let

$$\{O^*\} = \{O_i : i \in I\} \setminus \bigcup_{k=1}^{\infty} (\{O_n\}^k \cup \{O_m\}^k)$$

and let $\{O_n^* \ : n \in \mathbb{Z}^+\}$ be an enumeration of the sets of $\{O^*\}$ so that the connectivity of Y implies that there must exist an $N \in \mathbb{Z}^+$ such that the distance between $\bigcup_{k=1}^{N}(O_k^*)$ and both $\bigcup_{k=1}^{\infty} O_n^{(k)}$ and $\bigcup_{k=1}^{\infty} O_m^{(k)}$ is zero, for otherwise we have Y as the disjoint union of non-empty open subsets.

Therefore, since the distance between $\bigcup_{k=1}^{N}(O_k^*)$ and both $\bigcup_{k=1}^{\infty} O_n^{(k)}$ and $\bigcup_{k=1}^{\infty} O_m^{(k)}$is zero, the definition of the sets $\{O_m\}^k$ and $\{O_n\}^k$ implies that we can find a chain connecting x and y that consists of members of $\bigcup_{k=1}^{\infty}(\{O_n\}^k \cup \{O_m\}^k), \{O_1^*, O_2^*, ...\}$, and spheres $\{S_k : k \in \mathbb{Z}^+\}$ where each S_k has radius $\frac{\epsilon}{2^k}$, so that we are done.

QED

In proving this, we actually proved an interesting modification (which is stated in corollary 1) of a well known theorem which states that if $\{O_\alpha\}$ is an open cover of a connected metric space X , then for any two points $x, y \in X$, there exists a finite simple chain connecting x and y consisting of sets of $\{O_\alpha\}$(*see* [1].)

Corollary 1 *If X is a connected metric space and $\{O_\alpha\}$ is any cover of X, then there exists a sequence of spheres (open or closed) $\{S_n\}$, each of which has radius as small as desired, such that for any two elements $x, y \in X$, there exists a simple chain connecting x and y that consists of sets of $\{O_\alpha\}$ and $\{S_n\}$.*

PROOF: Immediately follows from the proof of the previous lemma. QED

Now we can prove main theorem of this section

Theorem 1 *Let X be an infinite connected metric space with $\delta(X) = d$ and let $\{O_n\}$ be a countable family of subsets of X such that $\sum_{n=1}^{\infty} \delta(O_n) < d$. Then $X \neq \bigcup_{n=1}^{\infty} O_n$.*

PROOF: Clearly if $\bigcup_{n=1}^{\infty} O_n$ is not connected, then it can not be all of X so assume it is connected. From the previous lemma, the diameter is subadditive so that the result immediately follows. QED

Before moving on, we first state two simple but important applications of the above results. First, since $\mathbb{R}$ is connected and $\delta(\mathbb{R}) = \infty$, it cannot have a c.d.s. cover. Second, if there exists a covering $\{O_n\}$ of some infinite metric space X with $\sum_{n=1}^{\infty} \delta(O_n)$ being arbitrarily small (for example, the Cantor set or any arbitrary countable set,) then X must be totally disconnected, for otherwise it would have a connected subset of diameter > 0 which is a contradiction to Theorem 2. It should be noted however, that no countable connected Hausdorff topological space with more than one point exists, but that countable connected topological spaces do exist ([2].)

For the sake of completeness, we will now prove the simple fact that there is no infinite metric space X such that every c.s.d. sequence of subsets cover X, and provide an easy sufficient condition for a metric space to have a sequentially decreasing cover which may or may not be convergent.

