The Indeterminate Limit Rule

Jason Schanker (jas42@duke.edu)

15 Lesley Drive

Syosset, NY 11791

Duke University Class of 2005

About the Author

In his senior year in high school, Jason Schanker became very interested in mathematical theory and began to examine a number of concepts in calculus. Although calculus provided him with a broad new repot ware of problems to solve and tools to solve them with, Jason spent a lot of his time on division by zero and limits involving infinity. After much time and dedication, he put forward a new theorem to encompass his ideas regarding limits containing infinity. After completing the report in his senior year of high school, Jason received assistance from Dr. Kraines at Duke University in the beginning of his freshman year of college in making revisions to his report and in submitting his work.

Abstract

Not all limits in certain indeterminate forms can be solved easily by L'Hôpital's Rule. Specifically, the rule would be difficult to implement on limits of fractions where the numerator and denominator both approach infinity and both contain exponential functions. In these cases, differentiating the numerator and denominator to simplify the limit would generally be futile since the derivatives of these functions contain the original exponential functions themselves. To remedy this problem, I have derived and proved a limit rule, which eliminates these troublesome functions and simplifies the limits into something that can be solved using L'Hôpital's Rule. In my report, I state this Indeterminate Limit Rule and prove it. I then proceed to mention and prove corollaries that make this rule more versatile. I conclude with some examples and with the intuitive reasoning behind the theorem.

Introduction

While L’Hôpital’s Rule seems to work well for finding virtually all limits in the form 0/0, it is difficult if not impossible to find the solutions to some of the limits involving infinity over infinity using the theorem. For example, if L’Hôpital’s Rule was applied to the limit, $\lim_{x\to\infty}\frac{2.8^x}{10x^{50}e^x\ln(10x)}$,

$\lim_{x\to\infty}\frac{2.8^x\ln 2.8}{10x^{50}e^x\ln(10x)+10e^x(50x^{49}\ln(10x)+10x^{49})}$ would be received. This expression is far more complicated than the original one and the more L’Hôpital’s Rule is used on the limit, the more complicated it becomes. The situations where L’Hôpital’s Rule fails or is difficult to implement for solving limits involving infinity over infinity occur when there are exponential functions contained in the numerator and denominator. This is because their derivatives always contain the original exponential functions within them. Thus, they can never be gotten rid of no matter how many times their derivatives are taken.

To remedy this problem, I have derived and proved my own limit rule that I denoted the Indeterminate Limit Rule. The Indeterminate Limit Rule is used for solving limits where the numerator and denominator both approach infinity as their inputs approach infinity. It states that if the ratio of the natural logarithms of the numerator and denominator is less than 1, then the limit is zero and if it’s greater than 1, the limit is infinity. If the test results in 1, then the test is inconclusive. However, the Limit Rule should not be necessary in these cases.

I. **The Indeterminate Limit Rule**

Theorem:

If $\lim_{x\to a} f(x)=\infty \wedge \lim_{x\to a} g(x)=\infty$ (a can be a finite number or it can be infinity), then:

1) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}<1 \to \lim_{x\to a}\frac{f(x)}{g(x)}=0$
2) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}>1 \to \lim_{x\to a}\frac{f(x)}{g(x)}=\infty$

Proof:

1) **L < 1**: To prove the first part of this theorem, we first take a to be infinity. In this case, since $\lim_{x\to\infty}\frac{\ln(f(x))}{\ln(g(x))}=L$ where L is less than 1 and because the numerator and denominator both approach positive infinity, $\exists N>0:\forall x\ge N, 0\le\frac{\ln f(x)}{\ln g(x)}\le c$ where c is a constant located in the open interval (L,1). For some N, $\frac{\ln f(x)}{\ln g(x)}$ will have to be greater than or equal to 0 because the numerator and denominator both approach positive infinity, forcing the ratio between the two functions to approach a non – negative value. A 'c' that satisfies these conditions can be found because $\frac{\ln f(x)}{\ln g(x)}$ can be made to get as close to L as one chooses as long as a large enough x is selected. (This is the definition of a limit whose input approaches infinity.) Since L is less than 1, a 'c' whose value is less than 1 and greater than L can be obtained. (The 'c' is like $L+\varepsilon$ where ε is a small positive value. By the limit, there must exist a positive N for which $L-\varepsilon<\frac{\ln f(x)}{\ln g(x)}<L+\varepsilon$ is true for all values greater than N.)
Some algebra can now be applied to the inequality:

$$0\le \ln f(x)\le c\ln g(x)$$
$$-\ln g(x)\le \ln f(x)-\ln g(x)\le c\ln g(x)-\ln g(x)$$
$$\ln(\frac{1}{g(x)})\le \ln f(x)-\ln g(x)\le (c-1)\ln g(x)$$
$$\ln(\frac{1}{g(x)})\le \ln[\frac{f(x)}{g(x)}]\le \ln[g(x)^{c-1}]$$
$$\frac{1}{g(x)}\le\frac{f(x)}{g(x)}\le[g(x)]^{c-1}$$

Now, by taking the limit as x approaches infinity of both sides:

$$\lim_{x\to\infty}\frac{1}{g(x)} \le \lim_{x\to\infty}\frac{f(x)}{g(x)} \le \lim_{x\to\infty}[g(x)]^{c-1}$$

