The Taylor Expansion of a Riemannian Metric [1]

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Abstract

We give a description of a Taylor expansion of the Riemannian metric in terms of covariant derivatives of the curvature tensor. After constructing the appropriate coordinate system, we derive relations from the Gauss lemma that allow us to explicitly calculate the first few terms in the expansion. This process is generalized to terms of arbitrary order, although their explicit calculation is far more computationally intensive than enlightening.

Contents

Chapter 1

Normal Coordinates

The reader is assumed to be familiar with manifolds, tensors, connections, and covariant differentiation. Throughout this document, all manifolds are assumed to be smooth, all tensor fields will simply be called tensors, all metrics will be assumed to be Riemannian, and unless otherwise noted, the Einstein summation convention is used.

1.1 Metric Connections and the Levi-Civita Connection

We begin by recalling that there exist on a manifold as many connections as there are choices of smooth functions Γ^k_{ij}. If the manifold is endowed with a metric, however, there exists a unique canonical connection.

A connection ∇ that satisfies

$$(\nabla_k g)_{ij} = 0, \tag{1.1}$$

where g is the metric, is called a metric connection and is said to be compatible with g. An equivalent and useful way of expressing this is

$$\nabla_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \tag{1.2}$$

We can interpret this as the requirement that if two vector fields are "constant" on a path, then their inner product should be constant as well. It is, of course, the connection that allows us to determine whether and how a vector field changes. We can see this if we covariantly differentiate the inner product of two vector fields X and Y that are parallel transported along some path.

$$\begin{aligned} 0 &= \nabla_Z\big(g(X,Y)\big) \\ &= Z^k\big((\nabla_k g)(X,Y) + g(\nabla_k X, Y) + g(X, \nabla_k Y)\big) \\ &= Z^k X^i Y^j (\nabla_k g)_{ij} \end{aligned}$$

Were we have imposed that the covariant derivatives of X and Y are zero. For this to hold for all X, Y, Z would imply 1.1. This places some restrictions on our choice of connection, but to ensure uniqueness we need to evoke the rule that the torsion tensor defined as:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.3}$$

is zero everywhere. Such a connection is said to be torsion free. In coordinate bases, the Lie bracket is always zero, and the requirement that the connection is torsion free is equivalent to:

$$\nabla_X Y = \nabla_Y X.$$

Defining our connection coefficients as usual way by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{e_i} e_j = \nabla_i e_j = \Gamma^k_{ij} e_k, \tag{1.4}$$

we have that the torsion tensor (with Lie bracket zero) is expressed as

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.$$

Thus, we see that when the Lie bracket is zero, the torsion free condition is equivalent to having a symmetric connection, as below.

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

If a metric connection is torsion free, it is referred to as the Levi-Civita connection. We are ensured of the existence and uniqueness of the Levi-Civita connection on any Riemannian manifold by the fundamental theorem of Riemannian geometry. Throughout the rest of this document, all connections will be the Levi-Civita connection.

1.2 The Exponential Map

We begin by defining the neighborhood E_p of 0 in T_pM:

$$E_p = \{v | \exists \text{ a geodesic } \gamma_v \text{ s.t. } \gamma_v(0) = p,\ \gamma_v'(0) = v, \text{ and } \gamma_v(1) \text{ is defined.}\}$$

This allows us to then define the exponential map

$$\begin{aligned} exp_p \quad : \quad & E_p \to M \\ & v \mapsto \gamma_v(1) \end{aligned}$$

we know that this map is well defined in some neighborhood of 0 in T_pM, and is even a diffeomorphism if the neighborhood is small enough, [1]. We also have that given a vector w in the tangent space at p, the path $c_w(t) = exp_p(tw)$ has both $c_w(0) = p$ and $c_w'(0) = w$. These properties give us a natural way to define the coordinate chart $(U, B \circ exp_p^{-1})$ for a neighborhood $U \subset M$ using the fact that T_pM is isomorphic to $\mathcal{R}^N$ with choice of basis B. We refer to the coordinates produced by the projection of the basis vectors in the tangent space by the exponential map as normal coordinates.

1.3 Relations in Normal Coordinates

Specifically choosing an orthonormal basis $\{b_k\}$ of T_pM for the foundation of our normal coordinates yields the following priceless relations:

$$g_{ij}\Big|_p = \delta_{ij} \tag{1.5}$$

$$\Gamma^k_{ij}\Big|_p = 0 \tag{1.6}$$

$$\frac{\partial g_{ij}}{\partial x^k}\Big|_p = 0 \tag{1.7}$$

We reach 1.5 by noting that

$$g_{ij}\Big|_p = \langle \frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p \rangle = \langle b_i, b_j \rangle = \delta_{ij}.$$

by our choice of the basis vectors.

To prove 1.6 we recall the geodesic equation:

$$\frac{\mathrm{d}^2\gamma^k}{\mathrm{d}t^2} + \Gamma^k_{ij}\Big|_{\gamma(t)} \frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t} = 0. \tag{1.8}$$

In our coordinates we have that geodesics are given by $\gamma^k(t) = \epsilon^k t$ for some vector ϵ^k. The geodesic equation then gives

$$\Gamma^k_{ij}\Big|_{\gamma(t)} \epsilon^i \epsilon^j = 0 \ \forall \epsilon.$$

Evaluating this relation at the origin and noting that it must hold true for all n-tuples gives

$$\Gamma^k_{ij}\Big|_p + \Gamma^k_{ji}\Big|_p = 0.$$

From this and the fact that the Levi-Civita connection is symmetric, the connection coefficients must be zero at the origin p. Using the fact that the Levi-Civita connection respects the metric,

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x^k} &= \frac{\partial}{\partial x^k}\langle e_i, e_j\rangle = \nabla_k \langle e_i, e_j\rangle \\ &= \langle \nabla_k e_i, e_j\rangle + \langle e_i, \nabla_k e_j\rangle \\ &= \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il}. \end{aligned} \tag{1.9}$$

Since the connection coefficients are zero at the origin, we immediately have 1.7.

For further relations in the Taylor expansion, however, we develop new methods and use the relations derived from the Gauss lemma in the following chapter.

1.4 Curvature and Connection Coefficients

In this section we will use the coordinate basis and the Levi-Civita connection to compute the components of the curvature tensor, as well as compute the connection coefficients in terms of the partial derivatives of the metric tensor. We recall the definition of the curvature tensor as the map:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.10}$$

To examine the components, we take the inner product with the dual basis and sum.

$$\begin{aligned}
R^l{}_{ijk} &= \langle \mathrm{d}x^l, R(e_i, e_j)e_k \rangle \\
&= \langle \mathrm{d}x^l, \nabla_j \nabla_k e_i - \nabla_l \nabla_j e_i \rangle \\
&= \langle \mathrm{d}x^l, \frac{\partial \Gamma^\alpha_{ki}}{\partial x^j} e_\alpha + \Gamma^\alpha_{ki}\Gamma^\beta_{j\alpha} e_\beta - \left(\frac{\partial \Gamma^\alpha_{jk}}{\partial x^k} \right) e_\alpha - \Gamma^\alpha_{ki}\Gamma^\beta_{j\alpha} e_\beta \rangle \\
&= \frac{\partial \Gamma^l_{ki}}{\partial x^j} + \Gamma^\alpha_{ki}\Gamma^l_{j\alpha} - \frac{\partial \Gamma^l_{jk}}{\partial x^k} - \Gamma^\alpha_{ki}\Gamma^l_{j\alpha}.
\end{aligned}$$

To better understand the connection coefficients, we first recall that the compatibility of the metric with the connection was used to derive the following relation in 1.9.

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il}. \tag{1.11}$$

We then use the symmetry of the connection and the metric to derive.

