Quasi p- or not quasi p-? That is the Question.*

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Section Zero: Introduction

The question might not be as profound as Shakespeare's, but nevertheless, it is interesting. Because few people seem to be aware of quasi p-groups, we will begin with a bit of history and a definition; and then we will determine for each group of order less than 24 (and a few others) whether the group is a quasi p-group for some prime p or not. This paper is a prequel to [Hwd]. In [Hwd] we prove that $(Z_3 \times Z_3) \rtimes Z_2$ and $Z_5 \rtimes Z_4$ are quasi 2-groups. Those proofs now form a portion of Proposition (12.1) It should also be noted that [Hwd] may also be found in this journal.

Section One: Why should we be interested in quasi p-groups?

In a 1957 paper titled *Coverings of algebraic curves* [Abh2], Abhyankar conjectured that the algebraic fundamental group of the affine line over an algebraically closed field k of prime characteristic p is the set of quasi p-groups, where by the algebraic fundamental group of the affine line he meant the family of all Galois groups $\text{Gal}(L/k(X))$ as L varies over all finite normal extensions of $k(X)$ the function field of the affine line such that no point of the line is ramified in L, and where by a quasi p-group he meant a finite group that is generated by all of its p-Sylow subgroups. More generally, he conjectured that for the affine line minus t points the algebraic fundamental group is the set of quasi (p,t)-groups, where by a quasi (p,t)-group he meant a finite group G such that $G/p(G)$ is generated by t generators where $p(G)$ is the (normal) subgroup of G generated by all of its p-Sylow subgroups[1]. These conjectures became known as the **Abhyankar Conjecture**.

In [Abh2] Abhyankar showed that the algebraic fundamental group of the affine line over an algebraically closed field of prime characteristic p is contained in the set of quasi p-groups and also that the algebraic fundamental group of the affine line over an algebraically closed field of prime characteristic p minus t points is contained in the set of quasi (p,t)-groups.

In 1995, Harbater and Raynaud shared the Cole Prize in Algebra (which is awarded every five years by the American Mathematical Society) for showing the reverse containment. In a 1994 paper [Ray], Raynaud proved the first conjecture, and, in a paper [Har] also published in 1994, Harbater proved the second. The proofs of Raynaud and Harbater are entirely existential. However, some constructive proofs have been done. In a sequence of papers, Abhyankar (along with several of his students and other colleagues) has constructed specific coverings whose Galois groups are various quasi p-groups. These results are summarized in [Abh4] (before Raynaud and Harbater) and [Abh5] (after Raynaud and Harbater).

Notwithstanding the amount of attention the Abhyankar conjecture received, not much has been published about the group theoretical properties of quasi p- and quasi (p,t)-groups. I shall discuss, from an *elementary group theoretical* perspective, many of the elementary properties of quasi p-groups, as well as provide examples and determine for each group of order less than 24 whether the group is a quasi p-group for some prime p or not. I will, with a few exceptions, use only ideas and examples that would be found in an undergraduate abstract algebra course (based upon, for example, [Gal]).

Section Two: What is a quasi p-group?

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[1]Actually, his conjecture about quasi (p,t)-groups was stronger – for any nonsingular projective curve C_g of genus g over an algebraically closed field of prime characteristic p the algebraic fundamental group of $C_{g,t}$ is the set of quasi $(p, 2g+t)$-groups where $C_{g,t}$ is C_g minus t of its points.

The term *quasi p-group* first appeared in [Abh1]; however, the idea appeared earlier with a different name. (See [Abh3] and [Abh2].) Abhyankar defined a quasi p-group as follows:

Definition (2.1). *A finite group G is called a quasi p-group if it is equal to $p(G)$, the (normal) subgroup of G generated by all of its p-Sylow subgroups.*

We will prove that this definition is equivalent to two others, but first we prove two lemmas.