Since c is less than 1, c minus 1 will be less than zero. Since $\lim_{x\to\infty} g(x)=\infty$, and c – 1 is less than zero, $\lim_{x\to\infty}[g(x)]^{c-1} = 0$ and $\lim_{x\to\infty}\frac{1}{g(x)} = 0$ (The reciprocal of a limit whose result is infinity is zero). Because the inequality simplifies to $0 \le \lim_{x\to\infty}\frac{f(x)}{g(x)} \le 0$, the Squeeze Law can be used to finish up the first part of the proof and get the desired result: $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$. Therefore, if $\lim_{x\to\infty} f(x)=\infty$, $\lim_{x\to\infty} g(x)=\infty$, and $\lim_{x\to\infty}\frac{\ln(f(x))}{\ln(g(x))} = L$ where L is less than 1, $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ must equal zero. We can prove the first part of this theorem true for all finite a by noting that $\exists\partial > 0 : \forall x$ where $a-\partial < x < a+\partial, 0 \le \frac{\ln f(x)}{\ln g(x)} \le c$ and by following the same procedure that we did with proving it true for infinity. The only difference would be the value that x approaches in the limit, mainly a finite a value instead of infinity.

2) **L > 1**: If $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))} = L$, where L is greater than 1 (L can be infinity), than $\lim_{x\to a}\frac{\ln(g(x))}{\ln(f(x))} = \frac{1}{L}$. Since L is greater than 1, the reciprocal of L must be less than 1. Since $\lim_{x\to a}\frac{\ln(g(x))}{\ln(f(x))} = \frac{1}{L}$ where the reciprocal of L is less than 1, $\lim_{x\to a}\frac{g(x)}{f(x)} = 0$ by the first part of my proof to the Indeterminate Limit Rule. Well, if $\lim_{x\to a}\frac{g(x)}{f(x)} = 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \infty$. Therefore, if $\lim_{x\to a} f(x)=\infty$, $\lim_{x\to a} g(x)=\infty$, and $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))} = L$ where L is greater than 1, $\lim_{x\to a}\frac{f(x)}{g(x)}$ must equal infinity.

$\therefore$ The two parts of the Indeterminate Limit Rule hold true.

III. **The Extended Indeterminate Limit Rule**

Sometimes the Indeterminate Limit Rule needs to be used recursively. In these cases, the Extended Indeterminate Limit Rule becomes useful. If $\lim_{x\to a} f(x)=\infty$, $\lim_{x\to a} g(x)=\infty$, where a can be a finite number or infinity, and $\ln^{n}(h(x))$ is defined to be the n[th] natural logarithm of some function h(x), where n is defined to be a non – negative integer or whole number ($\ln^n(h(x)) = \ln(\ln^{n-1}(h(x)))$, then:

1) $\lim_{x\to a}\frac{\ln^n(f(x))}{\ln^n(g(x))} < 1 \to \lim_{x\to a}\frac{f(x)}{g(x)} = 0$

2) $\lim_{x\to a}\frac{\ln^n(f(x))}{\ln^n(g(x))} > 1 \to \lim_{x\to a}\frac{f(x)}{g(x)} = \infty$

The proof of this rule is as follows:

By the first Indeterminate Limit Rule, if $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))} < 1$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = 0$ so if $\lim_{x\to a}\frac{\ln^n(f(x))}{\ln^n(g(x))} < 1$ where n is a positive integer, then by letting $h(x) = \ln^{n-1}(f(x))$ and $j(x) = \ln^{n-1}(g(x))$, $\lim_{x\to a}\frac{\ln(h(x))}{\ln(j(x))} < 1$ which means that $\lim_{x\to a}\frac{h(x)}{j(x)} = 0$ by the Indeterminate Limit Rule. By substituting back for h(x) and j(x), $\lim_{x\to\infty}\frac{\ln^{n-1}(f(x))}{\ln^{n-1}(g(x))} = 0$ is received. Now if we let $h(x) = \ln^{n-2}(f(x))$ and $j(x) = \ln^{n-2}(g(x))$, the limit simplifies to $\lim_{x\to a}\frac{\ln(h(x))}{\ln(j(x))} = 0$. Since zero is clearly less than 1, we can apply the Indeterminate Limit Rule again and substitute for h(x) and j(x) to get $\lim_{x\to a}\frac{\ln^{n-2}(f(x))}{\ln^{n-2}(g(x))} = 0$. By continuing this process, we eventually can get $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))} = 0$ which by the Indeterminate Limit Rule implies that $\lim_{x\to a}\frac{f(x)}{g(x)} = 0$. Similarly if $\lim_{x\to a}\frac{\ln^n(f(x))}{\ln^n(g(x))} > 1$, then it is know that $\lim_{x\to a}\frac{\ln^{n-1}(f(x))}{\ln^{n-1}(g(x))} = \infty$ by the Indeterminate Limit Rule. Since infinity is greater than 1, we can say that $\lim_{x\to a}\frac{\ln^{n-2}(f(x))}{\ln^{n-2}(g(x))} = \infty$ by the Indeterminate Limit Rule. This process can be continued like the one mentioned above n times until $\lim_{x\to a}\frac{\ln^{n-n}(f(x))}{\ln^{n-n}(g(x))}$ which equals $\lim_{x\to a}\frac{\ln^0(f(x))}{\ln^0(g(x))}$, which is in turn equal to $\lim_{x\to a}\frac{f(x)}{g(x)}$ is shown to be infinity.

