

Quadratic Assignment Problems (QAP) and its Size Reduction Method

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Abstract

The Quadratic Assignment Problem (QAP) is a discrete optimization problem which can be found in economics, operations research, and engineering. It seeks to locate N facilities among N fixed locations in the most economical way. This paper gives a brief introduction to QAP and discusses how to reduce the problem size to $N-1$ if the original problem satisfies certain conditions.

1. Introduction

Suppose a manager wants two employees to finish two tasks in a minimum number of days, and each employee must do one and only one task. Employee A can finish task 1 in 3 days, task 2 in 5 days; employee B can finish task 1 in 4 days, task 2 in 2 days.

There is no trick. The two tasks can be done in 3 days, with A doing task 1 and B doing task 2.

This simple example is actually called the Linear Assignment Problem (LAP). We now take one step further to look at the Quadratic Assignment Problem (QAP).

The Quadratic Assignment Problem (QAP) is a discrete optimization problem which can be found in many fields of study, including economics, operations research, and engineering [1]. In its basic interpretation, the problem seeks to locate N facilities among N fixed locations, where for each pair of facilities (i,k) a certain flow of commodities $f(i,k)$ is known and for each pair of locations (j,n) a corresponding distance $d(j,n)$ is known. The two-way transportation costs between facilities i and k , given that i is assigned to location j and k is assigned to location n , are $f(i,k) \cdot d(j,n) + f(k,i) \cdot d(n,j)$. The objective is to find an assignment minimizing the sum of all such transportation costs [2].

From the problem definition, we can draw three matrices: an $N \times N$ Flow matrix, whose entries F_{ik} are equal to $f(i,k)$; an $N \times N$ Distance matrix, whose entries D_{jn} are equal to $d(j,n)$; and an $N^2 \times N^2$ block matrix, Cost. (A block matrix is a matrix that is defined using smaller matrices.) The entries of the Cost matrix, C_{ijkn} , are equal to $F_{ik}D_{jn}$, i.e., $f(i,k) \cdot d(j,n)$. (i, j, k, n are all positive integers and $i, j, k, n \leq N$.)

$$F = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} \quad D = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$$

Flow and Distance matrices for $N = 3$

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

$$= \begin{pmatrix} C_{1111} & * & * & * & C_{1212} & * & * & * & C_{1313} \\ * & C_{1122} & C_{1123} & C_{1221} & * & C_{1223} & C_{1321} & C_{1322} & * \\ * & C_{1132} & C_{1133} & C_{1231} & * & C_{1233} & C_{1331} & C_{1332} & * \\ * & C_{2112} & C_{2113} & C_{2211} & * & C_{2213} & C_{2311} & C_{2312} & * \\ C_{2121} & * & * & * & C_{2222} & * & * & * & C_{2323} \\ * & C_{2132} & C_{2133} & C_{2231} & * & C_{2233} & C_{2331} & C_{2332} & * \\ * & C_{3112} & C_{3113} & C_{3211} & * & C_{3213} & C_{3311} & C_{3312} & * \\ * & C_{3122} & C_{3123} & C_{3221} & * & C_{3223} & C_{3321} & C_{3322} & * \\ C_{3131} & * & * & * & C_{3232} & * & * & * & C_{3333} \end{pmatrix}$$

Cost matrix for $N = 3$

Note that there are some asterisks in the Cost matrix. For example, consider C_{2212} , which is $F_{21}D_{22}$ ($C_{ijkn} = F_{ik}D_{jn}$). This refers to the transportation costs between facility 2 and facility 1, given that facility 2 is assigned to location 2 and facility 1 is assigned to location 2. Since we can assign only one facility to one location, this cost is meaningless. In general, for C_{ijkj} ($= F_{ik}D_{jj}$) to be meaningful, i must equal k ; similarly, for C_{ijin} ($= F_{ii}D_{jn}$) to be meaningful, j must equal n , since one facility can be assigned to only one location. All meaningless costs are denoted by asterisks and can be left out of consideration.