Lemma (2.2). *Let G be a finite group such that p divides $|G|$ the order of G. Let N be a normal subgroup of G, which we denote by $N \trianglelefteq G$. Let $|G|_p$ denote the p-order of G; i.e., the maximum power of p that divides the order of G. Assume that $|N|_p = |G|_p$. Then N contains all elements of G whose order is a power of p. In particular, $p(G)$ is a subgroup of N which we denote by $p(G) \leq N$.*

Proof: Assume, by way of contradiction, that there exists $g \in G$ such that $|g| = p^\beta$ where $\beta > 0$ and $g \notin N$. Consider the coset gN. Because $(gN)^{p^\beta} = N$ the order of gN must divide p^β. Say, $|gN| = p^\gamma$, $1 \leq \gamma \leq \beta$. So, p^γ divides $|G/N|_p = |G|_p/|N|_p = 1$. This is a contradiction and completes the proof. □

Lemma (2.3). *Let G be a finite group whose order is divisible by p. Then $|G|_p = |p(G)|_p$.*

Proof: Let H be a p-Sylow subgroup of G. Then $H \leq p(G) \leq G$. So, $|H|_p \leq |p(G)|_p \leq |G|_p$. Because $|H|_p = |G|_p$, $|G|_p = |p(G)|_p$. □

Now that we have these two lemmas, we can show two alternative definitions of a quasi p-group.

Proposition (2.4). *If G is a finite group, then the following are equivalent: 1. G is a quasi p-group. 2. G is generated by all of its elements whose order is a power of p. 3. G has no nontrivial quotient group whose order is prime to p.*

Proof: Assume that G is a quasi p-group; i.e., G is generated by all of its p-Sylow subgroups. If $|G|_p = p^\alpha$, then every p-Sylow subgroup has order p^α, and every element of the p-Sylow subgroups has order p^β where $\beta \leq \alpha$. In particular, every element of the p-Sylow subgroups has order a power of p. Because G is generated by all of its p-Sylow subgroups, G is generated by elements whose orders are powers of p. Therefore (1) implies (2).

Assume that G is generated by all of its elements whose orders are powers of p. Because every element of G whose order is a power of p is contained in some p-Sylow subgroup of G, (2) implies (1).

Again, assume that G is a quasi p-group. Furthermore, assume, by way of contradiction, that for some proper normal subgroup N of G, the order of G/N is prime to p. So, $|G/N|_p = 1$. By Lemma (2.2), $p(G) \leq N$, which implies that $p(G)$ is a proper subgroup of G, which is denoted by $p(G) < G$. Because this is a contradiction, our assumption that for some proper normal subgroup N of G, the order of G/N is prime to p is false. Therefore, (1) implies (3).

Finally, assume that G has no nontrivial quotient group whose order is prime to p, and assume, by way of contradiction, that $p(G) \neq G$. Then $p(G)$ is a proper normal subgroup of G, which we denote by $p(G) \triangleleft G$, and by Lemma (2.3) $|G/p(G)|_p = 1$. Because this is a contradiction, our assumption that $p(G) \neq G$ is false. Therefore, (3) implies (1). □

The alternative definitions are what we will use in this paper. (2) will be used primarily to prove that a group is a quasi p-group by constructing generators whose orders are powers of p, and (3) will be used primarily to prove that a group is not a quasi p-group when we are able to capture all the power of p order elements in a proper normal subgroup.

Notice that quasi p-groups need to have "many" p-Sylow subgroups. At least by (3) if G were a group with a unique proper p-Sylow subgroup, then G would not be a quasi p-group.

Recall that a group G is a p-group if every element of G has order a power of p. Obviously every p-group is a quasi p-group. But, it is easy to see that the order of a finite p-group must be a power of p, and this is not the case with quasi p-groups. Here are two examples.

The **symmetric group of degree** n S_n is a quasi 2-group because for all n, S_n is generated by transpositions (2-cycles) which are elements of order 2. But, $|S_n| = n!$. We shall prove later that S_n is only a quasi 2-group.

The **alternating group of degree** $n > 2$ A_n is a quasi 3-group because for all $n > 2$, A_n is generated by 3-cycles. But, $|A_n| = n!/2$.

Section Three: What about homomorphic images and extensions?

First, we note that the homomorphic image of a quasi p-group is a quasi p-group.