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2} g^{k\alpha} g_{\beta\alpha} \left(\Gamma^\beta_{ij} + \Gamma^\beta_{ji} \right) \\
&= \frac{1}{2} g^{k\alpha} \left\{ g_{\beta\alpha} \left(\Gamma^\beta_{ij} + \Gamma^\beta_{ji} \right) + g_{j\beta} \left(\Gamma^\beta_{i\alpha} - \Gamma^\beta_{\alpha i} \right) + g_{i\beta} \left(\Gamma^\beta_{j\alpha} - \Gamma^\beta_{\alpha j} \right) \right\} \\
&= \frac{g^{k\alpha}}{2} \left\{ \left(\Gamma^\beta_{ij} g_{\beta\alpha} + \Gamma^\beta_{i\alpha} g_{j\beta} \right) + \left(\Gamma^\beta_{ji} g_{\beta\alpha} + \Gamma^\beta_{j\alpha} g_{i\beta} \right) - \left(\Gamma^\beta_{\alpha i} g_{\beta j} + \Gamma^\beta_{\alpha j} g_{i\beta} \right) \right\} \\
&= \frac{1}{2} g^{k\alpha} \left(\frac{\partial g_{j\alpha}}{\partial x^i} + \frac{\partial g_{\alpha i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\alpha} \right).
\end{aligned} \tag{1.12}$$

Chapter 2

The Gauss Lemma

2.1 Derivation of the Gauss Lemma

One of the reasons that normal coordinates are so useful in our calculation of the Taylor expansion is that they lead us to certain relations between the partial derivatives of g_{ij} . We investigate the geodesic $\gamma(t)$, which in normal coordinates is parametrized by the line $\gamma^k(t) = \epsilon^k t$. We recall the geodesic equation

$$\frac{\mathrm{d}^2\gamma^k(t)}{\mathrm{d}t^2} + \sum_{i,j=1}^{N} \Gamma^k_{ij}(\gamma(t)) \frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t} \frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t} = 0 \tag{2.1}$$

In normal coordinates, the geodesic equation becomes

$$\sum_{i,j=1}^{N} \Gamma^k_{ij}(\gamma(t))\epsilon^i\epsilon^j = 0 \tag{2.2}$$

We may multiply by t^2 on both sides to derive a more general formula at a point $\mathbf{x} = (x_1 \ldots x_n)$

$$\sum_{i,j=1}^{N} \Gamma^k_{ij} x^i x^j = 0 \tag{2.3}$$

Now, it is true that we may always parametrize a geodesic $\gamma(t)$ such that the value $\frac{\mathrm{d}\gamma}{\mathrm{d}t}$ is constant [3]. From Chapter 1 we know that $g_{ij} = \delta_{ij}$ at the origin. Therefore

$$\langle \frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\mathrm{d}\gamma}{\mathrm{d}t} \rangle = \langle \frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\mathrm{d}\gamma}{\mathrm{d}t} \rangle \Big|_p = \sum_{i,j=1}^{N} \epsilon^i\epsilon^j\delta_{ij} = \sum_{i=1}^{N} (\epsilon^i)^2$$

but it is also true that

$$\langle \frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\mathrm{d}\gamma}{\mathrm{d}t} \rangle = \sum_{i,j=1}^{N} g_{ij}\epsilon^i\epsilon^j$$

Equating the results and multiplying by t^2 on both sides, we get the important equation

$$\sum_{i,j=1}^{N} g_{ij}x^i x^j = \sum_{i=1}^{N}(x^i)^2 \tag{2.4}$$

for any point locally on the manifold. Using equation 2.3, we substitute in our expression for Γ^k_{ij} in terms of the first order partials of g_{ij} to get

$$\sum_{i,j=1}^{N} \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) x^i x^j = 0$$

Permuting the indices i and j , we get

$$\sum_{i,j=1}^{N} \left(\frac{\partial g_{ik}}{\partial x^j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\right) x^i x^j = 0 \tag{2.5}$$

Our aim is to take equations 2.3, 2.4 and 2.5 and derive a more general and useful result for normal coordinates.

Let us introduce the function

$$\overline{x}^{\beta} = \sum_{\alpha=1}^{N} g_{\alpha\beta}x^{\alpha} \tag{2.6}$$

Differentiating by $\frac{\partial}{\partial x^{\delta}}$ we get

$$\frac{\partial \overline{x}^{\beta}}{\partial x^{\delta}} = \sum_{\alpha=1}^{N} \frac{\partial g_{\beta\alpha}}{\partial x^{\delta}} x^{\alpha} + g_{\beta\delta} \tag{2.7}$$

We may use this in conjuction with equation 2.5 to obtain

$$\begin{aligned} 0 &= \sum_{j=1}^{N}\left(\frac{\partial \overline{x}^k}{\partial x^j} - g_{kj}\right)x^j - \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial \overline{x}^i}{\partial x^k} - g_{ik}\right)x^i \\ &= \sum_{j=1}^{N}\frac{\partial \overline{x}^k}{\partial x^j}x^j - \overline{x}^k - \frac{1}{2}\left(\sum_{i=1}^{N}\frac{\partial \overline{x}^i}{\partial x^k}x^i - \overline{x}^k\right) \\ &= \sum_{j=1}^{N}\frac{\partial \overline{x}^k}{\partial x^j}x^j - \frac{1}{2}\left(\sum_{i=1}^{N}\frac{\partial \overline{x}^i}{\partial x^k}x^i + \overline{x}^k\right) \\ &= \sum_{j=1}^{N}\frac{\partial \overline{x}^k}{\partial x^j}x^j - \frac{1}{2}\frac{\partial\left(\sum_{i=1}^{N} x^i\overline{x}^i\right)}{\partial x^k} \end{aligned}$$

Now, by equations 2.4 and 2.6 we have

$$\sum_{i=1}^{N} x^i\overline{x}^i = \sum_{i=1}^{N}(x^i)^2 \Rightarrow 0 = \sum_{j=1}^{N}\frac{\partial \overline{x}^k}{\partial x^j}x^j - x^k = \sum_{j=1}^{N}\frac{\partial(\overline{x}^k - x^k)}{\partial x^j}x^j$$

Which implies, for any geodesic $\gamma(t)$,

$$\frac{\mathrm{d}[\overline{x}^k - x^k](\gamma(t))}{\mathrm{d}t} = 0$$

Since, at the base point p, it is obvious that $\overline{x}^k(p) = x^k(p)$, we finally arrive at the Gauss Lemma

$$x^k = \sum_{j=1}^{N} g_{kj} x^j \tag{2.8}$$

2.2 Derivatives of the Gauss Lemma

The beauty of equation 2.8 is that it easily allows us to derive dependencies of the n^{th} order partials of g_{ij} at the base point. As we shall see, this will greatly simplify our attempts to calculate factors in the Taylor expansion of g_{ij}. We set the base point of our normal coordinate system to be $\mathbf{x} = (0 \ldots 0)$. Differentiating equation 2.8 by $\frac{\partial}{\partial x^i}$, we get

$$\delta_{ki} = \sum_{j=1}^{N} \frac{\partial g_{kj}}{\partial x^i} x^j + g_{ki}$$

Evaluating at the origin (base point), we get

$$g_{ki}(0) = \delta_{ki} \tag{2.9}$$

a result that is not terribly impressive, since it was a property of normal coordinates that we have already derived. Let us continue the process by differentiating again, this time by $\frac{\partial}{\partial x^l}$. We get

$$0 = \sum_{j=1}^{N} \frac{\partial^2 g_{kj}}{\partial x^i \partial x^l} x^j + \frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^l}$$

Again, we evaluate at the base point to get

$$\left(\frac{\partial g_{kl}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^l} \right) \Bigg|_p = 0 \tag{2.10}$$

We repeat the same process for the next order to get

$$\left(\frac{\partial^2 g_{km}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{ki}}{\partial x^l \partial x^m} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} \right) \Bigg|_p = 0 \tag{2.11}$$

We find that if we wish to obtain equations for the n^{th} order partials of g at the base point, we must differentiate the Gauss Lemma $n+1$ times and then evaluate at the origin. We introduce the useful notation

$$\frac{\partial^n g_{ij}}{\partial x^{i_1} \ldots \partial x^{i_n}} \Bigg|_p = (i_1, \ldots, i_n | i, j)$$

In our notation, equations 2.10 and 2.11 become

$$(i|l,k) + (l|i,k) = 0 \tag{2.12}$$

and

$$(i,l|m,k) + (l,m|i,k) + (m,i|l,k) = 0 \tag{2.13}$$

We have rearranged indices when permissible (using the facts that $g_{ij} = g_{ji}$ and that partial differentials commute) in order to make the pattern apparent. In order to generate an n^{th} order Gauss equation, pick $n+2$ indices. Fix one of these indices; this letter will always index an entry of g; that is, this entry will always stay on the right side of the horizontal bar in our tuple notation. Keep adding succesive lists, cyclically permuting the other $n+1$ indices. For example, a third order Gauss equation would be

$$(a,b,c|d,e) + (b,c,d|a,e) + (c,d,a|b,e) + (d,a,b|c,e) = 0 \tag{2.14}$$

Or rather

$$\left(\frac{\partial^3 g_{de}}{\partial x^a \partial x^b \partial x^c} + \frac{\partial^3 g_{ae}}{\partial x^b \partial x^c \partial x^d} + \frac{\partial^3 g_{be}}{\partial x^c \partial x^d \partial x^a} + \frac{\partial^3 g_{ce}}{\partial x^d \partial x^a \partial x^b} \right) \bigg|_p = 0 \tag{2.15}$$

