The Consecutive Integer Triangle and Exponentials

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The following research is an original topic that is the result of my fascination with mathematics. It explores natural arithmetic and geometric progressions with respect to factorization of integral powers and numbers. These patterns become noticeable through using and analyzing the consecutive integer triangle. The triangle is similar to Pascal's triangle, but lists the natural numbers consecutively as entries. Each row of the triangle corresponds to an integer. For example, row 1 corresponds to the integer 1 and row 2 corresponds to the integer 2 and so on. The basic idea is to fix a certain exponent and vary through the rows of the triangle to identify the patterns. The triangle that I have analyzed may be a triangular slice of a 3- dimensional lattice of numbers where certain relationships can be established between mathematical expressions and the chosen geometry.

Introduction

Many early mathematicians intuitively sensed that numbers are basis for all operations of nature. Philolaus believed that all things known have number. Several early mathematicians used triangles and other polygons to represent special properties of numbers in an attempt to demystify such abstract constructs. Pythagoras played with figurate numbers. Pascal used an arithmetic triangle that bares his namesake. Numbers are truly fascinating and their applications are endless. They are like characters in a story, but lack very developed personal traits. It is very rewarding to connect something so enigmatic as numbers to a physical shape such as a polygon because it is another step in the direction of understanding them. I am contributing my efforts to push the frontiers of understanding numbers further in that direction[1].

One of my greatest accomplishments is my own original research that I have been working on since I was eighteen. I have frequently given myself different mathematical puzzles to solve and a hot afternoon in Mississippi was no exception to the rule. It was the summer of my freshman year and I was at my annual family reunion. I was sitting at a

[1] Boyer, Carl B. and Uta C. Merzbach. The History of Mathematics. New York: John Wiley & Sons, 1991.

long table covered with a white cloth and cluttered with plates and cups; it was obvious that we ate well. My mind was wandering as I partially listened to one of my relatives speak. I just began to doodle by forming triangles by listing successive rows of the natural numbers. For example, the first row has one number (1), the second row has three numbers (234), the third row has five numbers (56789) and so on. After moments of concentration and some arithmetic manipulation, I noticed a pattern. I could represent certain powers of a natural number through geometric and algebraic patterns that existed in the triangle. The remainder of this paper will focus on this very interesting discovery.

Let me provide a diagram of the triangle:
Let R_i denote the i-th row

										1									
R_1																			
R_2									2	3	4								
R_3								5	6	7	8	9							
.							10	11	12	13	14	15	16						
.						17	18	19	20	21	22	23	24	25					
.					26	27	28	29	30	31	32	33	34	35	36				
.				37	38	39	40	41	42	43	44	45	46	47	48	49			
.			50	51	52	53	54	55	56	57	58	59	60	61	62	63	64		
.		65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	
R_{10}	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100

Figure 1. The Consecutive Integer Triangle. This figure illustrates the first ten rows.

Let $R_i(j)$ denote the j-th entry of the i-th row.
Let each row R_i begin with the number

$$R_i(1) = (i - 1)^2 + 1.$$

R_i has

$$2(i – 1) + 1 = 2i – 1$$

terms.
The last term in R_i is

$$R_i(2i-1) = i^2.$$

Let us look at R_3, which is representative of the number 3. The row begins with

$$R_3(1) = (3 - 1)^2 + 1 = 5.$$

It has

$$2*3 – 1 = 5$$

terms and the last entry is

$$R_3(5) = 3^2 = 9.$$

Each row can be built by using the above formulas. I have defined the endpoints and the number of entries of each row. Refer to the triangle in Figure 1 to verify that each row obeys the above formulas.

First Pattern

Look at the sum of the first five rows of the consecutive integer triangle:

				1					=1
			2+	3+	4				=9
		5+	6+	7+	8+	9			=35
	10+	11+	12+	13+	14+	15+	16		=91
17+	18+	19+	20+	21+	22+	23+	24+	25	=189

Row	Sum		Pattern
1	1	=	$1^3 + 0^3$
2	9	=	$2^3 + 1^3$
3	35	=	$3^3 + 2^3$
4	91	=	$4^3 + 3^3$
5	189	=	$5^3 + 4^3$

The first pattern is that the sum of each row is the sum of consecutive cubes.