Proposition (3.1). *Let G be a quasi p-group, and let $\phi : G \to H$ be a homomorphism onto H. Then H is a quasi p-group.*

Proof: If H is a trivial homomorphic image (i.e., if either $H = G$ or H is the identity), then the result is true. So, assume that H is a nontrivial homomorphic image of G. Now, assume, by way of contradiction, that there exists a proper normal subgroup N_H of H such that $|H/N_H|$ is prime to p. Consider the canonical homomorphism $\tau : H \to H/N_H$. Composing the two homomorphisms we obtain $\tau \circ \phi : G \to H/N_H$ where $|H/N_H|$ is prime to p. Notice that $\mathrm{Ker}(\tau \circ \phi) \lhd G$ and $G/\mathrm{Ker}(\tau \circ \phi) \cong H/N_H$ where $|H/N_H|$ is prime to p; therefore, $|G/\mathrm{Ker}(\tau \circ \phi)|$ is prime to p, which is a contradiction because G is a quasi p-group. $\square$

Next we consider extensions of quasi p-groups.

Proposition (3.2). *Assume that $N \trianglelefteq G$, N is a quasi p-group, and G/N is a quasi p-group; then G is also a quasi p-group.*

Proof: Assume, by way of contradiction, that G is not a quasi p-group. Then $N \lhd G$. Because N is a quasi p-group, it is contained in $p(G)$. Because $N \leq p(G) \lhd G$, $\langle 1 \rangle = N/N \leq p(G)/N \lhd G/N$. In particular, $p(G)/N \lhd G/N$. Now $|G/N|/|p(G)/N| = |G/p(G)|$ which is prime to p. This is a contradiction because G/N is a quasi p-group. $\square$

Section Four: Simple groups provide many examples of quasi p-groups.

Proposition (4.1). *Let G be a simple group. If a prime p divides the order of G, then G is a quasi p-group.*

Proof: Because p divides the order of G, $p(G) \neq \langle 1 \rangle$. Because $p(G)$ is normal in G and G is simple, $p(G) = G$;i.e., G is a quasi p-group. $\square$

Therefore, the finite simple groups provide many examples of quasi p-groups.

Recall that the **Classification of the Finite Simple Groups** claims that the finite simple groups consist of the cyclic groups Z_p, the alternating groups A_n for $n \geq 5$, 16 infinite families, and 26 sporadic groups. A good discussion of the 110 year history and the current status of the proof of the classification is given in [Sol].

For example, the Monster whose order is $2^{46}3^{50}5^{9}7^{6}11^{2}13^{3}17^{1}19^{1}23^{1}29^{1}31^{1}41^{1}47^{1}59^{1}71^{1}$ is, therefore, a quasi p-group for $p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 17, 59, 71$.

Section Five: What about symmetric groups and alternating groups?

We noted in Section Two that S_n the symmetric group of degree n is a quasi 2-group. We now show that it is only a quasi 2-group.

Proposition (5.1). *For $n \geq 2$, S_n the symmetric group of degree n is always a quasi 2-group, but it is not a quasi p-group for any prime $p \neq 2$.*

Proof: Because S_n is generated by 2-cyles, it is always a quasi 2-group. To prove the rest, consider S_n/A_n. The order of this quotient group is 2; so, S_n has a nontrivial quotient group whose order is prime to every prime except $p = 2$. Therefore, S_n can only be a quasi 2-group. $\square$

We also noted in Section Two that A_n the alternating group for $n > 2$ is a quasi 3-group. We can say a bit more than that.

Proposition (5.2). *Consider the alternating group A_n: 1. For $n > 2$, A_n is always a quasi 3-group. 2. A_3 is only a quasi 3-group. 3. A_4 is only a quasi 3-group. 4. If $n \geq 5$, A_n is a quasi p-group for all p dividing $|A_n|$.*

Proof: To prove (1), we know that if $n > 2$ then A_n is generated by 3-cycles; so, it is a quasi 3-group.

To prove (2), notice that $|A_3| = 3$. (So, $A_3 \cong Z_3$.) The only prime that divides 3 is 3 itself; so, A_3 can only be a quasi 3-group.

To prove (3), notice that $|A_4| = 12 = 2^2 3$. A_4 has only one 2-Sylow subgroup (see, for example, [DF] p. 112), say Q. Therefore, $Q \triangleleft A_4$. Notice that $|A_4/Q| = 3$. So, A_4 has a nontrivial quotient group whose order is prime to every prime except $p = 3$. Therefore, A_4 can only be a quasi 3-group.