An n^{th} order Gauss equation is

$$\begin{aligned} (i_1, i_2, \ldots, i_{n-3}, i_{n-2}|i_{n-1}, i_n) &+ (i_2, i_3, \ldots, i_{n-2}, i_{n-1}|i_1, i_n) + \ldots \\ +(i_{n-1}, i_1, \ldots, i_{n-4}, i_{n-3}|i_{n-2}, i_n) &= 0 \end{aligned} \tag{2.16}$$

2.3 Systems of Gauss Equations

Equation 2.13 is not the only second order Gauss equation. We can do a simple permutation of indices to get

$$(i,l|k,m) + (l,k|i,m) + (k,i|l,m) = 0 \tag{2.17}$$

We can find the number of distinct Gauss equations that must exist for each order n. There is one Gauss equation for each index that we choose to be fixed in our tuple notation. We simply cycle through the other indices to get the $n+1$ tuples for an equation. Since there are $n+2$ indices that we may choose to be fixed, there are $n+2$ independent Gauss equations.

Since, for any order, there are $\binom{n+2}{n}$ distinct tuples (from the $n+2$ indices, we must choose the n that we are partially differentiating by), the number of dependent partials is

$$\binom{n+2}{n} - (n+2) = \frac{(n+2)(n-1)}{2}$$

Therefore, for any order, the Gauss Lemma will allow us to find $n+2$ linear equations involving the partial derivatives of g at the origin. These equations can be used to eliminate $\frac{(n+2)(n-1)}{2}$ partials and simplify calculations involving the partial derivatives of g.

Chapter 3

The Orthonormal Frame

3.1 Motivation

Recall our equation for the Christoffel symbols in normal coordinates (Levi-Civita connection)

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\mu}(\partial_i g_{j\mu} + \partial_j g_{i\mu} - \partial_\mu g_{ij}) \tag{3.1}$$

The presence of elements of G^{-1} in formula 3.1 introduces unwanted complications. When we examine higher order covariant derivatives of the curvature, we will obtain formulas involving both partial derivatives of G and partial derivatives of G^{-1}. The Gauss Lemma will not provide us with an easy way of simplifying these terms. The above formula was derived by considering the coordinate frame as the bases for our tangent spaces. A natural question arrises: Is there another frame of bases that will give a more convenient equation for Γ^k_{ij}?

3.2 Non-coordinate Bases

In the coordinate basis, T_pM is spanned by $\{e_j\} = \{\partial/\partial x^j\}$. We introduce the transformation matrix $A = \{a_i{}^j\}$. Then

$$\hat{e_i} = a_i{}^\mu (\partial/\partial x^\mu) \tag{3.2}$$

and the set $\{\hat{e_i}\}$ forms another set of basis vectors at T_pM. If we define A as a function of the coordinates $\mathbf{x}$, then we have a new set of basis vectors locally for each tangent space on the manifold. We would like our new frame to be orthonormal at all points. That is,

$$g(\hat{e_i}, \hat{e_j}) = g(a_i{}^\mu e_\mu, a_j{}^\nu e_\nu) = a_i{}^\mu a_j{}^\nu g_{\mu\nu} = \delta_{ij} \tag{3.3}$$

This equation can be rearranged

$$a_i{}^\mu g_{\mu\nu}(a_\nu{}^j)^T = \delta_{ij} \Leftrightarrow AGA^T = I \tag{3.4}$$

We would like to solve for the matrix A in terms of the metric G. We choose A to be symmetric ($A = A^T$) to simplify the calculation. We get the matrix solution $A = G^{-\frac{1}{2}}$. We can expand this solution in a power series around $G = I$

$$A = G^{-\frac{1}{2}} = I + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \ldots (2n-1)}{n! \cdot 2^n} \widetilde{G}^n \tag{3.5}$$

where $\widetilde{G} = G - I$. In terms of components, the first few terms are

$$a_i{}^j = \delta_{ij} - \frac{1}{2}\widetilde{g}_{ij} + \frac{3}{8}\sum_{k=1}^{N} \widetilde{g}_{ik}\,\widetilde{g}_{kj} - \ldots \tag{3.6}$$

It should be noted that any transformation matrix B may be obtained by multiplying our symmetric tranformation matrix A by a suitable orthogonal rotation $C \in O(N)$.

3.3 The Lie Bracket and Christoffel Symbols

Recall the definition of the Lie Bracket of two vector fields in coordinates

$$[X, Y] = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) e_\nu \tag{3.7}$$

and that the Lie bracket of two coordinate basis vectors is zero. This, however, is not the case in the orthonormal frame. Indeed, a simple computation will show that for two basis vectors $\hat{e}_i, \hat{e}_j \in T_pM$

$$[\hat{e}_i, \hat{e}_j]\Big|_p \equiv c^\gamma_{ij}(p)\hat{e}_\gamma\Big|_p = a^\gamma{}_\nu \left(a_i{}^\mu \partial_\mu a_j{}^\nu - a_j{}^\mu \partial_\mu a_i{}^\nu\right)(p)\ \hat{e}_\gamma \tag{3.8}$$

We define the Christoffel symbols as before, with respect to a connection

$$\nabla_i \hat{e}_j = \nabla_{\hat{e}_i} \hat{e}_j = \widehat{\Gamma}^k_{ij} \hat{e}_k \tag{3.9}$$

If we take the inner product of both sides of this equation with $\hat{e}_k$, it is apparent that

$$\widehat{\Gamma}^k_{ij} = g(\nabla_i \hat{e}_j, \hat{e}_k) \tag{3.10}$$

3.4 Torsion, Curvature, and the Levi-Civita Connection

3.4.1 Torsion and Curvature

We introduce the dual basis $\{\hat{\theta}^i\}$ defined by $\langle \hat{\theta}^i, \hat{e}_j \rangle = \delta_{ij}$.

Recall the definitions of the torsion tensor T and curvature tensor R given in chapter 1

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{3.11}$$

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.12}$$

Once again, in components

$$\begin{aligned} T^i{}_{jk} &= \langle \hat{\theta}^i, T(\hat{e}_j, \hat{e}_k) \rangle \\ &= \langle \hat{\theta}^i, \nabla_j \hat{e}_k - \nabla_k \hat{e}_j - [\hat{e}_j, \hat{e}_k] \rangle \\ &= \widehat{\Gamma}^i_{jk} - \widehat{\Gamma}^i_{kj} - c^i_{jk} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} R^i{}_{jkl} &= \langle \hat{\theta}^i, \nabla_k \nabla_l \hat{e}_j - \nabla_l \nabla_k \hat{e}_j - \nabla_{[\hat{e}_k, \hat{e}_l]} \hat{e}_j \rangle \\ &= \langle \hat{\theta}^i, \nabla_k (\widehat{\Gamma}^\epsilon_{lj} \hat{e}_\epsilon) - \nabla_l (\widehat{\Gamma}^\epsilon_{kj} \hat{e}_\epsilon) - c^\epsilon_{kl} \nabla_\epsilon \hat{e}_j \rangle \\ &= \partial_k \widehat{\Gamma}^i_{lj} - \partial_l \widehat{\Gamma}^i_{kj} + \widehat{\Gamma}^\epsilon_{lj} \widehat{\Gamma}^i_{k\epsilon} - \widehat{\Gamma}^\epsilon_{kj} \widehat{\Gamma}^i_{l\epsilon} - c^\epsilon_{kl} \widehat{\Gamma}^i_{\epsilon j} \end{aligned} \tag{3.14}$$