Proof:

Focus attention on row $R_{i.}$ It has

$$2i - 1$$

terms. Row R_i begins with the number

$$(i - 1)^2 + 1.$$

The sum of row R_i is

$$[(i - 1)^2 + 1] + [(i – 1)^2 + 2] + [(i – 1)^2 + 3] ... + [(i – 1)^2 + 2i – 1].$$

The above expression can be written as

$$[2(i - 1)](i - 1)^2 + \sum_{1}^{2i-1} k$$

Remember that

$$\sum_{1}^{n} k = [n(n + 1)]/2.$$

The sum of the terms in R_i is

$$(2i - 1)(i - 1)^2 + [(2i-1)(2i)]/2.$$

The above simplifies to

$$i^3 + (i - 1)^3,$$

which is exactly what I am aiming to prove. Therefore, the sum of each row in the triangle is the sum of consecutive cubes.

I began to think that there were some other properties about this triangle that were worth investigation. I decided to work with higher powers.

The Fourth Power

I will come back to cubes, but let me look at higher exponents. Start with the fourth power, k = 4, and look at R_1 and R_2 in Figure1, which represent the numbers 1 and 2, respectively.

$$1^4 = (1*1)(1 + 0/1).$$
$$2^4 = (2*4)(1 + 2/2).$$

I propose that each fourth power can be calculated by multiplying the endpoints of each row times an adjusting factor. The adjusting factor is where the patterns lay. Look at the following values where the exponent, k, is 4.

Let k = 4

Row					Value	Adjusting Factor
1 →	1^4	=	(1*1)(1 + 0/1)	=	1	(1 + 0/1)
2 →	2^4	=	(2*4)(1 + 2/2)	=	16	(1 + 2/2)
3 →	3^4	=	(5*9)(1 + 4/5)	=	81	(1 + 4/5)
4 →	4^4	=	(10*16)(1 + 6/10)	=	256	(1 + 6/10)
5 →	5^4	=	(17*25)(1 + 8/17)	=	625	(1 + 8/17)

The interesting pattern to notice here is that the whole number of the adjusting factor is 1. The fraction of the adjusting factor follows a pattern because the numerators are multiples of 2. The denominator follows the left side of the triangle: 1, 2, 5, 10, 17… and merely duplicates the right end point of each row. I hypothesize that the fourth power can be expressed as the equation below through observing patterns:

$$i^4 = (i^2 - (2i-2))(i^2)\left[1 + \frac{2(i-1)}{i^2 - (2i-2)}\right] \text{ or}$$

$$i^4 = R_i(1)*R_i(2i-1)\left[1 + \frac{2(i-1)}{R_i(1)}\right].$$

The right side of the equation can be reduced to the left.
It would seem logical that the whole number should change along with the numerator values of the adjusting factor. Of course, this hypothesis has to be tested.

The Fifth Power

As stated earlier, I am looking for changes in the whole number and numerator. Look at Figure 1 to observe the following pattern for the fifth powers, k = 5, of certain positive integers :

k=5

Row					Value	Adjusting Factor
1 →	1^5	=	(1*1)(1 + 0/1)	=	1	(1 + 0/1)
2 →	2^5	=	(2*4)(2 + 4/2)	=	32	(2 + 4/2)
3 →	3^5	=	(5*9)(3 + 12/5)	=	243	(3 + 12/5)
4 →	4^5	=	(10*16)(4 + 24/10)	=	1024	(4 + 24/10)
5 →	5^5	=	(17*25)(5 + 40/17)	=	3125	(5 + 40/17)

The whole numbers are following the pattern of the natural numbers and the numerators are still multiples of 2. I formulate an equation for the fifth powers of numbers based on the observed patterns:

$$i^5 = (i^2 - (2i-2))(i^2)\left[i + \frac{2(i-1)i}{i^2 - (2i-2)}\right] \text{ or}$$

$$i^5 = R_i(1)*R_i(2i-1)\left[i + \frac{2(i-1)i}{R_i(1)}\right].$$

Simplification shows that both sides are equal.

The Sixth Power

I continue to move higher in the exponential ladder to determine if patterns continue to emerge and I am successful in finding them.