Finally, to prove (4), recall that for $n \geq 5$, A_n is a simple group. This result follows from Proposition (4.1). $\square$

Section Six: What about dihedral groups?

Dihedral groups are familiar and provide interesting examples of quasi 2-groups.

Proposition (6.1). *The dihedral group $D_{2n} = \{s, r : s^2 = 1, r^n = 1, rs = sr^{-1}\}$ is always a quasi 2-group, but it is not a quasi p-group for any prime $p \neq 2$.*

Proof: We know that D_{2n} contains a normal subgroup isomorphic to Z_n – the rotation subgroup. Notice that $|D_{2n}/Z_n| = 2$; so, D_{2n} has a nontrivial quotient group whose order is prime to every prime except $p = 2$. Therefore, D_{2n} can only be a quasi 2-group.

To prove that it is always a quasi 2-group, we consider D_{2n} to be given by generators and relations as in the statement of the proposition. D_{2n} consists of the identity 1; the rotations $r, r^2, ..., r^{n-1}$; and the reflections $s, sr, sr^2, ..., sr^{n-1}$. Each of the reflections has order 2. Consider the products of s and each of the reflections. For each $j = 1, 2, ..., n-1$; $s(sr^j) = r^j$. Therefore, the reflections, which are elements of order 2, generate D_{2n}, and D_{2n} is a quasi 2-group.

Thus, D_{2n} is always and only a quasi 2-group. $\square$

Section Seven: What about direct products?

In the next section we will examine abelian and nilpotent groups, but before doing that an analysis of direct products would be most beneficial.

Proposition (7.1). *Let $G = H \times K$. Then for a prime p, G is a quasi p-group if and only if both H and K are quasi p-groups.*

Proof: First, assume that G is a quasi p-group. Then, by Proposition (3.1), $K \cong G/H$ is a quasi p-group, and $H \cong G/K$ is a quasi p-group.

Now assume that H and K are both quasi p-groups. Notice that $p(H) \times \langle 1 \rangle \leq p(G)$ and $\langle 1 \rangle \times p(K) \leq p(G)$. Therefore, $G = H \times K = p(H) \times p(K) \leq p(G)$. This implies that $p(G) = G$; i.e., G is quasi p-group. $\square$

It is easy to find an example of a direct product that is not a quasi p-group for any prime p. For example, $Z_6 \cong Z_2 \times Z_3$ is not a quasi p-group for any prime p by Proposition (2.4)(3). The problem is that the $2(Z_6) \triangleleft Z_6$ and $3(Z_6) \triangleleft Z_6$; there are "too many" proper normal subgroups or equivalently there are "too few" 2-Sylow and 3-Sylow subgroups.

Section Eight: Only a few cyclic, abelian, and nilpotent groups are quasi p-groups.

Recall the following hierachy of finite groups: cyclic groups $\subset$ abelian groups $\subset$ nilpotent groups $\subset$ solvable groups $\subset$ all finite groups. We will consider cyclic, abelian, and nilpotent quasi p-groups in this section and solvable groups in the next.

Proposition (8.1). *The cyclic group Z_n is a quasi p-group if and only if it is a p-group.*

Proof: Because every p-group is a quasi p-group, it is only necessary to show that if Z_n is a quasi p-group then Z_n is a p-group. Assume, by way of contradiction, that p divides n but that n is not a power of p. Say, $|Z_n|_p = p^\alpha$. Then Z_n has a unique proper normal subgroup N of order p^α. Then $p(Z_n) = N \neq Z_n$ which is a contradiction. Therefore, if Z_n is a quasi p-group, then $n = p^\alpha$. $\square$

Combining this result with the Fundamental Theorem of Finite Abelian Groups and Proposition (7.1) yields the following:

Proposition (8.2). *A finite abelian group G is a quasi p-group if and only if it is a p-group.*

Proof: By the Fundmental Theorem of Finite Abelian Groups, $G \cong Z_{n_1} \times Z_{n_2} \times ... \times Z_{n_s}$.
Because every p-group is a quasi p-group, it is only necessary to show that if G is a quasi p-group then G is a p-group.
So, assume that G is a quasi p-group. By Proposition (7.1) each of the factors in the direct product must also be a quasi p-group. Then Proposition (8.1) tells us that each factor must in fact be a p-group; so, G is a p-group. □

There are several equivalent definitions of a nilpotent group. We will take as definition the one that is most applicable to our study of quasi p-groups.