3.4.2 The Levi-Civita Connection

The Levi-Civita connection is defined as respecting the metric and being torsion free. Referring back to equation 3.13 and 3.14 we see that the torsion free condition requires

$$c^k_{ij} = \widehat{\Gamma}^k_{ij} - \widehat{\Gamma}^k_{ji} \Rightarrow [\hat{e}_i, \hat{e}_j] = \nabla_i \hat{e}_j - \nabla_j \hat{e}_i \tag{3.15}$$

and

$$R^i{}_{jkl} = \partial_k \widehat{\Gamma}^i_{lj} - \partial_l \widehat{\Gamma}^i_{kj} + \widehat{\Gamma}^\epsilon_{lj} \widehat{\Gamma}^i_{k\epsilon} - \widehat{\Gamma}^\epsilon_{kj} \widehat{\Gamma}^i_{l\epsilon} - (\widehat{\Gamma}^\epsilon_{kl} - \widehat{\Gamma}^\epsilon_{lk}) \widehat{\Gamma}^i_{\epsilon j} \tag{3.16}$$

Also recall from chapter 1 that a metric connection demands

$$\begin{aligned} 0 &= (\nabla_i g)(\hat{e}_j, \hat{e}_k) = \partial_i g(\hat{e}_j, \hat{e}_k) - \widehat{\Gamma}^\mu_{ij} g(\hat{e}_\mu, \hat{e}_k) - \widehat{\Gamma}^\mu_{ik} g(\hat{e}_j, \hat{e}_\mu) \\ \Longrightarrow &\quad \partial_i g(\hat{e}_j, \hat{e}_k) = g(\nabla_i \hat{e}_j, \hat{e}_k) + g(\hat{e}_j, \nabla_i \hat{e}_k) \end{aligned} \tag{3.17}$$

It is straightforward to permute indices twice on the above equation, add, and determine

$$\begin{aligned} \widehat{\Gamma}^k_{ij} &= g(\nabla_i \hat{e}_j, \hat{e}_k) = \frac{1}{2}[\partial_i g(\hat{e}_j, \hat{e}_k) + \partial_j g(\hat{e}_i, \hat{e}_k) - \partial_k g(\hat{e}_i, \hat{e}_j) + \\ &\quad g([\hat{e}_i, \hat{e}_j], \hat{e}_k) - g([\hat{e}_j, \hat{e}_k], \hat{e}_i) + g([\hat{e}_k, \hat{e}_i], \hat{e}_j)] \end{aligned}$$

However, since $\partial_i g(\hat{e}_j, \hat{e}_k) = \partial_i \delta_{jk} = 0$,

$$\widehat{\Gamma}^k_{ij} = \frac{1}{2}\Big\{ g([\hat{e}_i, \hat{e}_j], \hat{e}_k) - g([\hat{e}_j, \hat{e}_k], \hat{e}_i) + g([\hat{e}_k, \hat{e}_i], \hat{e}_j) \Big\} \tag{3.18}$$

3.4.3 The Christoffel Symbols Computed

If we write $\hat{e}_\gamma = a_\gamma{}^\nu e_\nu$ equation 3.8 becomes

$$[\hat{e}_i, \hat{e}_j] = \left(a_i{}^\mu \partial_\mu a_j{}^\nu - a_j{}^\mu \partial_\mu a_i{}^\nu\right)\ e_\nu \tag{3.19}$$

Plugging into equation 3.18, we get

$$\begin{aligned}
\widehat{\Gamma}^k_{ij} &= \frac{1}{2}\Big\{g([\hat{e}_i, \hat{e}_j], \hat{e}_k) - g([\hat{e}_j, \hat{e}_k], \hat{e}_i) + g([\hat{e}_k, \hat{e}_i], \hat{e}_j)\Big\} \\
&= \frac{1}{2}\Big\{g\big(\left(a_i{}^\mu \partial_\mu a_j{}^\nu - a_j{}^\mu \partial_\mu a_i{}^\nu\right)\ e_\nu, a_k{}^\rho e_\rho\big) \\
&\quad -g\big(\left(a_j{}^\mu \partial_\mu a_k{}^\nu - a_k{}^\mu \partial_\mu a_j{}^\nu\right)\ e_\nu, a_i{}^\rho e_\rho\big) \\
&\quad +g\big(\left(a_k{}^\mu \partial_\mu a_i{}^\nu - a_i{}^\mu \partial_\mu a_k{}^\nu\right)\ e_\nu, a_j{}^\rho e_\rho\big)\Big\}
\end{aligned} \tag{3.20}$$

We use the linearity of g to derive an equation that will be of crucial importance to us

$$\begin{aligned}
\widehat{\Gamma}^k_{ij} &= \frac{1}{2}\Big\{a_i{}^\mu a_k{}^\rho \left(\partial_\mu a_j{}^\nu\right) g_{\nu\rho} - a_j{}^\mu a_k{}^\rho \left(\partial_\mu a_i{}^\nu\right) g_{\nu\rho} \\
&\quad -a_j{}^\mu a_i{}^\rho \left(\partial_\mu a_k{}^\nu\right) g_{\nu\rho} + a_k{}^\mu a_i{}^\rho \left(\partial_\mu a_j{}^\nu\right) g_{\nu\rho} \\
&\quad +a_k{}^\mu a_j{}^\rho \left(\partial_\mu a_i{}^\nu\right) g_{\nu\rho} - a_i{}^\mu a_j{}^\rho \left(\partial_\mu a_k{}^\nu\right) g_{\nu\rho}\Big\}.
\end{aligned} \tag{3.21}$$

Chapter 4

Low-Order Terms of the Expansion

We can expand the components of G in a Taylor series around the origin, p, as

$$g_{ij} = g_{ij}\Big|_p + \partial_k g_{ij}\Big|_p x^k + \frac{1}{2}\partial_l\partial_k g_{ij}\Big|_p x^k x^l + \frac{1}{6}\partial_m\partial_l\partial_k g_{ij}\Big|_p x^k x^l x^m + \cdots . \quad (4.1)$$

In normal coordinates, an explicit computation of the low-order terms is both interesting and feasible. From earlier

$$g_{ij}\Big|_p = \delta ij \quad (4.2)$$

$$\partial_k g_{ij}\Big|_p x^k = 0 \quad (4.3)$$

We will show

$$\frac{1}{2}\partial_l\partial_k g_{ij}\Big|_p x^k x^l = -\frac{1}{3}R_{ikjl}\Big|_p x^k x^l \quad (4.4)$$

$$\frac{1}{6}\partial_m\partial_l\partial_k g_{ij}\Big|_p x^k x^l x^m = -\frac{1}{6}\nabla_m R_{ikjl}\Big|_p x^k x^l x^m. \quad (4.5)$$

The computations follow a pattern whose generalizations we will explore in the final episode of this epic.

4.1 The Second-Order Term

To derive 4.4 we follow a soon-to-be-familiar scheme. We first examine the components of the curvature tensor at the origin and write them in terms of the Christoffel symbols at the origin. We then use nice properties of the ${a_i}^j$

to write the Christoffel symbols at the origin in terms of partial derivatives of g at the origin. Putting everything together, we look at the right-hand side of 4.4 and combine terms to write, for each particular choice of k and l, the coeffecient of $x^k x^l$ as a linear combination of partial derivatives of g (all evaluated at the origin). Finally, a derivative of the Gauss Lemma shows how the afforementioned linear combination can be rewritten as a multiple of a single partial derivative of g_{ij}.

4.1.1 The Curvature Tensor at the Origin

From Chapter 3, we have

$$R^i{}_{kjl} = \partial_j \widehat{\Gamma}^i_{lk} - \partial_l \widehat{\Gamma}^i_{jk} + \widehat{\Gamma}^\epsilon_{lk}\widehat{\Gamma}^i_{j\epsilon} - \widehat{\Gamma}^\epsilon_{jk}\widehat{\Gamma}^i_{l\epsilon} - c^\epsilon_{jl}\widehat{\Gamma}^i_{\epsilon k}$$

and in normal coordinates, the Christoffel symbols are

$$\begin{aligned} \widehat{\Gamma}^k_{ij} \;=\; & \frac{1}{2}\Big\{ a_i{}^\mu a_k{}^\rho \left(\partial_\mu a_j{}^\nu\right) g_{\nu\rho} - a_j{}^\mu a_k{}^\rho \left(\partial_\mu a_i{}^\nu\right) g_{\nu\rho} \\ & - a_j{}^\mu a_i{}^\rho \left(\partial_\mu a_k{}^\nu\right) g_{\nu\rho} + a_k{}^\mu a_i{}^\rho \left(\partial_\mu a_j{}^\nu\right) g_{\nu\rho} \\ & + a_k{}^\mu a_j{}^\rho \left(\partial_\mu a_i{}^\nu\right) g_{\nu\rho} - a_i{}^\mu a_j{}^\rho \left(\partial_\mu a_k{}^\nu\right) g_{\nu\rho}\Big\}. \end{aligned} \tag{4.6}$$