IV. **The Indeterminate Oblique Asymptote Limit Rules (Sum & Difference Rules)**

1) If $(\lim_{x\to a} f(x)+g(x)=\infty \wedge \lim_{x\to a}\frac{g(x)}{f(x)}<1 \wedge \lim_{x\to a} g(x)\ge 0) \wedge \lim_{x\to a} h(x)=\infty$, then:

a) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(h(x))}<1 \to \lim_{x\to a}\frac{f(x)+g(x)}{h(x)}=0$

b) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(h(x))}>1 \to \lim_{x\to a}\frac{f(x)+g(x)}{h(x)}=\infty$

2) If $(\lim_{x\to a} g(x)+h(x)=\infty \wedge \lim_{x\to a}\frac{h(x)}{g(x)}<1 \wedge \lim_{x\to a} h(x)\ge 0) \wedge \lim_{x\to a} f(x)=\infty$, then:

a) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}<1 \to \lim_{x\to a}\frac{f(x)}{g(x)+h(x)}=0$

b) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}>1 \to \lim_{x\to a}\frac{f(x)}{g(x)+h(x)}=\infty$

3) If $(\lim_{x\to a} f(x)-g(x)=\infty \wedge \lim_{x\to a}\frac{g(x)}{f(x)}<1 \wedge \lim_{x\to a} g(x)\ge 0) \wedge \lim_{x\to a} h(x)=\infty$, then:

a) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(h(x))}<1 \to \lim_{x\to a}\frac{f(x)-g(x)}{h(x)}=0$

b) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(h(x))}>1 \to \lim_{x\to a}\frac{f(x)-g(x)}{h(x)}=\infty$

4) If $(\lim_{x\to a} g(x)-h(x)=\infty \wedge \lim_{x\to a}\frac{h(x)}{g(x)}<1 \wedge \lim_{x\to a} g(x)\ge 0) \wedge \lim_{x\to a} h(x)=\infty$, then:

a) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}<1 \to \lim_{x\to a}\frac{f(x)}{g(x)-h(x)}=0$

b) $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}>1 \to \lim_{x\to a}\frac{f(x)}{g(x)-h(x)}=\infty$

Proof:

1) a) This part of the rule doesn't even require the use of formal limit definitions. If the prerequisites of condition 1 and part a are satisfied, then it is known that $\lim_{x\to a}\frac{f(x)}{h(x)}\ge \lim_{x\to a}\frac{g(x)}{h(x)}$ because the ratio between the limit as x approaches infinity of $g(x)$ to the limit as x approaches infinity of $f(x)$ is less than 1. Therefore,

$\lim_{x\to a}\frac{f(x)+f(x)}{h(x)} \geq \lim_{x\to a}\frac{f(x)+g(x)}{h(x)}$. Since $\lim_{x\to a}\frac{f(x)}{h(x)}=0$ by the Indeterminate Limit Rule, $\lim_{x\to a}\frac{f(x)+f(x)}{h(x)}$ can be expressed as $\lim_{x\to a}\frac{f(x)}{h(x)}+\lim_{x\to a}\frac{f(x)}{h(x)}$ or 0 + 0, which equals zero. Therefore, $0 \geq \lim_{x\to a}\frac{f(x)+g(x)}{h(x)}$. However since $\lim_{x\to a}\frac{f(x)+g(x)}{h(x)}$ can't be negative because $\lim_{x\to a} f(x)+g(x)$ and $\lim_{x\to a} h(x)$ are both positive infinity, $\lim_{x\to a}\frac{f(x)+g(x)}{h(x)}=0$.

b) Since $\lim_{x\to a}\frac{\ln(f(x))}{\ln(h(x))}>1$, $\lim_{x\to a}\frac{f(x)}{h(x)}=\infty$ by the Indeterminate Limit Rule. Since $\lim_{x\to a} g(x)\geq 0$, adding $\frac{g(x)}{h(x)}$ can only make the limit greater. Therefore, $\lim_{x\to a}\frac{f(x)+g(x)}{h(x)}=\infty$.

2) a) If $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}<1$, then the reciprocal of the limit, $\lim_{x\to a}\frac{\ln(g(x))}{\ln(f(x))}$ must be greater than 1. Thus by number 1 part b of this theorem, which was just proved, $\lim_{x\to a}\frac{g(x)+h(x)}{f(x)}=\infty$. Since $\lim_{x\to a}\frac{g(x)+h(x)}{f(x)}=\infty$, $\lim_{x\to a}\frac{f(x)}{g(x)+h(x)}=0$.

b) If $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}>1$, then the reciprocal of the limit, $\lim_{x\to a}\frac{\ln(g(x))}{\ln(f(x))}$ must be less than 1. Thus by number 1 part a of this theorem, which was just proved, $\lim_{x\to a}\frac{g(x)+h(x)}{f(x)}=0$. Since $\lim_{x\to a}\frac{g(x)+h(x)}{f(x)}=0$, $\lim_{x\to a}\frac{f(x)}{g(x)+h(x)}=\infty$.