Definition (8.3). *Let G be a finite group and let $p_1, p_2, ..., p_s$ be the distinct primes dividing the order of G and for each $i = 1, 2, ..., s$ let P_i be a p_i-Sylow subgroup of G. G is nilpotent if $G \cong P_1 \times P_2 \times ... \times P_s$. (See, for example, [DF] p. 193.)*

Proposition (8.4). *A finite nilpotent group G is a quasi p-group if and only if it is a p-group.*

Proof: Again it is only necessary to show that if G is a quasi p-group then G is a p-group. Let $p_1, p_2, ..., p_s$ be the distinct primes dividing the order of G. Because G is nilpotent, $G \cong P_1 \times P_2 \times ... \times P_s$ where for each $i = 1, 2, ..., s$ let P_i be a p_i-Sylow subgroup of G. By Proposition (7.1), because G is a quasi p-group, each P_i must be a p-group; therefore, there is only one prime p that divides the order of G. G is a p-group. □

Section Nine: But, solvable quasi p-groups need not be p-groups.

The next step in the hierarchy is solvable groups. Following the pattern we have had up to now with cyclic, abelian, and nilpotent groups, one might guess that a finite solvable group is a quasi p-group if and only if it is p-group. This is not the case, however.

Example (9.1). *S_3 is a finite solvable group, and it is a quasi 2-group. But, the order of S_3 is 6, which is not a power of 2. So, a solvable quasi p-group need not be a p-group.*

Also, a solvable group need not be a quasi p-group.

Example (9.2). *Z_6 is solvable, but it is neither a quasi 2-group nor a quasi 3-group.*

Finally, notice that:

Example (9.3). *By Propositions (2.4) and (4.1), the only simple solvable quasi p-group is Z_p.*

Section Ten: Semidirect products provide many interesting examples.

Although we have not made use of the fact so far, several of the groups that we have considered are semidirect products. We now examine the relationship between semidirect products and quasi p-groups.
Semidirect products are not a standard topic in a first course in abstract algebra, but they could be. [DF], [Hal], and [Rot] each discusses this topic. A quick introduction (which is enough for our needs) may be found in [AbC].

Proposition (10.1). *Let H and K be quasi p-groups. Then G, the semidirect product of H and K which we denote by $G \cong H \rtimes K$, is also a quasi p-group.*

Proof: Assume, by way of contradiction, that there exists a proper normal subgroup $N \triangleleft G$ such that $|G/N|$ is prime to p. Therefore, N contains all elements of G of p-power order, and, in particular, $p(H) \leq N$. But, because H is a quasi p-group and a subgroup of G, $H = p(H) \leq N$. Under the homomorphism $G \to G/H \cong K$, N is isomorphic to N/H which is normal in G/H which is isomorphic to K: $N \to N/H \triangleleft G/H \cong K$. Let N^* denote the image of N in K under this isomorphism. So, we have $N^* \triangleleft K$ and $K/N^* \cong (G/H)/(N/H) \cong G/N$. In particular, $|K/N^*| = |G/N|$. But $|G/N|$ is prime to p; so, K is not a quasi p-group. This is a contradiction, which completes the proof. □

Proposition (10.2). *If $G \cong H \rtimes K$ is q quasi p-group, then K is also a quasi p-group.*

Proof: Consider $G/H \cong K$. Because G is a quasi p-group, by Proposition (3.1) K is also a quasi p-group. □

Application (10.3). *Recall that in Example (2.5) we pointed out that S_n is a quasi p-group. As a result of Proposition (10.2) we see that S_n can only be a quasi 2-group because $S_n \cong A_n \rtimes Z_2$.*

Application (10.4). *Recall that in Proposition (6.1) we proved that D_{2n} is always a quasi 2-group but not a quasi p-group for any other prime p. The second half of this now follows from Proposition (10.2) by noting that $D_{2n} \cong Z_n \rtimes Z_2$.*

Example (10.5). *Notice that if $G \cong H \rtimes K$ is a quasi p-group then H need not be a quasi p-group. Just notice that $S_3 \cong Z_3 \rtimes Z_2$ is a quasi 2-group but Z_3 is not a quasi 2-group.*

Definition (10.6). *We will call a semidirect product a* **proper semidirect product** *if it is not a direct product.*

Notice that a direct product is a quasi p-group if and only if its factors are quasi p-groups. But, because proper semidirect products have one factor that is not normal, if that factor is a quasi p-group and if there are enough copies of that factor in the group, the semidirect product might be a quasi p-group.