Now taking a partial derivative of 3.6 gives

$$\partial_t a_i{}^j = -\frac{1}{2}\partial_t \widetilde{g}_{ij} + \frac{3}{8}\sum_{k=1}^{N}\big(\left(\partial_t \widetilde{g}_{ik}\right)\widetilde{g}_{kj} + \widetilde{g}_{ik}\left(\partial_t \widetilde{g}_{kj}\right)\big) - \ldots .$$

All of the $\widetilde{g}_{uv}$ are zero at the origin, so the above reduces to

$$\partial_t a_i{}^j\Big|_p = -\frac{1}{2}\partial_t \widetilde{g}_{ij}\Big|_p = -\frac{1}{2}\partial_t g_{ij}\Big|_p$$

since $\widetilde{G} = G - I$. But our computation in Chapter 1 showed each $\partial_t g_{ij}\Big|_p = 0$, so we have for all i, j, k

$$\widehat{\Gamma}^k_{ij}\Big|_p = 0$$

and hence

$$R^i{}_{kjl}\Big|_p = \partial_j \widehat{\Gamma}^i_{lk}\Big|_p - \partial_l \widehat{\Gamma}^i_{jk}\Big|_p .$$

Finally, since G is the identity at the origin, we have

$$\begin{aligned} R_{ikjl}\Big|_p &= g_{i\alpha}\Big|_p R^\alpha{}_{kjl}\Big|_p = R^i{}_{kjl}\Big|_p \\ &= \partial_j \widehat{\Gamma}^i_{lk}\Big|_p - \partial_l \widehat{\Gamma}^i_{jk}\Big|_p \end{aligned} \tag{4.7}$$

4.1.2 First-Order Partials of the Christoffel Symbols

The above reduces the components of R to derivatives of Christoffel symbols; we compute these now. Though a derivative of the symbols would involve much painful application of the product rule, we can ignore this when evaluating at the origin since, as computed above,

$$\partial_t a_i{}^j\Big|_p = -\frac{1}{2}\partial_t g_{ij}\Big|_p = 0.$$

Recalling that the matrices G and A are both the identity at the origin, we use 3.21 to write $\partial_l \widehat{\Gamma}^k_{ij}$ at the origin as

$$\begin{aligned}\partial_l \widehat{\Gamma}^k_{ij}\Big|_p &= \frac{1}{2}\Big\{\delta_i{}^\mu \delta_k{}^\rho \left(\partial_\mu a_j{}^\nu\right)\delta_{\nu\rho} - \delta_j{}^\mu \delta_k{}^\rho \left(\partial_\mu a_i{}^\nu\right)\delta_{\nu\rho} \\ &\quad -\delta_j{}^\mu \delta_i{}^\rho \left(\partial_\mu a_k{}^\nu\right)\delta_{\nu\rho} + \delta_k{}^\mu \delta_i{}^\rho \left(\partial_\mu a_j{}^\nu\right)\delta_{\nu\rho} \\ &\quad +\delta_k{}^\mu \delta_j{}^\rho \left(\partial_\mu a_i{}^\nu\right)\delta_{\nu\rho} - \delta_i{}^\mu \delta_j{}^\rho \left(\partial_\mu a_k{}^\nu\right)\delta_{\nu\rho}\}\end{aligned}$$

which simplifies to

$$\partial_l \widehat{\Gamma}^k_{ij}\Big|_p = \partial_l \partial_k a_i{}^j - \partial_l \partial_j a_i{}^k. \tag{4.8}$$

Returning again to 3.6, we differentiate twice to find a relationship between $\partial_l \partial_k a_i{}^j$ and $\partial_l \partial_k g_{ij}$ at the origin. Differentiating twice could be unpleasant, but remembering that the first partials of g (and hence a) are zero at p, we can skip straight to evaluating at the origin and find

$$\partial_l \partial_k a_i{}^j\Big|_p = -\frac{1}{2}\partial_l \partial_k g_{ij}\Big|_p. \tag{4.9}$$

Hence 4.8 can be written in tuple notation as

$$\partial_l \widehat{\Gamma}^k_{ij} = -\frac{1}{2}\big((l,k|i,j) - (l,j|i,k)\big). \tag{4.10}$$

4.1.3 Applying the Gauss Lemma

From 4.7, we find the coefficient of $x^k x^l$ in $R_{ikjl}\Big|_p x^k x^l$ to be

$$\begin{aligned}R_{ikjl}\Big|_p + R_{iljk}\Big|_p &= \left(\partial_j \widehat{\Gamma}^i_{lk} - \partial_l \widehat{\Gamma}^i_{jk}\right) + \left(\partial_j \widehat{\Gamma}^i_{kl} - \partial_k \widehat{\Gamma}^i_{jl}\right) \\ &= -\frac{1}{2}\big((j,i|l,k) - (j,k|l,i)\big) + \frac{1}{2}\big((l,i|j,k) - (l,k|j,i)\big) \\ &\quad +\frac{1}{2}\big((j,i|k,l) - (j,l|k,i)\big) - \frac{1}{2}\big((k,i|j,l) - (k,l|j,i)\big)\end{aligned}$$

which simplifies to

$$R_{kilj}\Big|_p + R_{likj}\Big|_p = -\frac{1}{2}\big\{2(k,l,|i,j) - \big((j,k|l,i) + (j,l|k,i)\big) - \big((i,k|l,j) + (i,l|k,j)\big) + 2(i,j|k,l)\big\} \quad (4.11)$$

Finally, the second-order Gauss Lemma yields the following equations:

$$(j,k|l,i) + (l,j|k,i) + (k,l|j,i) = 0 \quad (4.12)$$
$$(i,k|l,j) + (l,i|k,j) + (k,l|i,j) = 0 \quad (4.13)$$
$$(i,j|l,k) + (l,i|j,k) + (j,l|i,k) = 0 \quad (4.14)$$
$$(i,j|k,l) + (k,i|j,l) + (j,k|i,l) = 0 \quad (4.15)$$

Subtracting 4.12 and 4.13 from the sum of 4.14 and 4.15 shows that

$$2(i,j|k,l) = 2(k,l|i,j) \quad (4.16)$$

Applying 4.12 and 4.13 to the second and third terms of 4.11, respectively, together with 4.16 gives

$$R_{kilj} + R_{likj} = -3(k,l|i,j)$$

which shows that equation 4.4 holds term by term.

4.2 The Third-Order Term

In computing the third-order term, 4.5, we follow a program similar to that for the second-order term, noting with relief that the derivatives of a and the Christoffel symbols remain relatively simple (at the origin) since the first partials of g are zero at the origin. We begin by examining the first covariant derivative of R at the origin.

4.2.1 The First Covariant Derivative of the Curvature Tensor

We calculate the covariant derivative of the curvature tensor evaluated at our base point p.

$$\begin{aligned}\nabla_m R_{ijkl} = \nabla_m R^i{}_{jkl} &= \langle \hat{\theta}^i, \nabla_m \nabla_k \nabla_l \hat{e}_j - \nabla_m \nabla_l \nabla_k \hat{e}_j - \nabla_m \nabla_{[\hat{e}_k, \hat{e}_l]} \hat{e}_j \rangle \\ &= \langle \hat{\theta}^i, \nabla_m \nabla_k (\widehat{\Gamma}^{\epsilon}_{lj} \hat{e}_\epsilon) - \nabla_m \nabla_l (\widehat{\Gamma}^{\epsilon}_{kj} \hat{e}_\epsilon) - c^{\epsilon}_{kl} \nabla_\epsilon \hat{e}_j \rangle\end{aligned}$$

Now, if we evaluate this expression at p where $\mathbf{x} = 0$, all of the Christoffel symbols and c's are zero. Therefore, in the above equation, we may only concern ourselves with terms that contain no Γ's or c's

$$
\begin{aligned}
\nabla_m R_{ijkl}\Big|_p = \nabla_m R^i{}_{jkl}\Big|_p &= \langle \hat{\theta}^i, \nabla_m \nabla_k (\widehat{\Gamma}^{\epsilon}_{lj}\hat{e}_\epsilon) - \nabla_m \nabla_l (\widehat{\Gamma}^{\epsilon}_{kj}\hat{e}_\epsilon)\rangle\Big|_p \\
&= \langle \hat{\theta}^i, (\partial_m \partial_k \widehat{\Gamma}^{\epsilon}_{lj})\hat{e}_\epsilon - (\partial_m \partial_l \widehat{\Gamma}^{\epsilon}_{kj})\hat{e}_\epsilon\rangle\Big|_p \\
&= \partial_m \partial_k \widehat{\Gamma}^i_{lj}\Big|_p - \partial_m \partial_l \widehat{\Gamma}^i_{kj}\Big|_p \\
&= \partial_m R_{ijkl}\Big|_p \qquad (4.17)
\end{aligned}
$$