3) a) Since $\lim_{x\to a}\frac{\ln(f(x))}{\ln(h(x))}<1$, $\lim_{x\to a}\frac{f(x)}{h(x)}=0$ by the Indeterminate Limit Rule. Since $\lim_{x\to a} g(x)\geq 0$, subtracting $\frac{g(x)}{h(x)}$ can't increase the limit. Therefore, $\lim_{x\to a}\frac{f(x)-g(x)}{h(x)}\leq 0$. However, because both the numerator and denominator approach positive infinity as x approaches infinity, $\lim_{x\to a}\frac{f(x)-g(x)}{h(x)}$ cannot be negative. Thus $\lim_{x\to a}\frac{f(x)-g(x)}{h(x)}=0$.

b) To prove the first part of this theorem, we first take a to be infinity. In this case, since $\lim_{x\to\infty}\frac{g(x)}{f(x)}=L$, where L is less than 1 and because the numerator and denominator both approach a non – negative number or positive infinity, $\exists N>0:\forall x\geq N, 0\leq\frac{g(x)}{f(x)}\leq c$ where c is a constant located in the open interval (L,1). For some N, $\frac{g(x)}{f(x)}$ will have to be greater than or equal to 0 because the numerator and denominator both approach positive infinity or a non – negative number, forcing the ratio between the two functions to approach a non – negative value. A 'c' that satisfies these conditions can be found because $\frac{g(x)}{f(x)}$ can be made to get as close to L as one chooses as long as a large enough x is selected. (This is the definition of a limit whose input approaches infinity.) Since L is less than 1, a 'c' whose value is less than 1 and greater than L can be obtained. (The 'c' is like $L+\varepsilon$ where ε is a small positive value. By the limit, there must exist a positive N for which $L-\varepsilon<\frac{\ln f(x)}{\ln g(x)}<L+\varepsilon$ is true for all values greater than N.) Some algebra can now be applied to the inequality:

$$
\begin{aligned}
&0\leq g(x)\leq cf(x)\\
&-f(x)\leq g(x)-f(x)\leq cf(x)-f(x)\\
&f(x)\geq f(x)-g(x)\geq f(x)-cf(x)\\
&f(x)\geq f(x)-g(x)\geq(1-c)f(x)\\
&\frac{f(x)}{h(x)}\geq\frac{f(x)-g(x)}{h(x)}\geq\frac{(1-c)f(x)}{h(x)}
\end{aligned}
$$

Now, by taking the limit as x approaches infinity of both sides:

$$\lim_{x\to\infty}\frac{f(x)}{h(x)}\geq\lim_{x\to\infty}\frac{f(x)-g(x)}{h(x)}\geq\lim_{x\to\infty}\frac{(1-c)f(x)}{h(x)}$$

Since c is less than 1, 1 minus c will be greater than zero. Because $\lim_{x\to\infty}\frac{\ln(f(x))}{\ln(h(x))}>1$, $\lim_{x\to\infty}\frac{f(x)}{h(x)}=\infty$ by the Indeterminate Limit Rule. Because multiplying by a positive constant, 1 – c, will not change the result (If we assume p to be a positive constant, the Indeterminate Limit Rule will produce the same results for $\lim_{x\to\infty}\frac{f(x)}{h(x)}$ and

$\lim_{x\to\infty}\frac{pf(x)}{h(x)}$ because $\lim_{x\to\infty}\frac{\ln(f(x))}{\ln(h(x))}=\lim_{x\to\infty}\frac{(\frac{f'(x)}{f(x)})}{(\frac{h(x)}{h'(x)})}$ and

$\lim_{x\to\infty}\frac{\ln(pf(x))}{\ln(h(x))}=\lim_{x\to\infty}\frac{\ln(f(x))+\ln(p)}{\ln(h(x))}=\lim_{x\to\infty}\frac{(\frac{f'(x)}{f(x)})}{(\frac{h(x)}{h'(x)})}$.), $\lim_{x\to\infty}\frac{(1-c)f(x)}{h(x)}$ will also be infinity. Therefore, $\infty\geq\lim_{x\to\infty}\frac{f(x)-g(x)}{h(x)}\geq\infty$ by the above inequality. By the Squeeze Law, $\lim_{x\to\infty}\frac{f(x)-g(x)}{h(x)}=\infty$ is received. The part of this theorem for all finite a can be proved similarly by noting that $\exists\partial>0:\forall x$ where $a-\partial<x<a+\partial, 0\leq\frac{\ln g(x)}{\ln f(x)}\leq c$ and by following the same procedure that we did with proving it true for infinity. The only difference would be the value that x approaches in the limit, specifically a finite a value instead of infinity.

4) a) If $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}<1$, then the reciprocal of the limit, $\lim_{x\to a}\frac{\ln(g(x))}{\ln(f(x))}$ must be greater than 1. Thus by number 3 part b of this theorem, which was just proved, $\lim_{x\to a}\frac{g(x)-h(x)}{f(x)}=\infty$. Since $\lim_{x\to a}\frac{g(x)-h(x)}{f(x)}=\infty$, $\lim_{x\to a}\frac{f(x)}{g(x)-h(x)}=0$.

b) If $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}>1$, then the reciprocal of the limit, $\lim_{x\to a}\frac{\ln(g(x))}{\ln(f(x))}$ must be less than 1. Thus by number 3 part a of this theorem, which was just proved, $\lim_{x\to a}\frac{g(x)-h(x)}{f(x)}=0$. Since $\lim_{x\to a}\frac{g(x)-h(x)}{f(x)}=0$, $\lim_{x\to a}\frac{f(x)}{g(x)-h(x)}=\infty$.