Example (10.7). *Based upon examples, it was tempting to conjecture that all proper semidirect products whose non-normal factor is a quasi p-group are quasi p-groups. But, this is not the case. $S_3 \times Z_3$ which is a group of order 18 is "quasi nothing" by Proposition (7.1), but $S_3 \times Z_3$ is a wreath product $Z_3 \wr Z_2$, and therefore, is a proper semidirect product.*

Section Eleven: Is a group of order pq a quasi p-group?

Next we will consider all groups of order pq where p and q are distinct primes. Such groups are typically treated in graduate courses in abstract algebra.

Proposition (11.1). *Let $|G| = pq$, with $p < q$. 1. If p does not divide $q-1$, then G is not a quasi p-group. 2. If p divides $q-1$, then G is only a quasi p-group.*

Proof: To prove (1), we note that if p does not divide $q-1$, then G is cyclic of order pq. (See, for example, [DF] pp. 183 and 184 or [Hal] pp. 49 and 50.) Therefore, by Proposition (8.1) G is not a quasi p-group.

To prove (2), we note that $G \cong Z_q \rtimes Z_p$. Furthermore, this is the unique non-abelian group of order pq and there are exactly q p-Sylow subgroups. (See, for example, [Hal] pp. 49 and 50.) Let $\langle x \rangle$ be the unique normal subgroup isomorphic to Z_q, and let $\langle y \rangle$ be one of the p-Sylow subgroups. Notice that the p-Sylow subgroups account for the identity and $q(p-1) = qp - q$ elements of order p. The remaining elements of G are $\{x, x^2, ..., x^{q-1}\}$. We want to show that these elements can be generated by elements of order a power of p.

Consider the element yx. We claim that the order of yx is p. Because $|G| = pq$ the only possibilities for $|yx|$ are $pq, p, q, 1$. G is not cyclic; so, $|yx| \neq pq$. $|yx| = 1$ is not possible because this would imply that $|y| = |x^{-1}|$, but $|y| = p$ and $|x| = q$. If $|yx| = q$, then $yx \in \langle x \rangle$; i.e., for some $m \in \{1, ..., q-1\} yx = x^m$. This would imply that $y = x^{m-1}$, which is not possible (because for each $m \in \{1, ..., q-1\}$, x^m has order q whereas y has order p). Therefore, we can conclude that $|yx| = p$.

Because y has order p, y^{-1} also has order p. So, because yx and y^{-1} each have order p; $x = y^{-1}(yx)$ is generated by elements of order p. Therefore each of the elements in $\langle x \rangle$ is generated by elements of order a power of p, and we can conclude that G is a quasi p-group. □

Section Twelve: Classification of the groups of order less than 24.

For the groups of order less than 24, we present two tables (taken from [DF]) – one for the abelian groups and one for the non-abelian groups, and we classify those according to whether they are quasi p-groups for some prime p or not. For the abelian groups, all results follow from Proposition (8.2).

Abelian groups of order less than 24

Order	Group	Quasi p-group
1	Z_1	p-group for every prime p
2	Z_2	2-group
3	Z_3	3-group
4	Z_4	2-group
4	$Z_2 \times Z_2$	2-group
5	Z_5	5-group
6	Z_6	quasi nothing
7	Z_7	7-group
8	Z_8	2-group
8	$Z_4 \times Z_2$	2-group
8	$Z_2 \times Z_2 \times Z_2$	2-group
9	Z_9	3-group
9	$Z_3 \times Z_3$	3-group
10	Z_{10}	quasi nothing
11	Z_{11}	11-group
12	Z_{12}	quasi nothing
12	$Z_6 \times Z_2$	quasi nothing
13	Z_{13}	13-group
14	Z_{14}	quasi nothing
15	Z_{15}	quasi nothing
16	4 groups	2-groups
17	Z_{17}	17-group
18	Z_{18}	quasi nothing
18	$Z_6 \times Z_3$	quasi nothing
19	Z_{19}	19-group
20	Z_{20}	quasi nothing
20	$Z_{10} \times Z_2$	quasi nothing
21	Z_{21}	quasi nothing
22	Z_{22}	quasi nothing
23	Z_{23}	23-group