4.2.2 Second-Order Partials of the Christoffel Symbols

Because the terms $\widehat{\Gamma}^k_{ij}$ involve products of the form $a_i{}^\mu a_k{}^\rho \left(\partial_\mu a_j{}^\nu\right) g_{\nu\rho}$, it is relatively easy to compute second-order partial derivatives of the $\widehat{\Gamma}^k_{ij}$ evaluated at the origin. Recalling 4.9

$$
\partial_q \partial_r a_s{}^t\Big|_p = -\frac{1}{2}\partial_q \partial_r g_s{}^t\Big|_p
$$

we have

$$
\begin{aligned}
\partial_m \partial_l \widehat{\Gamma}^k_{ij}\Big|_p &= \partial_m \partial_l \partial_k a_i{}^j - \partial_m \partial_l \partial_j a_i{}^k\Big|_p \\
&= -\frac{1}{2}\left\{(m,l,k,|i,j) - (m,l,j|i,k)\right\}
\end{aligned}
$$

The simplicity of this result depends crucially on having relatively few derivatives to apply to the several products in the $\widehat{\Gamma}^k_{ij}$, so that the product rule for differentiation does not complicate things. The situation changes severely though when we consider higher order derivatives of the $\widehat{\Gamma}^k_{ij}$.

4.2.3 Applying the Gauss Lemma (again)

At the origin we have

$$
\begin{aligned}
\nabla_m R_{ikjl} = \partial_m R_{ikjl} &= \partial_m \partial_j \widehat{\Gamma}^i_{lk} - \partial_m \partial_l \widehat{\Gamma}^i_{jk} \qquad (4.18) \\
&= -\frac{1}{2}\big((m,j,i|l,k) - (m,j,k|l,i)\big) \qquad (4.19) \\
&\quad +\frac{1}{2}\big((m,l,i|j,k) - (m,l,k|j,i)\big) \qquad (4.20)
\end{aligned}
$$

To show equation 4.5, we combine terms and consider the coeffecient of $x^k x^l x^m$ for a particular choice of k, l, and m. This coeffecient is

$$
\frac{1}{6}\left\{\nabla_m R_{ikjl} + \nabla_m R_{iljk} + \nabla_l R_{ikjm} + \nabla_l R_{imjk} + \nabla_k R_{iljm} + \nabla_k R_{imjl}\right\} \qquad (4.21)
$$

Using 4.18, the above simplifies to

$$
\begin{aligned}
-\frac{1}{12}\Big\{6(m,l,k|i,j) &- 2\big((m,l,j|k,i)+(m,k,j|l,i)+(l,k,j|m,i)\big)\\
&- 2\big((m,l,i|k,j)+(m,k,i|l,j)+(l,k,i|m,j)\big)\\
&+ 2\big((m,i,j|l,k)+(l,i,j|l,k)+(k,i,j|l,m)\big)\Big\} \quad (4.22)
\end{aligned}
$$

and we can use the third-order Gauss Lemma to reduce each of the lines above to a single multiple of $(m,l,k|i,j)$ as follows. For third-order, the Gauss Lemma yeilds five independent equations. They are

$$
\begin{aligned}
(j,k,l|m,i)+(k,l,m|j,i)+(l,m,j|k,i)+(m,j,k|l,i) &= 0 \quad (4.23)\\
(l,k,i|m,j)+(k,i,m|l,j)+(i,m,l|k,j)+(m,k,l|i,j) &= 0 \quad (4.24)\\
(m,l,i|j,k)+(l,i,j|m,k)+(i,j,m|l,k)+(j,m,l|i,k) &= 0 \quad (4.25)\\
(m,k,i|j,l)+(k,i,j|m,l)+(i,j,m|k,l)+(j,m,k|i,l) &= 0 \quad (4.26)\\
(i,j,k|l,m)+(j,k,l|i,m)+(k,l,i|j,m)+(l,i,j|k,m) &= 0 \quad (4.27)
\end{aligned}
$$

We can immediately apply 4.23 and 4.24 to the second and third terms of 4.22, respectively. Subtracting 4.23 and 4.24 from the sum of 4.25, 4.26, and 4.27 shows how the last term of 4.22 reduces to a single multiple of $(m,l,k|i,j)$. So in the end we have the coeffecient 4.22 as $-(m,l,k|i,j)$, and hence equation 4.5.

So for $\mathbf{x}$ in a neighborhood of p, then, we have

$$g_{ij}(\mathbf{x}) = \delta_{ij} - \frac{1}{3}R_{ikjl}\Big|_p x^k x^l - \frac{1}{6}\nabla_m R_{ikjl}\Big|_p x^k x^l x^m + \cdots$$

4.3 Difficulties in Calculating the Fourth-Order Term

There are several factors which preclude the fourth-order term from being calculated simply. For the third-order term, we had that $\nabla_m R_{ijkl}|_p = \partial_m R_{ijkl}|_p$ which was a function of the second partials of $\widehat{\Gamma}$'s. It is no longer true that $\nabla_r \nabla_m R_{ijkl}|_p = \partial_r \partial_m R_{ijkl}|_p$ and the formula for $\partial_r \partial_m R_{ijkl}|_p$ involves terms that are not only the third order partials of the $\widehat{\Gamma}$'s, but also products of second and first order partials of the $\widehat{\Gamma}$'s. To say the least, this complicates calculation.

In addition, analytic expressions for the third order partials of the $\widehat{\Gamma}$'s become increasingly complicated. Recall

$$
\begin{aligned}
\widehat{\Gamma}^k_{ij} = \frac{1}{2}\Big\{ & a_i{}^\mu a_k{}^\rho \left(\partial_\mu a_j{}^\nu\right) g_{\nu\rho} - a_j{}^\mu a_k{}^\rho \left(\partial_\mu a_i{}^\nu\right) g_{\nu\rho}\\
& - a_j{}^\mu a_i{}^\rho \left(\partial_\mu a_k{}^\nu\right) g_{\nu\rho} + a_k{}^\mu a_i{}^\rho \left(\partial_\mu a_j{}^\nu\right) g_{\nu\rho}\\
& + a_k{}^\mu a_j{}^\rho \left(\partial_\mu a_i{}^\nu\right) g_{\nu\rho} - a_i{}^\mu a_j{}^\rho \left(\partial_\mu a_k{}^\nu\right) g_{\nu\rho}\Big\} \quad (4.28)
\end{aligned}
$$

Each term in the expression for $\widehat{\Gamma}^k_{ij}$ products of two a's, one g and one partial derivative of a. In our previous calculation we needed the second partial of $\widehat{\Gamma}^k_{ij}$.

When we twice differentiated each term, because $\partial_i {a_j}^k = \partial g_{jk} = 0$, the only terms that survived from the product rule were those that placed two partial derivatives on the a with a partial already on it. For a calculation of the fourth-order term, we must differentiate $\widehat{\Gamma}^k_{ij}$ three times. Because the second partials of a and g are not zero, many more terms survive the product rule. Indeed, before simplification, the expression for the third partial of $\widehat{\Gamma}^k_{ij}$ contains the sum of 60 terms. That's not all.

