A CHARACTERIZATION OF TREE TYPE

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Abstract. Let $L(G)$ be the Laplacian matrix of a simple graph G. The characteristic valuation associated with the algebraic connectivity $a(G)$ is used in classifying trees as Type I and Type II. We show a tree T is Type I if and only if its algebraic connectivity $a(T)$ belongs to the spectrum of some branch B of T.

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1. Introduction

1.1. **The Laplacian Matrix.** A *graph* $G(V,E)$ is an ordered pair of a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and a set of edges, E, which consists of unordered pairs of vertices. For the purposes of this paper, a graph G will mean a finite, simple (without multiple edges or loops) graph. The number of edges incident to a vertex v is called the *degree* of that vertex, and is denoted $d(v)$. The *Laplacian matrix* of a graph G with n vertices, denoted by $L(G)$, is the $n \times n$ matrix $[L_{ij}]$ where

$$L_{ij} = \begin{cases} d(v_i), & \text{if } i = j; \\ -1, & \text{if } \{v_i, v_j\} \in E; \\ 0, & \text{otherwise.} \end{cases}$$

A *tree* T is a connected graph that does not contain a cycle. A *branch* B of a tree T at vertex v is a connected component of the subgraph T_v of T obtained by deleting v and all edges incident to it. The vertex of B which is adjacent in T to v is called the *root* of B, and is denoted by $r(B)$. We also call B a *rooted* branch at $r(B)$.

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Let B be a rooted branch with vertex set $\{u_1, u_2, \ldots, u_r\}$, and let u_r be the root $r(B)$ of B. Denote by $\hat{L}(B) = [a_{ij}]$ the $r \times r$ matrix given by:

$$a_{ij} = \begin{cases} d(u_r) + 1, & \text{if } i = j = r; \\ d(u_i), & \text{if } i = j \neq r; \\ -1, & \text{if } \{u_i, u_j\} \text{ is an edge of } B; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose v is a vertex of a graph G. The *neighborhood* $N(v)$ of v in G is the set of all vertices of G that are adjacent to v. v is called a *pendant vertex* of G if and only if $|N(v)| = 1$.

We will say the *length* of a branch is the maximum length of paths from the root of the branch to a pendant vertex in the branch and define the *size* of a tree T, denoted $\|T\|$, as the maximum of the lengths of all branches at a designated center vertex of T.

Given a vertex v in a branch B of a tree T, another vertex u in T is said to be an *upper vertex* of v if the path from v to the root $r(B)$ of B goes through u. A vertex l is said to be a *lower vertex* of v if v is an upper vertex of l.

Remark 1.1. If a vertex v has an upper (lower) vertex u which belongs to $N(v)$, then u is said to be an upper (lower) *neighbor* of v. In a tree, an upper neighbor of a given vertex v must be unique.

1.2. **The Algebraic Connectivity and Characteristic Valuation.** The Laplacian matrix $L(G)$ is a symmetric, positive semidefinite, singular, matrix. The eigenvalues of $L(G)$ can be written as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. It can be shown that $\lambda_2 > 0$ if and only if G is connected. Because of this, λ_2 is called the *algebraic connectivity* of G and is denoted $a(G)$. More details about graphs and the Laplacian matrix can be found in [1], [7], and [8].

Let us assume that the graph G is connected. Then $a(G) > 0$. Let $\xi(G)$ denote the set of eigenvectors corresponding to $a(G)$. Fix an eigenvector $\bar{x} = (x_1, x_2, \ldots, x_n) \in \xi(G)$. Define $f\colon V \to \mathbb{R}$ by $f(v_i) = x_i, 1 \leq i \leq n$, and write $f \in \xi(G)$ to mean $(x_1, x_2, \ldots, x_n) \in \xi(G)$. The function f which assigns the coordinates of an eigenvector corresponding to $a(G)$ to the vertices of G is called a *characteristic valuation* or *Fiedler vector* of G. Note that $f \in \xi(G)$ is independent of the labeling of the vertices since relabeling results in a matrix that is permutation similar to the original matrix.