Nonabelian groups of order less than 24

Order	Group	Quasi p-group
6	S_3	quasi 2-group by 5.1 or 11.1
8	D_8	2-group or by 6.1
8	Q_8	2-group
10	D_{10}	quasi 2-group by 6.1 or 11.1
12	A_4	quasi 3-group by 5.2
12	D_{12}	quasi 2-group by 6.1
12	$Z_3 \rtimes Z_4$	quasi 2-group by 12.1
14	D_{14}	quasi 2-group by 6.1 or 11.1
16	9 groups	2-groups
18	D_{18}	quasi 2-group by 6.1
18	$S_3 \times Z_3$	quasi nothing by 7.1
18	$(Z_3 \times Z_3) \rtimes Z_2$	quasi 2-group by 12.1
20	D_{20}	quasi 2-group by 6.1
20	$Z_5 \rtimes Z_4$	quasi 2-group by 12.1
20	F_{20}	quasi 2-group by 12.2
21	$Z_7 \rtimes Z_3$	quasi 3-group by 11.1
22	D_{22}	quasi 2-group by 6.1 or 11.1

Proposition (12.1). *Each of $Z_3 \rtimes Z_4$, $(Z_3 \times Z_3) \rtimes Z_2$, and $Z_5 \rtimes Z_4$ is only a quasi 2-group.*

Proof: The proofs for $(Z_3 \times Z_3) \rtimes Z_2$, and $Z_5 \rtimes Z_4$ can be found in [Hwd]. Here we will prove that $Z_3 \rtimes Z_4$ is only a quasi 2-group; the proof is similar to the proof for $Z_5 \rtimes Z_4$.

In terms of generators and relations, $Z_3 \rtimes Z_4 = \langle x, y | x^4 = y^3 = 1, x^{-1}yx = y^{-1} \rangle$. So, $x \in 2(Z_3 \rtimes Z_4)$. If we can get $y \in 2(Z_3 \rtimes Z_4)$, we will be done because then $2(Z_3 \rtimes Z_4) = Z_3 \rtimes Z_4$. Notice that because $x^{-1}yx = y^{-1}$, $yx = xy^{-1} = xy^2$. Now consider the order of xy. $(xy)^2 = xyxy = xxy^2y = x^2$. So, the order of xy is 4, and, therefore, $xy \in 2(Z_3 \rtimes Z_4)$. Because $x, xy \in 2(Z_3 \rtimes Z_4)$, $y = x^{-1}xy \in 2(Z_3 \rtimes Z_4)$; and we can conclude that $Z_3 \rtimes Z_4$ is a quasi 2-group.

Because all the elements of order 3 in $Z_3 \rtimes Z_4$ are in the factor Z_3, $3(Z_3 \rtimes Z_4)$ is a proper subgroup of $Z_3 \rtimes Z_4$. Therefore, $Z_3 \rtimes Z_4$ is only a quasi 2-group. □

The group identified in the table of non-abelian groups as F_{20} is the **Frobenius group of order 20**. It has generators and relations (See, for example, [DF] p. 170.): $F_{20} = \langle x, y, | y^4 = x^5 = 1, yxy^{-1} = x^2 \rangle$, or we can think of it as a subgroup of S_5: $F_{20} = \langle (2354), (12345) \rangle$. We will use the permutation representation and show, using a brute force argument similar to the arguments in Proposition (11.1) and Proposition (12.1), that F_{20} is a quasi 2-group.