Unlike in previous terms, the fourth partials of a are not directly proportional to the fourth order partials of g. Recall

$$ {a_i}^j = \delta_{ij} - \frac{1}{2}\widetilde{g}_{ij} + \frac{3}{8}\sum_{k=1}^{N} \widetilde{g}_{ik}\,\widetilde{g}_{kj} - \dots $$

which implies

$$ \begin{aligned} \partial_a\partial_b\partial_c\partial_d {a_i}^j\Big|_p &= -\frac{1}{2}\partial_a\partial_b\partial_c\partial_d\widetilde{g}_{ij} \\ &+ \frac{3}{8}\sum_{k=1}^{N}\partial_a\partial_b\widetilde{g}_{ik}\,\partial_c\partial_d\widetilde{g}_{kj} \\ &+ \frac{3}{8}\sum_{k=1}^{N}\partial_a\partial_c\widetilde{g}_{ik}\,\partial_b\partial_d\widetilde{g}_{kj} \\ &+ \frac{3}{8}\sum_{k=1}^{N}\partial_a\partial_d\widetilde{g}_{ik}\,\partial_c\partial_b\widetilde{g}_{kj} \\ &+ \frac{3}{8}\sum_{k=1}^{N}\partial_c\partial_d\widetilde{g}_{ik}\,\partial_a\partial_b\widetilde{g}_{kj} \\ &+ \frac{3}{8}\sum_{k=1}^{N}\partial_b\partial_d\widetilde{g}_{ik}\,\partial_a\partial_c\widetilde{g}_{kj} \\ &+ \frac{3}{8}\sum_{k=1}^{N}\partial_b\partial_c\widetilde{g}_{ik}\,\partial_a\partial_d\widetilde{g}_{kj} \end{aligned} \tag{4.29} $$

Ugly indeed.

These three factors render calculating the fourth order term a daunting task. Fortunately, explicit computation of the fourth order term is of little consequence. We have developed a sound method for calculating terms in the Taylor expansion, assuming that term in the Taylor expansion may be written as a function of the curvature and covariant derivatives of the curvature. Computing higher order terms in the expansion would be a bothersome, but finite, task. This all assumes, however, that we can 'back-calculate'–reducing a covariant derivative of R to a single partial derivative of g. We now seek an inductive proof that will show that it is indeed possible to write the n^{th} term in the Taylor expansion of the Riemannian metric in terms of covariant derivatives of the curvature.

Chapter 5

The n^{th} Order Term

We give an inductive proof that, $\forall n \geq 2$,

$$\partial_{k_n} \cdots \partial_{k_1} \; g_{ij} x^{k_n} \cdots x^{k_1} \;=\; C\Bigg(\nabla_{k_n} \cdots \nabla_{k_3} R_{ik_2jk_1} + \begin{array}{c}\text{a polynomial in lower order}\\ \text{covariant derivatives of } R\end{array}\Bigg) x^{k_n} \cdots x^{k_1} \tag{5.1}$$

where C is a scalar constant.

Chapter 4 showed that the above is valid for our base step $n = 2$, and for $n = 3$. For our proof, we assume the statement is true for all integers less than n and show that it is true for n.

We accomplish this by showing

$$\nabla_{k_n} \cdots \nabla_{k_3} R_{ik_2jk_1} \; x^{k_n} \cdots x^{k_1} \;=\; \frac{1}{C}\Bigg(\partial_{k_n} \cdots \partial_{k_1} g_{ij} + \begin{array}{c}\text{a polynomial in lower}\\ \text{order partials of } g\end{array}\Bigg) x^{k_n} \cdots x^{k_1}. \tag{5.2}$$

Using the inductive hypothesis, we write "a polynomial in lower order partials of g" as "a polynomial in lower order covariant derivatives of R". By subtracting these terms from both sides we get 5.1. We refer the reader to Appendix A for details of the inductive argument.

5.1 The n^{th} Covariant Derivative of the Curvature Tensor

We begin by searching for the highest-order partials of g appearing in $(\nabla_{k_n} \cdots \nabla_{k_3} R_{ik_2jk_1})$. Using 3.14, we have

$$\nabla_{k_n} \cdots \nabla_{k_3} R_{ik_2jk_1}\Big|_p \;=\; \langle \hat{\theta}^i, \nabla_{k_n} \cdots \nabla_{k_3} \nabla_j \nabla_{k_1} \hat{e}_{k_2} \rangle \Big|_p$$

$$
\begin{aligned}
& -\langle \hat{\theta}^i, \nabla_{k_n} \cdots \nabla_{k_3} \nabla_{k_1} \nabla_j \hat{e}_{k_2} \rangle \Big|_p \\
& -\langle \hat{\theta}^i, \nabla_{k_n} \cdots \nabla_{k_3} (c^{\epsilon}_{jk_1} \nabla_{\epsilon} \hat{e}_{k_2}) \rangle \Big|_p \\
= \; & \partial_{k_n} \cdots \partial_{k_3} \partial_j \widehat{\Gamma}^i_{k_1 k_2} \Big|_p - \partial_{k_n} \cdots \partial_{k_3} \partial_{k_1} \widehat{\Gamma}^i_{jk_2} \Big|_p \\
& + \begin{array}{c} \text{a polynomial in the lower} \\ \text{order partials of the } \widehat{\Gamma}\text{'s} \end{array}
\end{aligned} \tag{5.3}
$$

Recalling equation 3.21, we get

$$
\begin{aligned}
\partial_{k_n} \cdots \partial_{k_3} \partial_j \widehat{\Gamma}^i_{k_1 k_2} \Big|_p = \; & \frac{1}{2} \Big\{ \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_{k_1} a_{k_2}{}^{i} \\
& - \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_{k_2} a_{k_1}{}^{i} \\
& - \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_{k_2} a_{i}{}^{k_1} \\
& + \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_i a_{k_2}{}^{k_1} \\
& + \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_i a_{k_1}{}^{k_2} \\
& - \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_{k_1} a_{i}{}^{k_2} \Big\} \Big|_p \\
& + \begin{array}{c} \text{a polynomial in lower} \\ \text{order partials of } a \end{array} \\
= \; & - \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_{k_2} a^i_{k_1} \Big|_p \\
& + \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_i a_{k_2}{}^{k_1} \Big|_p \\
& + \begin{array}{c} \text{a polynomial in lower} \\ \text{order partials of } a \end{array}
\end{aligned} \tag{5.4}
$$

since A is symmetric. Therefore

$$
\begin{aligned}
\nabla_{k_n} \cdots \nabla_{k_3} R_{ik_2jk_1} \Big|_p = \; & - \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_{k_2} a^i_{k_1} \Big|_p \\
& + \partial_{k_n} \cdots \partial_{k_3} \partial_j \partial_i a_{k_2}{}^{k_1} \Big|_p \\
& + \partial_{k_n} \cdots \partial_{k_3} \partial_{k_1} \partial_{k_2} a^j_i \Big|_p \\
& - \partial_{k_n} \cdots \partial_{k_3} \partial_{k_1} \partial_i a_{j}{}^{k_2} \Big|_p \\
& + \begin{array}{c} \text{a polynomial in lower} \\ \text{order partials of } a\text{'s} \end{array}
\end{aligned} \tag{5.5}
$$

Now recalling 3.6, we have

$$\partial_{k_n}\cdots\partial_{k_1}a_i{}^j\Big|_p = \left(-\frac{1}{2}\partial_{k_n}\cdots\partial_{k_1}g_{ij} + \begin{array}{c}\text{a polynomial in lower}\\ \text{order partials of } g\end{array}\right)\Big|_p \tag{5.6}$$

Hence

$$\begin{aligned}\nabla_{k_n}\cdots\nabla_{k_3}R_{ik_2jk_1}\Big|_p \quad = \quad & \frac{1}{2}\big(\partial_{k_n}\cdots\partial_{k_3}\partial_j\partial_{k_2}g_{k_1i}\\ & -\partial_{k_n}\cdots\partial_{k_3}\partial_j\partial_i g_{k_1k_2}\\ & -\partial_{k_n}\cdots\partial_{k_3}\partial_{k_1}\partial_{k_2}g_{ij}\\ & +\partial_{k_n}\cdots\partial_{k_3}\partial_{k_1}\partial_i g_{jk_2}\big)\Big|_p\\ & +\begin{array}{c}\text{a polynomial in lower}\\ \text{order partials of } g\end{array}\end{aligned} \tag{5.7}$$

Converting to tuple notation, this becomes

$$\begin{aligned}\nabla_{k_n}\cdots\nabla_{k_3}R_{ik_2jk_1}\Big|_p \quad = \quad & \frac{1}{2}\big((k_n,\ldots,k_3,j,k_2|k_1,i)\\ & -(k_n,\ldots,k_3,j,i|k_1,k_2)\\ & -(k_n,\ldots,k_3,k_1,k_2|i,j)\\ & +(k_n,\ldots,k_3,k_1,i|j,k_2)\big)\\ & +\begin{array}{c}\text{a polynomial in lower}\\ \text{order partials of } g\end{array}\end{aligned} \tag{5.8}$$

