

The Old Hats Problem: Solution by Binomial Inversion.

This is a nice expository paper that presents three ways to calculate a formula for the number of derangements in the symmetric group (especially impressive given that it was written by a high school sophomore). The changes suggested below are strictly stylistic.

Comments:

1. Section I, paragraph 2: missing period after “1) The usual approach using the inclusion-exclusion principle”
2. Section I, paragraph 2: Replace “2) Developing...” with “2) Developing and solving the recurrence relation  $P_n = P_{n-1} - \frac{1}{n}(P_{n-1} - P_{n-2})$ .” ( $P_n$  has been defined above so it doesn't need to be defined again here).
3. Section I, paragraph 3: Replace “This gives a chance...” with “We will also use an idea in this presentation to solve a related interesting question: What is the expected value of the number of men who receive their correct hats? We conclude with a derivation of the binomial inversion formula itself.”
4. Heading of Section II: Should read “Solution by the Inclusion-Exclusion Principle”
5. Suggested rewrite of Section II: Let  $N$  denote the total number of permutations of  $n$  hats. To calculate the number of derangements,  $D_n$ , we want to exclude all permutations possessing any of the attributes  $a_1, a_2, \dots, a_n$  where  $a_i$  is the attribute that man  $i$  gets his correct hat for  $1 \leq i \leq n$ . Let  $N(i)$  denote the number of permutations possessing attribute  $a_i$  (and possibly others),  $N(i, j)$  the number of permutations possessing attributes  $a_i$  and  $a_j$  (and possibly others), etc. Then the inclusion-exclusion principle states that

$$D_n = N - \sum_{1 \leq i \leq n} N(i) + \sum_{1 \leq i < j \leq n} N(i, j) - \sum_{1 \leq i < j < k \leq n} N(i, j, k) + \dots + (-1)^n N(1, 2, \dots, n)$$

Now, by symmetry,  $N(1) = N(2) = \dots = N(i)$ ,  $N(1, 2) = N(1, 3) = \dots = N(i, j)$ , and so on. So we have

$$D_n = N - \binom{n}{1}N(1) + \binom{n}{2}N(1, 2) - \binom{n}{3}N(1, 2, 3) + \dots + (-1)^n \binom{n}{n}N(1, 2, \dots, n)$$

Now,  $N(1)$ , the number of permutations where man 1 gets his correct hat, is simply  $(n - 1)!$  since the remaining  $n - 1$  hats can be distributed in any order. Similarly,  $N(1, 2) = (n - 2)!$ ,  $N(1, 2, 3) = (n - 3)!$ , and so forth. Therefore

$$D_n = N - \binom{n}{1}(n - 1)! + \binom{n}{2}(n - 2)! - \binom{n}{3}(n - 3)! + \dots + (-1)^n \binom{n}{n}(n - n)!$$

Replacing  $N$ , the total number of permutations of  $n$  hats, with  $n!$  and simplifying, we obtain

$$D_n = n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right)$$

To generate  $P_n$ , the probability of a derangement occurring for  $n$  hats, we simply divide the total number of derangements,  $D_n$ , by the total permutations of the  $n$  hats,  $n!$ , to get

$$P_n = \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right)$$

We easily recognize this series as approaching  $1/e$  as  $n$  approaches infinity.

6. Section I, first sentence: Change to “In this section we derive the formula for  $D_n$  using a recurrence relation, as follows.”
7. Section II, second sentence: Combine with first sentence to form a single paragraph.
8. Section II, sixth line: Need a period after “b) man #2 does not get hat #1.
9. Section II, paragraph 5: change  $(n - 1)$  to  $n - 1$
10. Section II, paragraph 5: change the word “Therefore” before  $D_n = (n - 1)(D_{n-1} + D_{n-2})$  to “So” (the preceding sentence also starts with “Therefore”).
11. Section II, following  $D_n = (n - 1)(D_{n-1} + D_{n-2})$ : Change “The probability of a derangement then becomes...” to “The probability of a derangement is again the number of derangements,  $D_n$ , divided by all possible outcomes,  $n!$ . So

$$\begin{aligned}
 P_n = \frac{D_n}{n!} &= \frac{n-1}{n!}(D_{n-1} + D_{n-2}) \\
 &= (n-1) \left[ \frac{1}{n} \frac{D_{n-1}}{(n-1)!} + \frac{1}{n(n-1)} \frac{D_{n-2}}{(n-2)!} \right] \\
 &= \left( 1 - \frac{1}{n} \right) P_{n-1} + \frac{1}{n} P_{n-2} \\
 &= P_{n-1} - \frac{1}{n} (P_{n-1} - P_{n-2})
 \end{aligned}$$