Theorem 2 *No infinite metric space X exists where every sequence of c.s.d. sequence of subsets covers X.*

PROOF: For any $x \in X$, let $\{x_n : n \in \mathbb{Z}^+\}$ be a proper countable subset of X that does not contain x, and let $r_n = min\{\frac{1}{2^n}, d(x, x_n)\}$, then the set of open spheres $O_n = S_{r_n}(x_n)$ clearly satisfies the condition that X $\neq \bigcup_{n=1}^{\infty} O_n$ and $\sum_{n=1}^{\infty} \delta(O_n) < \infty$. QED

Theorem 3 *Let X be a locally compact Lindeloff metric space, then X has a sequentially decreasing cover.*

PROOF: Let cl denote the closure of a set. Since X is locally compact, there exists an open set O_x about each point $x \in X$ such that $cl(O_x)$ is compact. Since X is Lindeloff, we can find a countable subcover $\{O_{x_n} : n \in \mathbb{Z}^+\}$ of the cover $\{O_x : x \in X\}$. For each $n \in \mathbb{Z}^+$, choose a sphere of radius $\frac{1}{n^2}$ around each point $y \in cl(O_{x_n})$ and for each n, choose a finite subcover of these spheres of radius $\frac{1}{n^2}$, then these spheres comprise a sequentially decreasing cover.

QED

It should be noted that this sequentially decreasing cover might diverge if the number of spheres needed to cover each $cl(O_{x_n})$ is too large.

3 Metric Spaces With Outer Measures

Despite the fact that outer measures on metric spaces may seem very different from the topics discussed in the first section, as a third and more interesting application of our previous theorems, we will discuss outer measures on metric spaces. Furthermore, in doing so, we will prove a useful generalization of Theorem 2 and give an interesting sufficient condition for a subset of a connected metric space to be totally disconnected.

First, however we will state and prove an interesting variant of the well known fact that if m is any outer measure generated by a countably subadditive and monotone set function f on any family of sets $\{O_\alpha\}$, then for each $S \in \{O_\alpha\}$, $m(S) = f(S)$ (see [3]).

Theorem 4 *Let $\{A_\alpha\}$ be any family of subsets of X and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and subadditive real function, and for any $S \subseteq X$, define the outer measure m as $m(S) = inf\ \{\sum_{n=1}^{\infty} f(\delta(A_n)) : A_n \in \{A_\alpha\}$ and $S \subseteq \bigcup_{n=1}^{\infty} A_n\ \}$ where we denote $m(S)$ as ∞ if no countable cover by members of $\{A_\alpha\}$ exists (and for the sake of consistency denote $f(\infty) = \infty$.) Then if $A_m \in \{A_\alpha\}$ is connected, we have $m(A_m) = f(\delta(A_m))$.*

PROOF: For any $A_m \in \{A_\alpha\}$, A_m covers itself so clearly $m(A_m) \le f(\delta(A_m))$. Now for any $\epsilon > 0$ let $\{A'_n\} \subseteq \{A_\alpha\}$ be any countable cover of A_m such that $m(A_m) + \epsilon > \sum_{n=1}^{\infty} f(\delta(A'_n))$. Since A_m is connected, $\bigcup_{n=1}^{\infty}(A'_n \cap A_m)$ must be connected so that since δ as a set function is countably subadditive on connected unions,

$$f(\delta(A_m)) \le f(\sum_{n=1}^{\infty} \delta(A'_n \cap A_m)) \le \sum_{n=1}^{\infty} f(\delta(A'_n \cap A_m))$$

$$\le \sum_{n=1}^{\infty} f(\delta(A'_n)) < m(A_m) + \epsilon,$$

which implies that $f(\delta(A_m)) \le m(A_m)$ whence that $m(A_m) = f(\delta(A_m))$. QED

Now we are in a position to prove a strong generalization of our first theorem which will easily follow from our previous work.

Theorem 5 *Let $\{A_\alpha\}$ be a family of subsets of a connected metric space X and suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the conditions in the previous theorem. Define the outer measure m as in Theorem 4. Then X cannot have a countable cover $\{A_n : n \in \mathbb{Z}^+\}$, where each $A_n \in \{A_\alpha\}$ is connected and $\sum_{n=1}^{\infty} f(\delta(A_n)) < m(X)$, nor can X have an arbitrary countable cover $\{O_n : n \in \mathbb{Z}^+\}$ such that $\sum_{n=1}^{\infty} f(\delta(O_n)) < f(\delta(X))$.*

PROOF: For the first case, assume the contrary and let $\{A_n : n \in \mathbb{Z}^+\} \subseteq \{A_\alpha\}$ be a countable cover of X such that $\sum_{n=1}^{\infty} f(\delta(A_n)) < m(X)$. Now since each A_n is connected, $m(A_n) = f(\delta(A_n))$ so that

$$m(X) = m(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m(A_n)$$

$$= \sum_{n=1}^{\infty} f(\delta(A_n)) < m(X).$$

which is an obvious contradiction.