5.2 The n^{th} Order Coefficient

We now combine terms on the left-hand side of 5.2 and examine the coeffecient of $x^{k_n}\cdots x^{k_1}$ for a particular choice of indices $k_1,\ldots,k_n$. It is

$$\sum_\sigma \nabla_{k_{\sigma(n)}}\cdots\nabla_{k_{\sigma(3)}}R_{ik_{\sigma(2)}jk_{\sigma(1)}}\Big|_p \tag{5.9}$$

with the summation over all $\sigma\in S_n$. This coeffecient becomes

$$\begin{aligned}-\frac{1}{2}\Big\{n!(k_n,\ldots,k_1|i,j) \quad & - \quad (n-1)!\sum_{l=1}^{n}(k_n,\ldots,\hat{k}_l,\ldots,k_1,j|k_l,i)\\ & - \quad (n-1)!\sum_{l=1}^{n}(k_n,\ldots,\hat{k}_l,\ldots,k_1,i|k_l,j)\\ & + \quad 2(n-2)!\sum_{l_2=1}^{n}\sum_{l_1=1}^{l_2-1}(k_n,\ldots,\hat{k}_{l_1},\ldots,\hat{k}_{l_2},\ldots,k_1,j,i|k_{l_2},k_{l_1})\Big\}\\ & + \quad \begin{array}{c}\text{a polynomial in lower}\\ \text{order partials of } g\end{array}\end{aligned} \tag{5.10}$$

where hatted indices are omitted.

The term

$$n!(k_n, \dots, k_1|i,j) = \sum_{\sigma \in S_n} (k_\sigma(n), \dots, k_\sigma(3), k_\sigma(2), k_\sigma(1)|i,j), \tag{5.11}$$

is from the third term of 5.8, since partial derivatives commute.

Similarly,

$$(n-1)! \sum_{l=1}^{n} (k_n, \dots, \hat{k}_l, \dots, k_1, j|k_l, i) \tag{5.12}$$

appears from the first term of 5.8. This is because for each l there are $(n-1)!$ elements of S_n which fix l and hence $(n-1)!$ terms of the form $(k_n, \dots, \hat{k}_l, \dots, k_1, j|k_l, i)$ in the summation.

In the same way,

$$(n-1)! \sum_{l=1}^{n} (k_n, \dots, \hat{k}_l, \dots, k_1, i|k_l, j) \tag{5.13}$$

is from the fourth term of 5.8.

The second term of 5.8 yields

$$2(n-2)! \sum_{l_2=1}^{n} \sum_{l_1=1}^{l_2-1} (k_n, \dots, \hat{k}_{l_1}, \dots, \hat{k}_{l_2}, \dots, k_1, j, i|k_{l_2}, k_{l_1}). \tag{5.14}$$

This is slightly more complicated, but is still derived by counting permutations. For a particular l_1 and l_2, there are $(n-2)!$ permutations of S_n which fix both, and $(n-2)!$ permutations which transpose them. Hence there are $2(n-2)!$ terms of the form $(k_n, \dots, \hat{k}_{l_1}, \dots, \hat{k}_{l_2}, \dots, k_1, j, i|k_{l_2}, k_{l_1})$, recalling that partial derivatives commute and the tuples are symmetric in the last two indices.

5.3 Applying the n^{th} Order Gauss Lemma

We use the n^{th}-order Gauss Lemma to condense the n^{th} order partials of 5.10 to a single multiple of $(k_n, \dots, k_1|i,j)$. The process we follow is similar to that used at the end of sections 4.1.3 and 4.2.3. Terms 5.12 and 5.13 can be dealt with directly by two of the n^{th}-order Gauss Lemma equations:

$$\begin{aligned} (k_n, \dots, k_3, k_2, j|k_1, i) &+ (k_{n-1} \dots, k_2, j, k_1|k_n, i) + \dots \\ +(j, k_1, k_n, \dots, k_4, k_3|k_2, i) &= -(k_1, k_n, \dots, k_3, k_2|j, i) \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} (k_n, \dots, k_3, k_2, i|k_1, j) &+ (k_{n-1} \dots, k_2, i, k_1|k_n, j) + \dots \\ +(i, k_1, k_n, \dots, k_4, k_3|k_2, j) &= -(k_1, k_n, \dots, k_3, k_2|i, j). \end{aligned} \tag{5.16}$$

Simplification of term 5.14 is more involved, but straightforward. The reader can check that the sum of all $\binom{n+2}{2}$ tuples is zero. Subtracting 5.15 and 5.16 from this sum shows that the sum of all terms that have both i and j to the left of "|" is equal to the term having neither i nor j to the left of "|". That is,

$$\sum_{l_2=1}^{n}\sum_{l_1=1}^{l_2-1}(k_n,\ldots,\hat{k}_{l_1},\ldots,\hat{k}_{l_2},\ldots,k_1,j,i|k_{l_2},k_{l_1}) = (k_n,\ldots,k_1|i,j). \tag{5.17}$$

All together this gives that 5.10 is equal to

$$-\frac{1}{2}[n!+2(n-1)!+2(n-2)!](k_n,\ldots,k_1|i,j) + \begin{matrix}\text{a polynomial in lower}\\ \text{order partials of } g\end{matrix} \tag{5.18}$$

This is equivalent to 5.2 and the inductive proof is complete.

$$C = \frac{-2n!}{n!+2(n-1)!+2(n-2)!}$$

where the $n!$ in the numerator comes from the summation over $x^{k_n}\cdots x^{k_1}$.

5.4 Reflections

We have derived that

$$\partial_{k_n}\cdots\partial_{k_1}g_{ij}\,x^{k_n}\cdots x^{k_1} = \frac{-2n!}{n!+2(n-1)!+2(n-2)!}\Bigg(\nabla_{k_n}\cdots\nabla_{k_3}R_{ik_2jk_1} + \begin{matrix}\text{a polynomial in lower order}\\ \text{covariant derivatives of } R\end{matrix}\Bigg)x^{k_n}\cdots x^{k_1}$$

As a consequence, the curvature determines the metric since the Taylor expansion of the metric may be written entirely in terms of the curvature and covariant derivatives of the curvature.

Appendix A

Inductive Details

Specifically, our inductive hypothesis is that for any $\{l_1, \ldots, l_r\} \subsetneq \{k_1, \ldots, k_n\}$, and any i, j,

$$\partial_{l_r} \cdots \partial_{l_1} g_{ij}\ x^{l_r} \cdots x^{l_1} = \sum_{l_1,\ldots,l_r} \begin{pmatrix} \text{a polynomial in lower order} \\ \text{covariant derivatives of } R \end{pmatrix} x^{l_r} \cdots x^{l_1}. \tag{A.1}$$

Thus for any lower order partial of g,

$$\partial_{l_r} \cdots \partial_{l_1} g_{ij},$$

with $\{l_1, \ldots, l_r\} \subsetneq \{k_1, \ldots, k_n\}$, we write

$$\{m_1, \ldots, m_s\} = \{k_1, \ldots, k_n\} - \{l_1, \ldots, l_n\}$$

and have, by rearranging terms,

$$\sum_{k_1,\ldots,k_n} \partial_{l_r} \cdots \partial_{l_1} g_{ij}\ x^{k_n} \cdots x^{k_1} = \sum_{m_1,\ldots,m_s} \big(\partial_{l_r} \cdots \partial_{l_1} g_{ij}\ x^{l_1} \cdots x^{l_r}\big) x^{m_1} \cdots x^{m_s}$$

where summation over $l_1, \ldots, l_r$ on the right hand side is implicit, as in A.1. We apply A.1 to the above and have

$$\sum_{k_1,\ldots,k_n} \partial_{l_r} \cdots \partial_{l_1} g_{ij}\ x^{k_n} \cdots x^{k_1} = \sum_{k_1,\ldots,k_n} \begin{pmatrix} \text{a polynomial in lower order} \\ \text{covariant derivatives of } R \end{pmatrix} x^{k_1} \cdots x^{k_n}.$$

Hence

$$\sum_{k_1,\ldots,k_n} \begin{pmatrix} \text{a polynomial in lower} \\ \text{order partials of } g \end{pmatrix} x^{k_1} \cdots x^{k_n} = \sum_{k_1,\ldots,k_n} \begin{pmatrix} \text{a polynomial in lower order} \\ \text{covariant derivatives of } R \end{pmatrix} x^{k_1} \cdots x^{k_n}$$

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