Theorem 1.2. [3], [6] *If $T(V, E)$ is a tree, then, for a characteristic valuation $f \in \xi(G)$, only one of two cases may occur:*

Case 1. *For $V^f = \{v \in V | f(v) = 0\} \neq \emptyset$, the graph $T^f = (V^f, E^f)$ induced by T on V^f is connected and there is exactly one vertex $u \in V^f$ which is adjacent in T to a vertex not belonging to V^f. Moreover, the values of f along any path in T starting at u are either increasing, decreasing, or identically zero.*

Case 2. *For $f(v) \neq 0$ for all $v \in V$, T contains exactly one edge $\{u, w\}$ such that $f(u) > 0$ and $f(w) < 0$. Moreover, the values of f along any path which starts at u and does not contain w increase while the values of f along any path starting at w and not containing u decrease.*

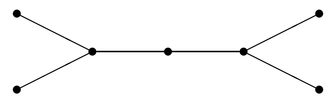

FIGURE 1. A Symmetric Tree

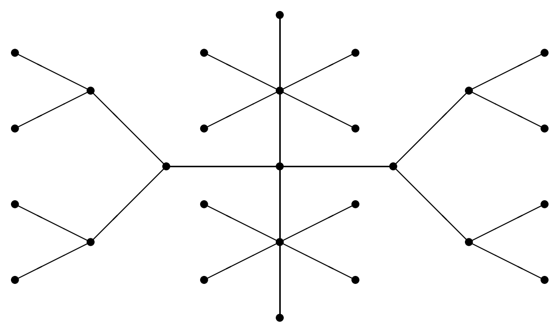

FIGURE 2. A Multi-symmetric Tree

In the first case, the tree is said to be *Type I* and to have a unique *characteristic vertex*, u, denoted as u_{ch}. In the second case, the tree is said to be *Type II* and to have a unique *characteristic edge*, $\{u, w\}$.

Theorem 1.3. [2] *Let $T(V, E)$ be a tree. Suppose $f \in \xi(T)$. If $a(T)$ is a multiple eigenvalue of $L(T)$, then there is a vertex $v \in V$ such that $f(v) = 0$.*

2. Symmetric Trees

The ideas and results of this section started with the work of Rong Zhang who looked at symmetric trees as part of a master's thesis at Central Michigan University with Sivaram K. Narayan [9]. The results of our development of this idea are given, but the proofs for some are omitted as they are corollaries of later results.

Definition 2.1. A tree $T(V, E)$ is said to be *symmetric* if there exists a vertex v such that all the branches of T at v are isomorphic. The vertex v is called the *center* of T and is denoted v_{ce}.

Definition 2.2. A tree $T(V, E)$ is said to be *multi-symmetric* if there exists a vertex v, called the center of T, such that all the branches of T at v can be partitioned into r classes $C_1, C_2, \ldots, C_r$ which satisfy the following conditions:

(1) Each class consists of two or more branches.
(2) Any two branches from the same class are isomorphic.
(3) Any two branches from different classes are not isomorphic.

Theorem 2.3. *Every multi-symmetric tree is Type I. Moreover, the center is the characteristic vertex of the tree.*

Proof. Let $T(V,E)$ be a multi-symmetric tree that can be partitioned into r classes. Let $B_{i1}, B_{i2}, \dots, B_{is_i}$ denote the s_i isomorphic branches of the class C_i, each of which contains n_i vertices, $1 \le i \le r$. Since $s_i \ge 2$ for $1 \le i \le r$, the Laplacian matrix of T, $L(T)$, can be written in the following form:

$$\begin{bmatrix} \mathcal{L}(C_1) & 0 & \dots & \mathcal{X}_1^t \\ 0 & \mathcal{L}(C_2) & \dots & \mathcal{X}_2^t \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}_1 & \mathcal{X}_2 & \dots & d(v_{ce}) \end{bmatrix}$$

where for $1 \le i \le r$

$$\mathcal{L}(C_i) = \begin{bmatrix} \hat{L}(B_{i1}) & 0 & \dots & 0 \\ 0 & \hat{L}(B_{i2}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{L}(B_{is_i}) \end{bmatrix}$$

$$\text{and} \quad \mathcal{X}_i = \begin{bmatrix} \bar{c}_{i1} & \bar{c}_{i2} & \dots & \bar{c}_{is_i} \end{bmatrix}$$

and, labeling the last vertex of each branch as its root, the row vector $\bar{c}_{ij}$ for $1 \le j \le s_i$ is