V. **Some Examples Of Limits Where The Indeterminate Limit Rule Can Be Applied**

The following examples are limits where the Indeterminate Limit Rule can be applied. Solutions to the limit problems follow (There are many variations in the order that my theorems, L'Hôpital's Rule, and the logarithm rules can be used. In the first problem, for example, one of the logarithm rules is used to simplify the fraction resulting from the Indeterminate Limit Rule. However, in the second problem, L'Hôpital's Rule is used to solve the limit resulting from the Indeterminate Limit Rule):

1) $\lim_{x\to\infty}\frac{2^x}{3^x}$

$$\lim_{x\to\infty}\frac{\ln(2^x)}{\ln(3^x)}=\lim_{x\to\infty}\frac{x\ln(2)}{x\ln(3)}=\lim_{x\to\infty}\frac{\ln(2)}{\ln(3)}=\frac{\ln(2)}{\ln(3)}<1 \text{ so } \lim_{x\to\infty}\frac{2^x}{3^x}=0$$

2) $\lim_{x\to\infty}\frac{5x^{10}2^x}{3^x}$

$$\lim_{x\to\infty}\frac{\ln(5x^{10}2^x)}{\ln(3^x)}=\lim_{x\to\infty}\frac{(\frac{(\ln(2)\bullet 5x^{10}2^x+50x^9 2^x)}{5x^{10}2^x})}{(\frac{(3^x\ln(3))}{3^x})}=\lim_{x\to\infty}\frac{(\ln(2)\bullet 5x^{10}2^x+50x^9 2^x)(3^x)}{(3^x\ln(3))(5x^{10}2^x)}$$

$$=\lim_{x\to\infty}\frac{(5\ln(2)x+50)}{\ln(3)(5x)}=\lim_{x\to\infty}\frac{x\ln(2)+10}{x\ln(3)}=\lim_{x\to\infty}\frac{\ln(2)}{\ln(3)}=\frac{\ln(2)}{\ln(3)}<1 \text{ so } \lim_{x\to\infty}\frac{5x^{10}2^x}{3^x}=0$$

3) $\lim_{x\to\infty}\frac{2.8^x}{e^x}$

$$\lim_{x\to\infty}\frac{\ln(2.8^x)}{\ln(e^x)}=\lim_{x\to\infty}\frac{x\ln(2.8)}{x\ln(e)}=\lim_{x\to\infty}\frac{\ln(2.8)}{1}=\ln(2.8)\approx 1.02962>1 \text{ so } \lim_{x\to\infty}\frac{2.8^x}{e^x}=\infty$$

4) $\lim_{x\to\infty}\frac{2.8^x}{10x^{50}e^x\ln(10x)}$

$$\lim_{x\to\infty}\frac{\ln(2.8^x)}{\ln(10x^{50}e^x\ln(10x))}=\lim_{x\to\infty}\frac{\ln(2.8^x)}{\ln(10)+\ln(x^{50})+\ln(e^x)+\ln(\ln(10x))}=$$

$$\lim_{x\to\infty}\frac{x\ln(2.8)}{\ln(10)+50\ln(x)+x\ln(e)+\ln(\ln(10x))}=\lim_{x\to\infty}\frac{\ln(2.8)}{\frac{50}{x}+1+\frac{1}{x\ln(10x)}}=$$

$$\lim_{x\to\infty}\frac{x\ln(2.8)}{50+x+\frac{1}{\ln(10x)}}=\lim_{x\to\infty}\frac{\ln(2.8)}{1-\frac{1}{x(\ln(10x))^2}}=\lim_{x\to\infty}\frac{x(\ln(10x))^2\ln(2.8)}{x(\ln(10x))^2-1}=$$

$$\lim_{x\to\infty}\frac{(x(\ln(10x))^2\ln(2.8))'}{(x(\ln(10x))^2)'}=\lim_{x\to\infty}\frac{\ln(2.8)(x(\ln(10x))^2)'}{(x(\ln(10x))^2)'}=\lim_{x\to\infty}\ln(2.8)\approx 1.02962>1$$

so $\lim_{x\to\infty}\frac{2.8^x}{10x^{50}e^x\ln(10x)}=\infty$

5) $\lim_{x\to\infty}\frac{3^{e^x}x^x(\ln x)^x}{e^{3^x}}\to\lim_{x\to\infty}\frac{\ln(3^{e^x}x^x(\ln x)^x)}{\ln(e^{3^x})}=\lim_{x\to\infty}\frac{\ln(3^{e^x})+\ln(x^x)+\ln((\ln x)^x)}{\ln(e^{3^x})}=$

$$\lim_{x\to\infty}\frac{e^x\ln(3)+x\ln(x)+x\ln((\ln x))}{3^x\ln(e)}=\lim_{x\to\infty}\frac{e^x\ln(3)}{3^x}+\lim_{x\to\infty}\frac{x\ln(x)}{3^x}+\lim_{x\to\infty}\frac{x\ln(\ln(x))}{3^x}=$$

$$\lim_{x\to\infty}\frac{e^x\ln(3)}{3^x}+\lim_{x\to\infty}\frac{\ln(x)+1}{3^x\ln(3)}+\lim_{x\to\infty}\frac{\ln(\ln(x))+1/\ln(x)}{3^x\ln(3)}=\lim_{x\to\infty}\frac{e^x\ln(3)}{3^x}+\lim_{x\to\infty}\frac{1}{x3^x(\ln(3))^2}+$$

$$\lim_{x\to\infty}\frac{\ln(x)\ln(\ln(x))+1}{3^x\ln(3)\ln(x)}=\lim_{x\to\infty}\frac{e^x\ln(3)}{3^x}+0+0=\lim_{x\to\infty}\frac{e^x\ln(3)}{3^x}\to\lim_{x\to\infty}\frac{\ln(e^x\ln(3))}{\ln(3^x)}=$$