For the second case, again assume the contrary and let $\{A_n : n \in \mathbb{Z}^+\}$ be a countable cover such that $\sum_{n=1}^{\infty} f(\delta(O_n)) < f(\delta(X))$, so that

$$f(\sum_{n=1}^{\infty} \delta(O_n)) < \sum_{n=1}^{\infty} f(\delta(O_n)) < f(\delta(X))$$

which is a contradiction to Theorem 1 by virtue of the monotonicity of f. QED

Theorem 5 of course generalizes Theorem 1 in the sense that if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is the identity function, then Theorem 1 immediately follows from Theorem 5. We will now use Theorem 5 to prove an interesting result concerning total disconnectedness in connected metric spaces.

Corollary 2 *Let X be a connected metric space and let m be an outer measure as in Theorem 5. Then any $S \subset X$ such that $m(S) = 0$ must be totally disconnected.*

PROOF: If we assume the contrary so that S is not totally disconnected, then it has a connected subset $C \subseteq S$ such that $\delta(C) > 0$. However, $m(C) = 0$ so that we can cover C by a sequence of sets $\{A_n : n \in \mathbb{Z}^+\} \subseteq \{A_\alpha\}$ where $\sum_{n=1}^{\infty} f(\delta(A_n))$ is arbitrarily small, which is a contradiction to Theorem 5. QED

It should be noted that $m(S) = 0$ is not a necessary condition for total disconnectedness, for if m is Lebesgue measure on $\mathbb{R}$, then $m(\mathbb{R}\backslash\mathbb{Q}) = \infty$ but $\mathbb{R}\backslash\mathbb{Q}$ is totally disconnected.

Now for the sake of completeness, we will make a comment about the measurable sets of the outer measure above. It can be shown that if m is any outer measure on a metric space X such that if $A, B \subseteq X : d(A, B) > 0 \Rightarrow$

$m(A \cup B) = m(A) + m(B)$, then every open set of X is measurable (such a metric space endowed with an outer measure with this property is called a "metric outer measure," see [3].) We now prove that with suitable conditions, our above outer measure is a metric outer measure.

Theorem 6 *Let X be some metric space and let m be an outer measure as in Theorem 3. Let X also satisfy the following property: If $A, B \subset X$ and $d(A,B) > 0$, and there exists an $\epsilon > 0$ such that if $\sum_{n=1}^{\infty} f(\delta(A_n)) \leq m(A) + \epsilon$ and $A \subset \bigcup_{n=1}^{\infty} A_n$, then each A_n is disjoint from A (or B respectively,) then m an outer metric measure.*

PROOF: For any $\epsilon > 0$, let $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and $B \subseteq \bigcup_{n=1}^{\infty} A'_n$ such that A_n and A'_n are both in $\{A_\alpha\}$ for any $n \in \mathbb{Z}^+$ where

$$\sum_{n=1}^{\infty} f(\delta(A_n)) \leq m(A) + \frac{\epsilon}{2}$$

and

$$\sum_{n=1}^{\infty} f(\delta(A'_n)) \leq m(B) + \frac{\epsilon}{2}.$$

Since $(\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} A'_n)$ covers A $\cup$ B,

$$m(A \cup B) \leq \sum_{n=1}^{\infty} (f(\delta(A_n)) + f(\delta(A'_n))) \leq m(A) + m(B) + \epsilon$$

so that

$$m(A \cup B) \leq m(A) + m(B).$$

Now for any $\epsilon > 0$ and $n \in \mathbb{Z}^+$, let $A \cup B \subseteq \bigcup_{n=1}^{\infty} A_n$ such that $A_n \in \{A_\alpha\}$ and