$$\bar{c}_{ij} = \begin{bmatrix} 0 & 0 & \dots & 0 & -1 \end{bmatrix}$$

Take $f \in \xi(T)$ such that $f = \left(\bar{x}^{11}, \dots, \bar{x}^{1s_1}, \bar{x}^{21}, \dots, \bar{x}^{2s_2}, \dots \bar{x}^{rs_r}, x^c\right)$ where x^c is a scalar and $\bar{x}^{ij}$ is a row vector of length n_i for $1 \le i \le r$ and $1 \le j \le s_i$. By definition, $L(T)f^t = a(t)f^t$. Using block multiplication of matrices, we have:

$$\left(\hat{L}(B_{ij}) - a(T)\right)\left(\bar{x}^{ij}\right)^t = (0,0,\dots,0,x^c)^t, 1 \le i \le r \text{ and } 1 \le j \le s_i \tag{1}$$

and

$$\sum_{i=1}^{i=r}\sum_{j=1}^{j=s_i} f(r(B_{ij})) = (d(v_{ce}) - a(t))\, x^c \tag{2}$$

Suppose that $T(V,E)$ is a Type II tree. Therefore, $f(v) \neq 0$ for all $v \in V$. By Theorem 1.3, $a(T)$ is simple and f is unique apart from its multiples. We first prove the following:

Claim. *Suppose* $f \in \xi(T) = \left(\bar{x}^{11}, \dots, \bar{x}^{1s_1}, \bar{x}^{21}, \dots, \bar{x}^{2s_2}, \dots \bar{x}^{rs_r}, x^c\right)$. *Then for each* i, $1 \le i \le r$, $\bar{x}^{ij} = \bar{x}^{ik}$ *for all* j,k, $1 \le j,k \le s_i$.

Proof of Claim. Assume for some i there exists j and k, $1 \le j,k \le s_i$, such that $\bar{x}^{ij} \neq \bar{x}^{ik}$. Now take g which is the same as f except that $\bar{x}^{ij}$ and $\bar{x}^{ik}$ are interchanged. Then f and g are two linearly independent vectors satisfying (1) and (2) above. This implies $a(T)$ is not simple, which is a contradiction of the assumption that $a(T)$ is simple. Therefore, the Claim must be true.

Since we have assumed $T(V,E)$ is a Type II tree, there must be a unique characteristic edge $\{u,w\}$ such that $f(u)f(w) < 0$. This edge may only be found in one of two possible places: either both vertices are part of a branch, or one of the vertices is v_{ce}.

Suppose that the edge belongs to some branch B_{ij}. But, as we have seen above, $\bar{x}^{ij} = \bar{x}^{ik}, 1 \leq j, k \leq s_i$. This means each B_{ik} must contain this edge as well. Since every class C_i has at least two branches, this contradicts the uniqueness of this edge.

The edge then must have v_{ce} as one of its vertices, and the root of some branch, $r(B_{ij})$, as the other. Then $f(v_{ce})f(r(B_{ij})) < 0$. But, because there exists another branch B_{ik} where $f(r(B_{ik}))f(r(B_{ij})) > 0$, $f(v_{ce})f(r(B_{ik})) < 0$ is also true, again contradicting the uniqueness of the characteristic edge.

As the characteristic edge cannot exist uniquely, the tree cannot be Type II. Therefore, every multi-symmetric tree is Type I.

To prove the center must be the characteristic vertex, a similar argument may be used, as assuming the vertex was in one of the branches would contradict its uniqueness using Theorem 1.2 . ☐

2.1. **Multiplicity of the algebraic connectivity.** Let T be a Type I tree, and let $f \in \xi(T)$ be a characteristic valuation of T. A branch B of T is said to be *passive* if $f(r(B)) = 0$. Otherwise, the branch is said to be *active*.

Theorem 2.4. [4] *Let T be a Type I tree with characteristic vertex v_{ch} and algebraic connectivity $a(T)$. Let B be a branch at v_{ch} with root $r(B)$. Then B is active if and only if $a(T)$ is an eigenvalue of $\hat{L}(B)$. Moreover, if $a(T)$ is an eigenvalue of $\hat{L}(B)$, then it is simple and it is the smallest eigenvalue of $\hat{L}(B)$.*

Theorem 2.5. [4] *Let T be a Type I tree with characteristic vertex v_{ch} and algebraic connectivity $a(T)$. Let m be the multiplictiy of $a(T)$ as an eigenvalue of $L(T)$. Then exactly $m+1$ branches at v_{ch} are active.*

In the following theorem, we prove the converse of Theorem 2.5 for a Type I tree. First, however, we note the following:

Remark 2.6. Let T be a tree with center vertex v_c. Denote the branches at v_c by $B_1, B_2, \ldots, B_{d(v_c)}$. By labeling v_c as the last vertex and labeling the other vertices of T so that those of B_i precede those of B_{i+1}, $i = 1, \ldots, d(v_c) - 1$, $L(T)$ takes the following partitioned form

$$\begin{bmatrix} \hat{L}(B_1) & 0 & \ldots & 0 & \bar{c}_1 \\ 0 & \hat{L}(B_2) & \ldots & 0 & \bar{c}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \hat{L}(B_{d(v_c)}) & \bar{c}_{d(v_c)} \\ \bar{c}_1^t & \bar{c}_2^t & \ldots & \bar{c}_{d(v_c)}^t & d(v_c) \end{bmatrix} \tag{3}$$

where $\bar{c}_i = \begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \end{pmatrix}^t$ is a 1 by $|V(B_i)|$–vector, $1 \leq i \leq d(v_c)$.

Definition 2.7. For a tree T, if $L(T)$ is of the form (3) and if each vertex is labeled before its upper neighbor, then we shall say $L(T)$ is in *standard form*.

Theorem 2.8. *Let T be a Type I tree with characteristic vertex v_{ch} and algebraic connectivity $a(T)$. Let m be the number of active branches of T. Then the multiplicity of $a(T)$ is equal to $m-1$.*

Proof. With $L(T)$ in standard form, and using $p(M)$ to mean the characteristic polynomial of matrix M, we can write the characteristic polynomial of $L(T)$ as:

$$p(L(T)) = (d(v_{ch}) - \lambda)\prod_{j=1}^{q} p(\hat{L}(B_j)) + \sum_{k=1}^{q}\left(\bar{c}_k^t \bar{c}_k \prod_{j=1, j\neq k}^{q} p(\hat{L}(B_j))\right) \tag{4}$$

By Theorem 2.4, $a(T)$ is a simple eigenvalue of $\hat{L}(B)$ for every active branch B of T. Therefore, $(\lambda - a(T))$ is a factor of the characteristic polynomial of $\hat{L}(B)$ for every active branch B of T. If there are m active branches of T, where $2 \leq m \leq q$, it follows from (4) that $(\lambda - a(T))^{m-1}$ is a factor of the characteristic polynomial of $L(T)$. Since $a(T)$ is an eigenvalue of $\hat{L}(B)$ if and only if B is an active branch, we conclude that the multiplicity of $a(T)$ is $m - 1$. □

Remark 2.9. Just as we are able to factor $(\lambda - a(T))$ from the characteristic polynomial of a tree T $(m - 1)$ times from m active branches, if q is the number of branches B for which λ_B is a simple eigenvalue of $\hat{L}(B)$, we are able to factor $(\lambda - \lambda_B)$ from the characteristic polynomial of $L(T)$ $(q - 1)$ times.

Note. Theorem 2.3 and Theorem 2.8 for the special case where T is a symmetric tree can be found in [9].

2.2. **Algebraic Symmetry.** Although, from Theorem 2.3, multi-symmetric trees must be Type I, it is not the case that a tree with no visual symmetry must be Type II. In fact, a tree with two branches which are not isomorphic may be either Type I or Type II. So, even though two branches may not be isomorphic, they may "act isomorphic" in their influence on tree classificaiton. Because the authors like the idea of Type I trees having some sort of symmetry, we introduce the following defintions:

Definition 2.10. Two branches B_1, B_2 are said to be *algebraically equivalent* if, using $\sigma(M)$ to denote the set of eigenvalues (spectrum) of a matrix M,

$$\min\left(\sigma\left(\hat{L}(B_1)\right)\right) = \min\left(\sigma\left(\hat{L}(B_2)\right)\right).$$

Definition 2.11. A tree $T(V, E)$ is said to be *algebraically symmetric* if there exists a center vertex v of T such that all the branches of T at v can be partitioned into r classes $C_1, C_2, \ldots, C_r$ which satisfy the following conditions:

(1) Each class consists of two or more branches.
(2) Any two branches from the same class are algebraically equivalent.
(3) Any two branches from different classes are not algebraically equivalent.

Example 2.12. The tree found in Figure 3 is Type I [4]. Calling the vertex whose characteristic valuation is 0 the center vertex v of the tree, the two branches B_1, B_2 which are rooted at v satisfy

$$\min\left(\sigma\left(\hat{L}(B_1)\right)\right) = \min\left(\sigma\left(\hat{L}(B_2)\right)\right) \approx 0.139149$$

Therefore it is algebraically symmetric.