$$\lim_{x\to\infty}\frac{x\ln(e)+\ln(\ln(3))}{x\ln(3)}=\lim_{x\to\infty}\frac{\ln(e)}{\ln(3)}=\frac{1}{\ln(3)}=.91024<1$$

$$\therefore \lim_{x\to\infty}\frac{e^x\ln(3)}{3^x}=0<1$$

$$\therefore \lim_{x\to\infty}\frac{3^{e^x}x^x(\ln x)^x}{e^{3^x}}=0$$

6) $\lim_{x\to\infty}\frac{e^x}{(\ln(x))^{\ln(x)}}$

$$\lim_{x\to\infty}\frac{\ln(e^x)}{\ln((\ln x)^{\ln(x)})}=\lim_{x\to\infty}\frac{x}{\ln(x)\ln((\ln x))}=\lim_{x\to\infty}\frac{1}{\frac{\ln((\ln x))}{x}+\frac{1}{x}}=\lim_{x\to\infty}\frac{x}{\ln(\ln(x))+1}=$$

$$\lim_{x\to\infty}\frac{1}{\frac{1}{x\ln(x)}}=\lim_{x\to\infty}x\ln x=\infty>1\ so \lim_{x\to\infty}\frac{e^x}{(\ln(x))^{\ln(x)}}=\infty$$

7) $\lim_{x\to\infty}\frac{x^x-e^x}{e^x+x^{\ln(x)}}$

$$\lim_{x\to\infty}\frac{e^x}{x^x}\to\lim_{x\to\infty}\frac{\ln(e^x)}{\ln(x^x)}=\lim_{x\to\infty}\frac{x\ln(e)}{x\ln(x)}=\lim_{x\to\infty}\frac{1}{\ln(x)}=0<1\text{ so }\lim_{x\to\infty}\frac{e^x}{x^x}=0<1$$

$$\lim_{x\to\infty}\frac{x^{\ln(x)}}{e^x}\to\lim_{x\to\infty}\frac{\ln(x^{\ln(x)})}{\ln(e^x)}=\lim_{x\to\infty}\frac{\ln(x)\ln(x)}{x\ln(e)}=\lim_{x\to\infty}\frac{(\ln(x))^2}{x}=\lim_{x\to\infty}\frac{(\frac{2(\ln(x))}{x})}{1}$$

$$=\lim_{x\to\infty}\frac{2\ln(x)}{x}=\lim_{x\to\infty}\frac{(\frac{2}{x})}{1}=0<1$$

$$\lim_{x\to\infty}\frac{\ln(x^x)}{\ln(e^x)}=\lim_{x\to\infty}\frac{x\ln(x)}{x\ln(e)}=\lim_{x\to\infty}\frac{\ln(x)}{1}=\infty>1\text{ so }\lim_{x\to\infty}\frac{x^x-e^x}{e^x+x^{\ln(x)}}=\infty$$

8) $\lim_{x\to 4}\frac{2^{\frac{1}{x-4}}}{e^{\frac{\cos(x-3)}{x-4}}}$

$$\lim_{x\to 4}\frac{2^{\frac{1}{x-4}}}{e^{\frac{\cos(x-3)}{x-4}}}\to\lim_{x\to 4}\frac{\ln(2^{\frac{1}{x-4}})}{\ln(e^{\frac{\cos(x-3)}{x-4}})}=\lim_{x\to 4}\frac{\frac{1}{x-4}\ln(2)}{\frac{\cos(x-3)}{x-4}}=\lim_{x\to 4}\frac{\ln(2)}{\cos(x-3)}=\frac{\ln(2)}{\cos(1)}\approx 1.282887>1$$

$$\therefore \lim_{x\to 4}\frac{2^{\frac{1}{x-4}}}{e^{\frac{\cos(x-3)}{x-4}}}=\infty$$

VI. **Dominating Terms: An Intuitive Approach To Solving Limits Involving Infinity Over Infinity And Why The Indeterminate Limit Rule Works**

In the past few sections of my report, I have stated and proved my theorems and shown how to use them. In this section, I will be taking a step back from the mechanical part of my theorem and will be attempting to explain the concepts and reasoning behind it.

When mathematicians and physicists attempt to solve limits where the numerator and denominator both approach infinity with the numerator and/or denominator containing sums or differences of functions, they may be inclined to use L'Hôpital's Rule or to find the oblique asymptotes of the function. For example, when solving a limit such as $\lim_{x\to\infty}\frac{4x^{10}+3x^{9}-7x^{2}+2x-1}{2x^{10}-5x^{3}-10x+7}$, one can use L'Hôpital's Rule 10 times and come up with $\lim_{x\to\infty}\frac{4\bullet 10!}{2\bullet 10!}=2$. One can also reason that as x becomes very large, the numerator and denominator behave like, $4x^{10}$ and $2x^{10}$, respectively. The individual could then just replace the limit, $\lim_{x\to\infty}\frac{4x^{10}+3x^{9}-7x^{2}+2x-1}{2x^{10}-5x^{3}-10x+7}$ with $\lim_{x\to\infty}\frac{4x^{10}}{2x^{10}}$ and easily cancel the variable terms. The second method can only be used when there is one term in the numerator (and/or denominator), $f(x)$, for which $f(x)$ will divide infinitely into each and every one of the other terms in the numerator as x approaches infinity. This is true because if $f(x)$ divided by any other terms is infinity, then every other term divided by $f(x)$ is zero. To illustrate this point, consider the previous example, $\lim_{x\to\infty}\frac{4x^{10}+3x^{9}-7x^{2}+2x-1}{2x^{10}-5x^{3}-10x+7}$. The term that was denoted $f(x)$ would be $4x^{10}$ in the numerator and $2x^{10}$ in the denominator because $4x^{10}$ divided by any one of the other terms in the numerator is infinity as x approaches infinity, and $2x^{10}$ divided by any one of the other terms in the denominator is also infinity as x approaches infinity. If the limit was to be expressed with the leading terms in the numerator and denominator factored out, the limit could be rewritten as $\lim_{x\to\infty}\frac{4x^{10}(1+\frac{3x^9}{4x^{10}}-\frac{7x^2}{4x^{10}}+\frac{2x}{4x^{10}}-\frac{1}{4x^{10}})}{2x^{10}(1-\frac{5x^3}{2x^{10}}-\frac{10x}{2x^{10}}+\frac{7}{2x^{10}})}$ or