Proposition (12.2). *F_{20} is only a quasi 2-group.*

Proof: Let $x = (12345)$ and $y = (2354)$. It is clear from the definition by generators and relations or by constructing the subgroup lattice that $H = \langle x \rangle = \langle (12345) \rangle$ is a normal subgroup of G. Therefore, we remark that $G = F_{20}$ is not a quasi 5-group. Now H contains the identity and 4 elements of order 5. Using x and y we can construct the remaining 15 elements of G. The subgroup lattice of G contains 5 2-Sylow subgroups – each of order 4. These account for the remaining 15 elements of G; each of these 15 nonidentity elements has order a power of 2. Let $G_1 = \langle (2345) \rangle$ which contains the subgroup $\langle (25)(34) \rangle$ of order 2. Let $G_2 = \langle (1435) \rangle$ which contains the subgroup $\langle (13)(45) \rangle$ of order 2. Let $G_3 = \langle (1254) \rangle$ which contains the subgroup $\langle (15)(24) \rangle$ of order 2. Let $G_4 = \langle (1325) \rangle$ which contains the subgroup $\langle (12)(35) \rangle$ of order 2. Let $G_5 = \langle (1243) \rangle$ which contains the subgroup $\langle (14)(23) \rangle$ of order 2. Following the pattern used in Proposition (11.1) that $(yx)(y^{4-1}x^{1-2}) = (yxy^3)x^{-1} = (yxy^{-1})x^{-1} = x^2x^{-1} = x$, we see that $(1325)(1342) = [(2354)(12345)][(2453)(15432)] = (12345)$ which is a generator for H. Notice that each of $(1325) \in G_4$ and $(1342) \in G_5$ has order a power of 2. So the elements of $G = F_{20}$ whose order is a power of 2 generate H. Therefore, they generate G. So $G = F_{20}$ is a quasi 2-group, and it is only a quasi 2-group by the remark at the beginning of the proof. □

Definition (12.3). *A finite group G is called a* **Frobenius group** *with Frobenius kernel Q if Q is a proper, nontrivial normal subgroup of G and $C_G(x) \leq Q$, where $C_G(x)$ denotes the centralizer of x in G, for all nonidentity elements x of Q. (See, for example, [DF] p. 862.)*

Although this definition is made for abstract finite groups, Frobenius groups come from the tradition of permutation groups. (A permutation group is a group which is a subgroup of S_n for some n). For this discussion, we think of Frobenius groups as permutation groups.

Comment (12.4). *In 1901, Frobenius showed that the Frobenius kernel is always a regular normal subgroup and that it consists of the identity together with those elements that fix no element of the permuted set.*

For F_{20}, the Frobenius kernel is $H = \langle (12345) \rangle = \{(12345), (13542), (14253), (15432), (1)\}$.

Comment (12.5). *A Frobenius group is the semidirect product of the Frobenius kernel by a Frobenius complement.*

For F_{20}, the Frobenius complements are G_1, G_2, G_3, G_4, G_5. They are isomorphic, and for $i = 1, 2, 3, 4, 5$ G_i is the stabilizer of i – the subgroup of permutations that stabilize i.

More about Frobenius groups may be found in [Abh2] pp. 78 and 79, [DF], [Hal], [Pas], [Rot], and [Wie].

Section Thirteen: How about infinite quasi p-groups?

Abhyankar's definition of a quasi p-group requires that the group be finite. Before ending, although we will not in this paper examine any of their properties, we would like to suggest a definition for infinite quasi p-groups.

Definition (13.1). *A group G of infinite order is said to be a quasi p-group if $G = p(G)$.*

$G = S_2 \times S_3 \times S_4 \times ...$ is an example of an infinite quasi 2-group.

Section Fourteen: Conclusions.

Normal p-Sylow subgroups are bad – at least as far as quasi p-groups are concerned. If $p(G)$ is a proper normal subgroup of G, then G cannot be a quasi p-group (Proposition (2.4)). On the other hand, simple groups provide lots of examples of quasi p-groups (Proposition (4.1)). Cyclic, abelian, and nilpotent groups are quasi p-groups if and only if they are p-groups (Propositions (8.1), (8.2), and (8.4)). But, solvable quasi p-groups need not be p-groups (Proposition (9.1)); however, simple, solvable quasi p-groups are p-groups (Proposition (9.3)). Proper semidirect products (e.g., D_{2n}, S_n, and F_{20}) that are quasi p-groups have a normal subgroup, but they have lots of p-Sylow subgroups. Quasi p-groups are "almost simple."

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