Definition 2.13. Given a tree T whose branches $\{B_j\}$ are partitioned into classes $\{C_i\}$, the spectrum of the class C_i, written $\sigma(C_i)$, is defined as

$$\sigma(C_i) = \bigcup_{B_j \in C_i} \sigma(\hat{L}(B_j))$$

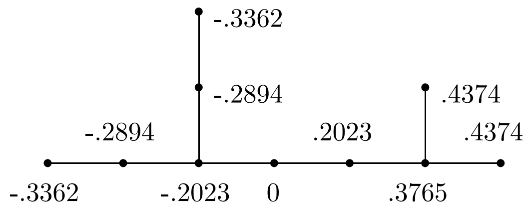

FIGURE 3. Algebraically Symmetric Tree

Lemma 2.14. *Given an algebraically-symmetric tree T, the algebraic connectivity of T, $a(T)$, will belong to the spectrum of one class. Furthermore, it will be the minimum eigenvalue of that spectrum.*

Proof. Suppose that T is an algebraically symmetric tree whose branches are partitioned into m classes $C_1, C_2, \ldots, C_m$. Consider a new tree T' created from T as follows: if $B_{i1}, B_{i2}, \ldots, B_{is_i}$ are the branches of T in class C_i, $1 \leq i \leq m$, then root an isomorphic copy of each branch to the center vertex v of T. The tree T' now consists of classes $\{C'_{ij}\}$ where two copies of B_{ij} of T are in C'_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq s_i$. Therefore, T' is a multi-symmetric tree associated with the given algebraically-symmetric tree T. By Theorem 2.3, T' is Type I.

Suppose B is a branch of T and $\lambda_B \in \sigma(\hat{L}(B))$. By Remark 2.9, we can factor $(\lambda - \lambda_B)$ from the characteristic polynomial of $L(T')$. Therefore, the characteristic polynomial of $L(T')$ is related to the characteristic polynomial of $L(T)$ as follows:

$$p(L(T')) = \left(\prod_{\lambda_B \in \sigma(\hat{L}(B))} (\lambda - \lambda_B) \right) p(L(T)) \tag{5}$$

when the product is taken over all branches B of T.

Since T' is Type I, by Theorem 2.4, $a(T')$ is a simple eigenvalue of $\hat{L}(B)$ for every active branch B of T'. Because T is algebraically symmetric, T' has at least four active branches that belong to two classes of T'. Suppose $a(T') \in \sigma(C'_{pq})$ and $a(T') \in \sigma(C'_{rs})$. Because T is algebraically symmetric, $a(T') \in \sigma(C_p)$ and $a(T') \in \sigma(C_r)$. Finally, by (5), $a(T')$ is the smallest positive eigenvalue of $\sigma(L(T))$. Hence $a(T') = a(T)$ and so $a(T)$ belongs to the spectrum of a class of T. Moreover, $a(T)$ is the minimum of the spectrum of that class. □

The following theorem follows from Lemma 2.14 and Theorem 3.2.

Theorem 2.15. *Every algebraically symmetric tree is Type I.*

3. TYPE I TREES

Although we have not yet given a necessary and sufficent condition for a tree to be Type I, it turns out that algebraic symmetry is closer than one might guess.

In what follows, if $L(T)$ is the Laplacian matrix of a tree and $\bar{x}$ is an eigenvector corresponding to an eigenvalue λ of $L(T)$, then we denote by x_v the coordinate of $\bar{x}$ corresponding to a vertex v of T with respect to some labeling of the vertices.

Theorem 3.1. *Suppose $\bar{x}$ is an eigenvector corresponding to an eigenvalue λ of $L(T)$. Let $p(\hat{L}(B))$ denote the characteristic polynomial of $\hat{L}(B)$, and let $p(\hat{L}(B))\big|_\lambda$ denote the value of $p(\hat{L}(B))$ evaluated at λ. For any vertex u of a tree T with (designated) center vertex v_c, if B_v is the branch at vertex u with root v, then*

$$\left[\prod_{i=1}^{n_v} p(\hat{L}(B_{l_i}))\big|_\lambda\right] x_u = p(\hat{L}(B_v))\big|_\lambda x_v \tag{6}$$

where l_i are the n_v lower neighbors of v in T.