$\lim_{x\to\infty}\frac{4x^{10}(1+\frac{3}{4x}-\frac{7}{4x^8}+\frac{1}{2x^9}-\frac{1}{4x^{10}})}{2x^{10}(1-\frac{5}{2x^7}-\frac{5}{x^9}+\frac{7}{2x^{10}})}$ or $\lim_{x\to\infty}\frac{4x^{10}}{2x^{10}}\bullet\lim_{x\to\infty}\frac{1+\frac{3}{4x}-\frac{7}{4x^8}+\frac{1}{2x^9}-\frac{1}{4x^{10}}}{1-\frac{5}{2x^7}-\frac{5}{x^9}+\frac{7}{2x^{10}}}$ or

because each of the variable terms in the limit approach 0 as x approaches infinity, $\lim_{x\to\infty}\frac{4x^{10}}{2x^{10}}\bullet\lim_{x\to\infty}\frac{1}{1}$ which is equivalent to $\lim_{x\to\infty}\frac{4x^{10}}{2x^{10}}$. If I were to describe this method in somewhat layman's terms, I would say that it entails finding the dominating terms in the numerator and denominator and then just solving the limit as if these largest terms were the only ones in it. In the example above, I'd identify the dominating terms in the numerator and denominator to be $4x^{10}$ and $2x^{10}$, respectively. This method can actually be used whenever the numerator and denominator are both simplified. For example, $\lim_{x\to\infty}\frac{5x+2x-1}{3x+2}$ is not simplified, but $\lim_{x\to\infty}\frac{7x-1}{3x+2}$ is. $\lim_{x\to\infty}\frac{\ln(5x)+\ln(2x)+7}{\ln(8x^2)-\ln(4x)}$ is not simplified, but $\lim_{x\to\infty}\frac{\ln(10x^2)+7}{\ln(2x)}$ is. The entire fraction does not have to be simplified, just the numerator and denominator individually. In other words, the numerator and denominator should each contain the least number of terms possible.

The reason I bring up this dominating term concept is because when I derived my limit rule, this is precisely the concept that I wanted to encompass in it. Only, instead of worrying solely about the dominating terms in sums and differences, I wanted my rule to also concern itself with the dominating terms when dealing with products. This is because a limit like $\lim_{x\to\infty}\frac{2.8^x}{e^x}$ would be considerably easier to solve than a limit like $\lim_{x\to\infty}\frac{2.8^x}{10x^{50}e^x\ln(10x)}$. A limit with a simple ratio between two exponential functions could be solved by factoring out the exponents, but limits with products of other functions in addition to the exponential functions potentially make the limit difficult to solve.

When I examined limits in the form of infinity divided by infinity containing products of functions, I found that only the “dominating terms in the numerator and denominator played a role in determining the limit. I found that if the fraction was simplified and if the numerator and denominator both approached infinity, then it was possible to “dispose” of part of the numerator and/or denominator and obtain the same results. If there is a function, $f(x)$, in the numerator and another function, $g(x)$, whose limit as x approaches some finite or infinite value, a, is greater than 0, in the numerator as well, such that $\lim_{x\to a}\frac{f(x)}{[g(x)]^n}=\infty$ for all positive integers, n, then $g(x)$ could be eliminated from the limit without changing the results. The same would be true if these functions were located in the denominator. This of course was contingent on the fact that the numerator and denominator were both expressed as a monomial by only including the dominating terms in each respective part of the fraction and that the fraction itself is simplified. (The former part of this sentence can be achieved by using the method in the Indeterminate Oblique Asymptote Rules.) It’s also assuming that the numerator and denominator both approached infinity. For example, $\lim_{x\to\infty}\frac{2.8^x}{10x^{50}e^x\ln(10x)}$ could be rewritten as $\lim_{x\to\infty}\frac{2.8^x}{e^x}$ without changing the limit because the fraction is simplified, the numerator and denominator both approach infinity, $\lim_{x\to\infty}\frac{e^x}{10^n}=\infty$ (The denominator is finite for all positive integers, n), $\lim_{x\to\infty}\frac{e^x}{x^{50n}}=\infty$ (Use L’Hôpital’s Rule 50n times to simplify the limit to $\lim_{x\to\infty}\frac{e^x}{(50n)!}$), and $\lim_{x\to\infty}\frac{e^x}{(\ln(10x))^n}=\infty$ for all positive integers, n. (Please note that the $g(x)$ here was first 10, followed by x^{50}, and concluding with ln(10x).) The discarding of functions in this situation is very similar to the method of disposing that I described in the above paragraph with sums and differences of functions. In both cases, simplifying is necessary. (With the method for sums and differences, only the numerator and the denominator need to be simplified, individually, but with the method for products, the fraction as a whole must be simplified as well.) In addition, the function being discarded in both situations must be dominated by the one being kept as the input approaches infinity. However, to dispose of functions when they are being multiplied with other functions in the situation described in this paragraph, one of the other functions must be infinitely greater than each of the ones being disposed of no matter what positive finite power the discarded ones are raised to.