12. Section III, last sentence of page 3: Change “Now, using our formula,...” to “So, using our formula,...” (the previous sentence also begins with “Now”).
13. Section III, top of page 4: Replace the line of dots with  $\dot{}$
14. Section III, page 4: last line of the equation immediately before the sentence beginning “Once again”: add a + sign before  $\frac{(-1)^{n-1}}{n!}$ .
15. Section III, page 4: Change “Once again, we recognize this as approaching  $1/e$ , as demonstrated before” to “Once again, we recognize this as approaching  $1/e$  as  $n$  approaches infinity.”
16. Section III, page 4: Change the bottom half of the page, beginning with “More formally, ...” to “More formally, we can write

$$P_n = P_{n-1} - \frac{1}{n}(P_{n-1} - P_{n-2})$$

as

$$P_n - P_{n-1} = -\frac{1}{n}(P_{n-1} - P_{n-2})$$

Now let  $Q_n = P_n - P_{n-1}$ . Then  $Q_n = -\frac{1}{n}Q_{n-1}$ , so

$$\begin{aligned}
 Q_n &= -\frac{1}{n} \left( -\frac{1}{n-1} \right) Q_{n-2} \\
 &= -\frac{1}{n} \left( -\frac{1}{n-1} \right) \left( -\frac{1}{n-2} \right) Q_{n-3} \\
 &= \dots = (-1)^{n-2} \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \dots \frac{1}{3} Q_2
 \end{aligned}$$

17. Section III, page 5, line 1: need a period after “ $Q_n = (-1)^n \frac{1}{n!}$ ”.
18. Section III, page 5: Replace the line of dots after “ $Q_{n-1} = P_{n-1} - P_{n-2}$ ” with  $\dot{}$ .

19. Section III, page 5: Remove comma after “get” in the line “Therefore, summing both sides, we get,”
20. Section IV, page 5: Change first and second paragraphs to: “The old hats problem can also be solved using binomial inversion. Specifically, given that

$$b_n = \sum_{i=0}^n \binom{n}{i} a_i$$

we can retrieve the  $a_i$  coefficients using the formula

$$a_n = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} b_i$$

This is the binomial inversion formula. For now, we assume it to be true, and use it to solve our problem. We will prove the formula using exponential generating functions in Section VI.

Now, the set of permutations of the hats can be expressed as the disjoint union of  $n+1$  subsets,  $A_0, A_1, A_2, \dots, A_n$ , where  $A_i$  is the set of permutations where exactly  $i$  men receive their correct hats.”

Note:  $S_n$  is typically used to denote the set of all permutations, so I changed  $S_i$ ’s to  $A_i$ ’s.

21. Section IV: page 5, last paragraph: Change  $S_{n-2}$  to  $A_{n-2}$ , replace  $C(n, 2)$  with  $\binom{n}{2}$  since this notation is used throughout the rest of the paper, change “within set  $S_{n-2}$ ” in the last sentence to “in  $A_{n-2}$ .”
22. Section IV: page 6, first line: Change  $S_{n-2}$  to  $A_{n-2}$ .
23. Section IV: page 6, third line: Change “our total permutations as:” to “the total number of permutations as”
24. Section IV: page 6, fifth line: Change “i.e.  $n! = b_n = \sum_{i=0}^n \binom{n}{i} D_i$ .” to “Setting  $b_n = n!$ , we have  $b_n = \sum_{i=0}^n \binom{n}{i} D_i$ .”
25. Section IV: page 6, last sentence: Change to “Dividing by  $n!$ , letting  $j = n - i$ , and reversing the order of summation, we once again obtain  $P_n = \sum_{j=0}^n (-1)^j \frac{1}{j!}$ .”
26. Section V: page 7, last line of first paragraph: change “walk into the restaurant” to “leave the restaurant”
27. Section V: page 7, second to last sentence: change “Now, we let  $j = n - i$ ...” to “Now, if we let  $j = n - i$  and recall that  $\binom{n}{j} = \binom{n}{n-i} = \binom{n}{i}$ , we can write”
28. Section V: page 8, first line: remove “, or  $E(n)$ ” so that there is just one equation.
29. Section V: page 8, third line: remove “, or  $(n-1)! E(n)$ ” so that there is just one equation.
30. Section VI: page 8: Add a first sentence: “We now derive the binomial inversion formula used in Section IV.”
31. Section VI: page 8: Remove “as follows” from the line before Eq 3 (the previous sentence also ends in “as follows”).
32. Section VI: page 9, first line: remove “for”
33. Section VI: page 9, following Eq 9: Change “since the things indexed with an  $i$ ” to “since the terms indexed by  $i$ ”

34. Section VI: page 10, following Eq 13: Change “Now, we can introduce that  $B(x) = \sum_{i=0}^{\infty} \frac{b_i x_i}{i!}$ , and that  $e^{-x} = \sum_{j=0}^{\infty} \frac{(-1)^j x_j}{j!}$  to get” to “Now, since  $B(x) = \sum_{i=0}^{\infty} \frac{b_i x_i}{i!}$  and  $e^{-x} = \sum_{j=0}^{\infty} \frac{(-1)^j x_j}{j!}$ , we have”
35. Section VI: page 10, following Eq 14: change “indexes” to “indices”
36. Section VI: page 10, following Eq 15: Change ”Now, we can introduce  $n = i + j$ , and once again re-index to change the order in which we sum over the  $i$ - $j$  plane:” to “Letting  $n = i + j$  and once again re-indexing to change the order in which we sum over the  $i$ - $j$  plane, we have” (the sentences before and after this one both begin with “Now”)
37. Section VI: General comment: all of the equation numbers are not needed since only a few of them are referenced.