$$\sum_{n=1}^{\infty} f(\delta(A_n)) \leq m(A \cup B) + \epsilon.$$

Divide $\{A_n\}$ into $\{A'_n\}$ and $\{A''_n\}$ where each set of the former intersects only A and each set of the latter intersects only B. Therefore, $m(A) \leq \sum_{n=1}^{\infty} f(\delta(A'_n))$ and $m(B) \leq \sum_{n=1}^{\infty} f(\delta(A''_n))$ so that

$$m(A) + m(B) \leq \sum_{n=1}^{\infty} (f(A''_n) + f(A'_n)) = \sum_{n=1}^{\infty} f(\delta(A_n)) \leq m(A \cup B) + \epsilon.$$

Consequently, it follows that $m(A) + m(B) \leq m(A \cup B)$, so that $m(A \cup B) = m(A) + m(B)$. QED

4 Divergent Sequentially Decreasing Subsets

We now briefly consider divergent sequentially decreasing covers. Again, the example regarding the real line that was previously discussed illustrates the difficulty in determining whether a metric space has such a cover. However, removing the requirement that the sum of the diameter of the sets converge makes the situation much more difficult than in the case when the sum converges. Despite this, we will prove a somewhat startling result concerning such subsets, but first we prove a very easy result.

Theorem 7 *Let X be a metric space such that $\delta(X) = \infty$ with the property that there exists a $k \in \mathbb{Z}^+$ such that for any open or closed sphere S with radius $r, \delta(S) \geq k \cdot r$. Then there exists spheres with the following properties:*
1) $\lim_{n\to\infty} \delta(S_n) = 0$
2) $\sum_{n=1}^{\infty} \delta(S_n)$ *diverges*
3) *the sets of $\{S_n\}$ are pairwise disjoint.*

Proof: For any $x \in X$, place a sphere around x with radius 1. Clearly there exists an element $x' \in X$ not in this sphere since $\delta(X) = \infty$, so place a sphere of radius $\frac{1}{2}$ around x'. Again, there exists an $x'' \in X$ not in the union of these two spheres so we can inductively construct spheres of radii $1, \frac{1}{2}, \frac{1}{3}, ...$ around $x, x', x'', ...$ respectively, that satisfy the above conditions since the spheres are chosen so that they are pairwise disjoint. QED

Theorem 8 *Let X be an infinite metric space with the property that there exists a $k \in \mathbb{Z}^+$ such that for any open or closed sphere S with radius $r, \delta(S) \geq k \cdot r$. Then if $\{x_n\}$ is any countable subset of X, there exist spheres (open or closed) S_n with the following properties:*
1) $\lim_{n\to\infty} \delta(S_n) = 0$
2) $\sum_{n=1}^{\infty} \delta(S_n)$ *diverges*
3) *for each $n \in \mathbb{Z}^+$ except possibly for a single positive integer $n', x_n \in O_n$*
4) $X \neq \bigcup_{n=1}^{\infty} S_n$.

Proof: Let y not be in $\{x_n\}$. Since a countable subset of more than a single element is disconnected, we can write $\{y\} \cup \{x_n\}$ as the disjoint union of two closed subsets A and B. Without loss of generality, let $y \in A$ and be a limit point of A so that y is not a limit point of B, and that B has infinitely many points. Let N be some positive integer such that the sphere of radius $\frac{1}{N}$ around y is disjoint from B. Put a sphere (closed or open) of radius $\frac{1}{N+1}$, $\frac{1}{N+2}$, ... around each element of B so that the collection of these spheres satisfies our desired property. Now if B is finite, let N' be some positive integer such that $d(A, B) > \frac{1}{N'}$, and place spheres (open or closed) of radius radius $\frac{1}{N'}$, $\frac{1}{N'+1}$, ... around each point in A, where we denote this collection of spheres by S. For some point $x' \in B$ and each $x \in B$ such that $x \neq x'$, there exists a sphere around x with radius r_x that does not contain x', so that the collection of these spheres together with the collection of spheres in S satisfy our desired properties.