The Indeterminate Limit Rule encompasses this idea. If $\lim_{x\to a}\frac{f(x)}{[g(x)]^n}=\infty$ for all positive integers, n, then $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}=\infty$. Instead of going through a rigorous formal proof of this, I'll provide an intuitive explanation. The limit as x approaches infinity of $f(x)$ is clearly much larger than $[g(x)]^n$ for any positive integer n. Thus by substituting $f(x)$ with $[g(x)]^n$, in the limit, $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}$, the limit is only being made smaller. By performing this replacement, $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}$ becomes $\lim_{x\to a}\frac{\ln([g(x)]^n)}{\ln(g(x))}$ which can also be expressed as $\lim_{x\to a}\frac{n\ln(g(x))}{\ln(g(x))}$ or simply n. Therefore, $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}>n$. However, since n can be made to be any positive integer, n can be made extremely large without bounds. Therefore since $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}>n$, $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}=\infty$. Now consider the limit, $\lim_{x\to a}\frac{f(x)g(x)}{h(x)}$. If the Indeterminate Limit Rule is performed on this limit, $\lim_{x\to a}\frac{\ln(f(x)g(x))}{\ln(h(x))}$ is obtained. This limit can also be expressed as $\lim_{x\to a}\frac{\ln(f(x))+\ln(g(x))}{\ln(h(x))}$. Factoring out $\ln(f(x))$ in the numerator results in $\lim_{x\to a}\frac{\ln(f(x))(1+\frac{\ln(g(x))}{\ln(f(x))})}{\ln(h(x))}$. Since $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}=\infty$, $\lim_{x\to a}\frac{\ln(g(x))}{\ln(f(x))}=0$, which means that the limit simplifies to $\lim_{x\to a}\frac{\ln(f(x))(1+0)}{\ln(h(x))}$ or $\lim_{x\to a}\frac{\ln(f(x))}{\ln(h(x))}$ which is the same result that would have been obtained if the Indeterminate Limit Rule was being performed on $\lim_{x\to a}\frac{f(x)}{h(x)}$. This is why the Indeterminate Limit Rule yielded the same results for $\lim_{x\to\infty}\frac{2^x}{3^x}$ and $\lim_{x\to\infty}\frac{5x^{10}2^x}{3^x}$ and for $\lim_{x\to\infty}\frac{2.8^x}{10x^{50}e^x\ln(10x)}$ and $\lim_{x\to\infty}\frac{2.8^x}{e^x}$ in the examples shown above. Also consistent with the dominant term method is the fact that in some places where the limit is not simplified such as $\lim_{x\to\infty}\frac{xe^x}{e^x}$, the Indeterminate Limit Rule will fail (result in 1). Here, the dominant term method of disposing of the x in the limit fails because once the exponential terms are cancelled, the x becomes the dominant term in the limit.

To conclude this long section, I'd like to end on a note about my original rule. My original rule was the same as the rule I presented in this paper except for the fact that instead of taking the ratio of the natural logarithms of the functions in the numerator and denominator, it took the ratio of the ratio between the numerator's derivative and original function and the ratio between the denominator's derivative and original function (and it was broadened to account for x approaching any value rather than just infinity). In other words, instead of taking $\lim_{x\to a}\frac{\ln(f(x))}{\ln(g(x))}$, the rule involved finding $\lim_{x\to\infty}\frac{(\frac{f'(x)}{f(x)})}{(\frac{g'(x)}{g(x)})}$. I derived my new limit rule from my original one by recognizing that the new one becomes the old one after L'Hôpital's Rule is applied to it (and infinity is changed to a). My original rule came from my realization that the faster a function approaches infinity, the greater the ratio between its derivative and the original function ($\ln(x) \to \lim_{x\to\infty}\frac{(\frac{1}{x})}{\ln x} = \lim_{x\to\infty}\frac{1}{x\ln x} = 0$,

$x^n \to \lim_{x\to\infty}\frac{nx^{n-1}}{x^n} = \lim_{x\to\infty}\frac{n}{x} = 0$, $b^x \to \lim_{x\to\infty}\frac{b^x\ln(b)}{b^x} = \lim_{x\to\infty}\ln(b) = \ln b$,

$x^x \to \lim_{x\to\infty}\frac{x^x(\ln(x)+1)}{x^x} = \lim_{x\to\infty}(\ln(x)+1) = \infty$).

Conclusion

The Indeterminate Limit Rule simplifies limits with the indeterminate form, $\frac{\infty}{\infty}$, that contain exponential functions (and functions raised to various functions' power such as $(\ln x)^{\ln(x)}$) into limits containing monomials, logarithms, finite numbers, or sums/products of these functions that can then be solved by L'Hôpital's Rule (assuming logarithmic rules are applied). The Indeterminate Limit Rule can be used to show that if the fraction contains monomial terms in the numerator and denominator and it is simplified, then limits where the ratio of the natural logarithm of one function in either the numerator or denominator to another one in the same place is 0, can be simplified to exclude the function approaching infinity slower.