Here we define one more $N \times N$ matrix, the Linear Cost matrix L . The entries L_{ij} are equal to C_{ijij} , i.e. $F_{ii}D_{jj}$ ($i, j \leq N$). It refers to the cost of placing facility i in location j . This is called a “linear” cost since only one facility is being assigned to a location. When two facilities are simultaneously assigned to locations, the associated cost is called a “quadratic” cost.

$$L = \begin{pmatrix} F_{11}D_{11} & F_{11}D_{22} & F_{11}D_{33} \\ F_{22}D_{11} & F_{22}D_{22} & F_{22}D_{33} \\ F_{33}D_{11} & F_{33}D_{22} & F_{33}D_{33} \end{pmatrix}$$

Linear Cost matrix for N = 3

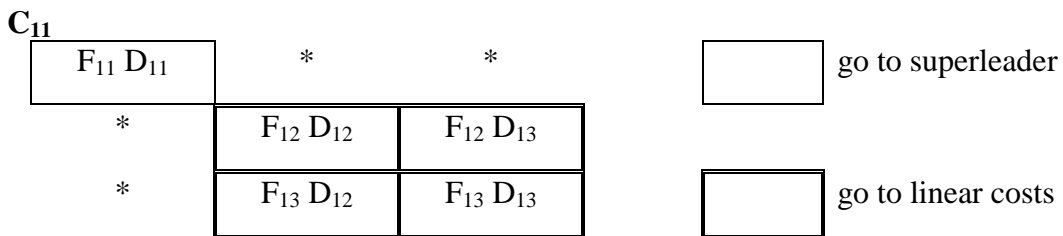
We have a 9 x 9 Cost matrix when there are 3 facilities to be assigned to 3 locations. The size of the cost matrix is already quite large for a very small QAP. Therefore, it is still considered a computationally nontrivial task to solve QAP. It takes about 90 days for a 360MHz computer to solve a size N = 30 problem using one of the currently available algorithms [3].

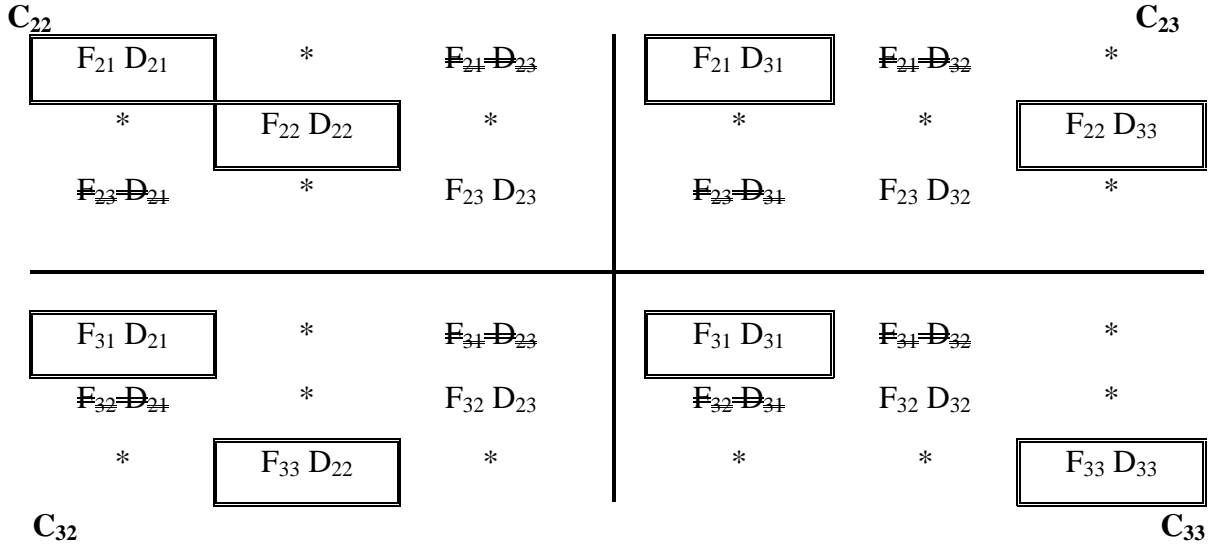
2. Size Reduction Method

However, if the original problem satisfies certain conditions, the size of the QAP can be reduced by 1. We will consider a size N = 3 problem. This will explain how the size reduction method works and will also illustrate how to tackle a small QAP.

We first make an arbitrary assignment. To keep things simple, facility 1 is assigned to location 1. Then C_{12} , C_{13} , C_{21} , and C_{31} can be crossed out. This is similar to getting a minor matrix by deleting the first row and the first column. (It differs from a minor because we still keep C_{11} .) The elements in C_{12} , one of the sub-matrices that are eliminated, are equal to $F_{1k}D_{2n}$. These refer to assigning facility 1 to location 2. However, since we have already made the assignment of facility 1 to location 1, these costs are not useful now. Similarly, the costs in C_{13} , C_{21} , and C_{31} are not useful either.