Proof. We begin by using elementary row operations to reduce $L(T) - \lambda I$ to an upper-triangular form. Let D_v represent the diagonal element of the upper-triangular form of $L(T) - \lambda I$ corresponding to v. We show that

$$D_v = p(\hat{L}(B_v)) \tag{7}$$

for every v in T, with the exception of the center vertex v_c, where $D_{v_c} = p(L(T))$.

We shall consider a vertex v to be *pure* when the row of $L(T)-\lambda I$ corresponding to v has only zeros to the left of its diagonal entry. Writing $L(T)$ in standard form, we note that pendant vertices are pure. Also, pendant vertices trivially satisfy (7). Consider the equation resulting from multiplying the row corresponding to any pendant vertex with the vector $\bar{x}$. This gives (6) for the coordinate of the pendant vertex.

We proceed by induction on the distance from the center vertex. At distance $\|T\|$, all vertices are pendant, so we are done from above. Assume, then, that all vertices of distance $\|T\|-k$, $k \geq 0$ from the center satisfy (7) and are pure. Consider any vertex v of distance $\|T\| - (k+1)$ from the center. If v is a pendant vertex, we are done. If not, by the assumption, all n_v lower neighbors $l_1, \ldots, l_{n_v}$ satisfy (7) and are pure. Proceed (without division) using elementary row operations to purify v. The new diagonal entry for v becomes

$$(d(v) - \lambda)\prod_{i=1}^{n_v} p(\hat{L}(B_{l_i})) - \sum_{j=1}^{n_v} \frac{\prod_{i=1}^{n_{l_j}} p(\hat{L}(B_{l_{j,i}}))\prod_{i=1}^{n_v} p(\hat{L}(B_{l_i}))}{p(\hat{L}(B_{l_j}))}$$

where the $l_{j,i}$ are the n_{l_j} lower neighbors of l_j.

As the above is the characteristic polynomial of $\hat{L}(B_v)$, it is easily seen that (7) holds for v. Because v is pure, consider the equation resulting from multiplying the row corresponding to v with the vector $\bar{x}$. This gives (6) for x_v. By induction, the lemma holds. □

Although Theorem 3.1 is true for any choice of vertex v_c, we shall find that for some trees, the right choice of v_c and application of the theorem will show them to be Type I, in which case v_c is indeed the characteristic vertex and deserves the subscript.

Theorem 3.2. *A tree T is Type I if and only if its algebraic connectivity $a(T)$ belongs to the spectrum of $\hat{L}(B)$ for some branch B of T.*

Proof. If T is a Type I tree, $a(T)$ must belong to $\hat{L}(B)$ for some active branch B of T by Theorem 2.4.

Suppose $a(T)$ belongs to the spectrum of $\hat{L}(B)$ for some branch B of T at vertex v_c. From Lemma 3.1, $x_{v_c} = 0$ if $a(T)$ does not belong to the spectrum of any sub-branch of B. If $a(T)$ belongs to a sub-branch of B, apply Lemma 3.1 to this sub-branch. If repeated, this process must eventually reach a pendant vertex. Here, however, we find that, since the pendant vertex has no sub-branches, the characteristic valuation of the upper neighbor must be zero. Hence, by Theorem 1.2, we conclude that T must be a Type I tree. □

Remark 3.3. The method of the proof of Theorem 3.1 also gives the following stronger result. Let T be a tree and $M(T)$ a Hermitian matrix whose graph is T. If B is a branch of T, let $M[B]$ denote the principal submatrix of $M(T)$ resulting from retention of the rows and columns corresponding to vertices of B. Suppose $\bar{x}$ is an eigenvector corresponding to an eigenvalue λ of $M[B]$. Let $p(M[B])$ denote the characteristic polynomial of $M[B]$, and let $p(M[B])\big|_\lambda$ denote the value of $p(M[B])$ evaluated at λ. For any vertex u of a tree T with center vertex v_c, if B_v is the branch at vertex u with root v, then

$$\left[\prod_{i=1}^{n_v} p(M[B_{l_i}])\big|_\lambda\right] x_u = p(M[B_v])\big|_\lambda x_v \tag{8}$$

where l_i are the n_v lower neighbors of v in T.

Thus we may conclude that valuations other than the characteristic valuation need not satisfy Theorem 1.2.

For a result which characterizes trees in terms of the spectra of the inverses of submatrices corresponding to the branches of T, see Corollary 2.1 in [5].

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