Now if y is not a limit point of $\{x_n\}$, then let N'' be some positive integer such that $\delta(y, \{x_n\}) > \frac{1}{N''}$ so that placing spheres of radius $\frac{1}{N''+1}$, $\frac{1}{N''+2}$, ...

around each x_1, x_2, ... produces a collection of spheres that satisfy our desired properties. QED

What makes this theorem startling is the fact that if X is a separable metric space satisfying the hypothesis of Theorem 8, and if $\{r_n\}$ is some countable dense subset, then it is somewhat counterintuitive that Theorem 8 should hold if we use $\{r_n\}$ as our set.

It should be noted however, that covers of metric spaces with the first two properties of the above theorem may not even exist. For example, if X is any uncountable set with the discrete metric $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$, then if $\lim_{n\to\infty} \delta(O_n) = 0$ and each of these sets are non-empty, they each must eventually only contain one point so that $\sum_{n=1}^{\infty} \delta(O_n)$ converges.

Finally, we construct a d.s.d. for the metric space $\mathbb{R}^n$. In the case where $n = 1$, let $M_k = \sum_{j=1}^{k} \frac{1}{j}$ for all $k \in \mathbb{Z}^+$ and let $A_0 = [-1, 1]$ and $A_k = [M_k, M_{k+1}] \cup [-M_{k+1}, -M_k]$ for all $k \in \mathbb{Z}^+$, then clearly $\{A_k : k \in \mathbb{Z}^+\}$ is a d.s.d. cover of $\mathbb{R}$. Now letting A_0 be the n - dimensional cube centered at the origin with side length 1, we can place a finite number of n - dimensional squares of side length $M_2 - M_1$ on each side of A_0 so that the union of these squares is a square of side M_2 which we call A_1. Continuing this process for each $k \in \mathbb{Z}^+$, the squares A_k provide a d.s.d. cover for $\mathbb{R}^n$. However, for most metric spaces, this approach clearly does not make sense, and hence does not give us much insight into finding any really general sufficient conditions for a metric space to have a d.s.d. cover. Using Theorem 3 and Theorem 1, however, we can immediately conclude that a sufficient condition for a metric space to have a d.s.d. cover is that it be connected, Lindeloff, locally compact, and have infinite diameter, although this is extremely restrictive and does not provide much information about the general nature of d.s.d. covers.

5 Conclusion

In the course of this paper we gave necessary conditions, both topological and measure theoretic, for a metric space X to not have a c.s.d cover, and gave sufficient conditions, both topological and measure theoretic, for a metric space and subsets to be totally disconnected. We also found a sufficient condition for a specific outer measure to be an outer metric measure, proved a surprising result concerning d.s.d. subsets of specific class of metric spaces, gave a sufficient condition for a metric space to have a d.s.d., and proved various other theorems and lemmas. However the topic is so broad that our treatment is in no way exhaustive, so that only a few of the many possible results on this subject have been obtained.

Now although we addressed the problem of giving necessary conditions for a metric space X to never have a countable covering $\{A_n\}$ where $\sum_{n=1}^{\infty} \delta(A_n) < \delta(X)$, it is not immediately clear how or if one would address this problem using methods that differ from our 'chain' approach. Moreover, despite the fact that we gave a sufficient condition for a metric space X to have a c.s.d. or d.s.d.

cover, it is not clear how to find stronger necessary and sufficient conditions for d.s.d. covers or how to find strong sufficient conditions for a c.s.d. cover of a metric space to exist. Clearly we can not adopt the 'chain' approach used in Section 1 to prove results about when $\sum_{n=1}^{\infty} \delta(A_n)$ diverges, since this approach is based on $\sum_{n=1}^{\infty} \delta(A_n)$ converging.

Another open question regarding d.s.d. and c.s.d. subsets and covers is its applications to fields other than metric topology. More specifically, despite the fact that we were able apply our theorems to the topics of total disconnectedness and outer measures on metric spaces, it is not clear if, or how, we can apply our ideas and theorems to other topics in mathematics.

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