In each of the remaining sub-matrices, i.e. C_{11} , C_{22} , C_{23} , C_{32} , and C_{33} , we get the minor matrices by deleting the first row and the first column, but again, the element (1,1) in each sub-matrix is kept. We verify this by looking at two examples. Consider C_{2213} , an element that is eliminated in this process. It is $F_{21}D_{23}$, which refers to assigning facility 2 to location 2 and facility 1 to location 3. This should be crossed out. However, we cannot eliminate C_{2311} , the element (1,1) in C_{23} . It is $F_{21}D_{31}$, which has facility 1 assigned to location 1.





Cost sub-matrices after assigning facility 1 to location 1

Note that the element C_{1111} deserves some special treatment. It is the cost of assigning facility 1 to location 1 and thus a cost that must be paid after our arbitrary assignment. It will go to what we call “superleader” in QAP literature. This can be viewed as the accumulated costs when operations are performed on the cost matrix.

From the remaining sub-matrices we see that some elements “go to linear costs”. We now define a new $(N-1) \times (N-1)$ Linear Cost matrix L' . In our case, it is a $(3-1) \times (3-1)$, i.e. 2×2 matrix. This L' matrix is achieved by adding the elements that come from the cost matrix.

$$L' = \begin{pmatrix} F_{12}D_{12} + F_{21}D_{21} + F_{22}D_{22} & F_{12}D_{13} + F_{21}D_{31} + F_{22}D_{33} \\ F_{13}D_{12} + F_{31}D_{21} + F_{33}D_{22} & F_{13}D_{13} + F_{31}D_{31} + F_{33}D_{33} \end{pmatrix}$$

New Linear Cost matrix after assigning facility 1 to location 1

In this matrix, the elements $L'_{i-1,j-1}$ are the costs of assigning facility 1 to location 1 and facility i to location j ($2 \leq i, j \leq N$). This is almost the same as the definition in the original Linear Cost matrix L , where L_{ij} refers to the cost of assigning facility i to location j , except in here we add the cost of assigning facility 1 to location 1, a cost already incurred, and shift the matrix indices by 1.

A general $(N-1) \times (N-1)$ Linear Cost matrix L' is given by:

$$L' = \begin{pmatrix} F_{12}D_{12} + F_{21}D_{21} + F_{22}D_{22} & F_{12}D_{13} + F_{21}D_{31} + F_{22}D_{33} & F_{12}D_{14} + F_{21}D_{41} + F_{22}D_{44} & \dots \\ F_{13}D_{12} + F_{31}D_{21} + F_{33}D_{22} & F_{13}D_{13} + F_{31}D_{31} + F_{33}D_{33} & F_{13}D_{14} + F_{31}D_{41} + F_{33}D_{44} & \dots \\ F_{14}D_{12} + F_{41}D_{21} + F_{44}D_{22} & F_{14}D_{13} + F_{41}D_{31} + F_{44}D_{33} & F_{14}D_{14} + F_{41}D_{41} + F_{44}D_{44} & \dots \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

We can see the following:

$$\begin{aligned} L'_{11} &= F_{12} D_{12} + F_{21} D_{21} + F_{22} D_{22} \\ L'_{12} &= F_{12} D_{13} + F_{21} D_{31} + F_{22} D_{33} \\ L'_{22} &= F_{13} D_{13} + F_{31} D_{31} + F_{33} D_{33} \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

Then we can deduce that

$$\begin{aligned} L'_{i-1,j-1} &= F_{1i} D_{1j} + F_{i1} D_{j1} + F_{ii} D_{jj}, & 2 \leq i, j \leq N \\ &= 2F_{1i} D_{1j}, & \text{if } F_{i1} D_{j1} = F_{1i} D_{1j}, \text{ and } F_{ii} D_{jj} = 0 \end{aligned}$$

When we first define the $N \times N$ Linear Cost matrix in a size N problem before the arbitrary assignment, the entries L_{ij} are equal to C_{ij} , i.e. $F_{ii} D_{jj}$. We now want to find a new $(N-1) \times (N-1)$ Flow matrix F' and a new $(N-1) \times (N-1)$ Distance matrix D' such that $L'_{ij} = F'_{ii} D'_{jj}$. ($i, j \leq N-1$.) This will complete the size reduction method. (An $(N-1)^2 \times (N-1)^2$ Cost block matrix C' can always be drawn from the F' and D' matrices. Therefore, we will have all the four matrices in the size $N-1$ problem.)

If $F_{i1} D_{j1} = F_{1i} D_{1j}$ (this is true when the F and D matrices are symmetric) and $F_{ii} D_{jj} = 0$, (this happens when $F_{ii} = 0$ or $D_{jj} = 0$, i.e. the diagonal entries of either the F matrix or the D matrix are zero) for all $i, j \leq N$, we can create the diagonal entries of F' and D' by letting $F'_{i-1,i-1} = 2F_{1i}$, and $D'_{j-1,j-1} = D_{1j}$ ($2 \leq i, j \leq N$) so that $L'_{i-1,j-1} = 2F_{1i} D_{1j} = F'_{i-1,i-1} D'_{j-1,j-1}$. The other entries of F' and D' are determined by the elements left in the cost matrix, since they must be included in the new cost matrix C' . They are essentially the off-diagonal entries of the minor matrices obtained by removing the appropriate row and column (first row and first column in our case) of the original Flow and Distance matrices.

$$F' = \begin{pmatrix} 2F_{12} & F_{23} \\ F_{32} & 2F_{13} \end{pmatrix} \qquad D' = \begin{pmatrix} D_{12} & D_{23} \\ D_{32} & D_{13} \end{pmatrix}$$

New Flow and Distance matrices after assigning facility 1 to location 1

Theorem 1: The problem size of a QAP can be reduced from N to $N-1$ if the original problem satisfies the following two conditions:

1. The Flow (F) and the Distance (D) matrices are symmetric.
2. $F_{ii} D_{jj} = 0$, i.e. the diagonal entries of either the F matrix or the D matrix are zero, for all $i, j \leq N$.

3. Example illustrating the Size Reduction Method

Consider the following size 16 QAP:

$$F = \begin{pmatrix} 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 \\ | 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 \\ | 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 \\ | 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 \\ | 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 \\ | 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 \\ | 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 \\ | 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 \\ | 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 \\ | 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 \\ | 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 \\ | 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 \\ | 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 & 5 & 8 & 3 & 2 \\ | 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 & 8 & 7 & 4 & 6 \\ | 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 & 3 & 4 & 6 & 1 \\ | 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 & 2 & 6 & 1 & 9 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ | 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ | 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 \\ | 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 \\ | 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 \\ | 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 \\ | 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 \\ | 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ | 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ | 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ | 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 \\ | 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 \\ | 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ | 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 \\ | 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 \\ | 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

The Flow and the Distance matrices are symmetric. The diagonal entries of the D matrix are all zero, which automatically means that $F_{ii}D_{jj} = 0$, for all $i, j \leq 16$. Both conditions are satisfied. We now apply the size reduction method and make the assignment of facility 1 to location 1. There is something special about the D matrix in this example. Every row of D is a cyclic

Both problems are solved using an algorithm described in [3], using the Sun Ultra 10 with a single 360MHz Computer Processing Unit. The size 16 problem takes 3,650.48 seconds to solve. The size 15 problem takes only 154.755 seconds, which is about 24 times faster.

4. Conclusion

Note that the size reduction method does not solve QAP, it only reduces the problem size by 1. If we assign facility 1 to location 1, the problem now changes from assigning facilities 1, 2, 3, ..., N among locations 1, 2, 3, ..., N to assigning facilities 2, 3, ..., N among locations 2, 3, ..., N. This method, however, can be generalized such that it can arbitrarily assign any one facility to any location, if the original problem satisfies the two conditions stated in Theorem 1.

One caveat of the size reduction method is that we do not know whether the arbitrary assignment is in the solution of the original QAP. In our case, the best assignment of facility 1 may not be location 1, and thus we may change the problem by assigning facility 1 to location 1. However, in some quadratic assignment problems, such as the example of size 16 above, the first assignment is purely arbitrary. In that case, reducing problem size does help to solve the QAP. An important application is the balancing of fan blades, which is a classical QAP. If the N fan blades are on the periphery of a circle, it does not matter where the first fan blade is assigned. Since the solving procedure runtime increases exponentially with size, it saves a lot of time and resources if we solve the reduced size problem instead of the original problem.

Another occasion for applying the size reduction method is when we want to study a large symmetric QAP more closely before solving it. The first arbitrary assignment is made and the reduced size problem is examined. This is always done in the branch-and-bound solution scheme, one of the currently available algorithms for solving QAP.

References

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