Let k = 6

Row					Value	Adjusting Value
1 →	1^6	=	(1*1)(1 + 0/1)	=	1	(1 + 0/1)
2 →	2^6	=	(2*4)(4 + 8/2)	=	64	(4 + 8/2)
3 →	3^6	=	(5*9)(9 + 36/5)	=	729	(9 + 36/5)
4 →	4^6	=	(10*16)(16 + 96/10)	=	4096	(16 + 96/10)
5 →	5^6	=	(17*25)(25 + 200/17)	=	15 625	(25 + 200/17)

Now the whole number of the adjusting factor begins to follow the pattern of squares. I formulate another equation to test whether this idea is correct:

$$i^6 = (i^2 - (2i-2))(i^2)\left[i^2 + \frac{2(i-1)i^2}{i^2 - (2i-2)}\right] \text{ or}$$

$$i^6 = R_i(1)*R_i(2i-1)\left[i^2 + \frac{2(i-1)i^2}{R_i(1)}\right].$$

Of course, both sides simplify and equate.

A pattern becomes noticeable from dealing with the exponents k = 4, 5 and 6. The formula that models the exponential function is as follows:

$$i^n = (i^2 - (2i-2))(i^2)\left[i^{n-4} + \frac{2(i-1)i^{n-4}}{i^2 - (2i-2)}\right] \text{ or}$$

$$i^n = R_i(1)*R_i(2i-1)\left[i^{n-4} + \frac{2(i-1)i^{n-4}}{R_i(1)}\right].$$

The pattern is evident in the adjusting factor. The whole number in the product is an exponent of degree n - 4 and the numerators are multiples of 2.

The pattern is not so obvious with cubes, so I worked with the higher exponents first and now I return to the cubes.

Cubic Pattern

k=3

Row						Value	Adjusting Factor
1 →	1^3	=	(1*1)(1 + 0/1)	=	1		(1 + 0/1)
2 →	2^3	=	(2*4)(1/2 + 1/2)	=	8		(1/2 + 1/2)
3 →	3^3	=	(5*9)(1/3 + (4/3)/5)	=	27		(1/3 + (4/3)/5)
4 →	4^3	=	(10*16)(1/4 + (6/4)/10)	=	64		(1/4 + (6/4)/10)
5 →	5^3	=	(17*25)(1/5 + (8/5)/17)	=	125		(1/5 + (8/5)/17)

Focus on the patterns developed by the products of the endpoints and adjusting factor. The adjusting factor in each of the product has i^{-1} in what is usually the whole number spot. The numerator involves fractional multiples of 2 - the numerator can be shown to be a product of two times the reciprocal of the representative row number. The formula that I propose is:

$$i^3 = (i^2 - (2i-2))(i^2)\left[i^{-1} + \frac{2(i-1)i^{-1}}{i^2 - (2i-2)} \right] \text{ or}$$

$$i^3 = R_i(1) * R_i(2i-1)\left[i^{-1} + \frac{2(i-1)i^{-1}}{R_i(1)} \right].$$

The equation reduces to i^3 on both sides, so my hypothesis is correct.

Alternative Patterns

The powers of numbers do not have to follow one pattern. Below I have identified other patterns that equate to the fifth and sixth powers of numbers.

k=5

Row					Value	Adjusting Factor
1 →	1^5	=	(1*1)(3 + -2/1)	=	1	(3 + -2/1)
2 →	2^5	=	(2*4)(4 + 0/2)	=	32	(4 + 0/2)
3 →	3^5	=	(5*9)(5 + 2/5)	=	243	(5 + 2/5)
4 →	4^5	=	(10*16)(6 + 4/10)	=	1024	(6 + 4/10)
5 →	5^5	=	(17*25)(7 + 6/17)	=	3125	(7 + 6/17)

The whole numbers are following the pattern of the natural numbers, but the whole numbers has started at 3 instead of 1. The numerators are still multiples of 2. I formulate an equation for the fifth powers of the integers based on the observed patterns:

$$i^5 = (i^2 - (2i-2))(i^2)\left[\, i + 2 + \frac{2(i-2)}{i^2 - (2i-2)} \,\right] \text{ or}$$

$$i^5 = R_i(1) * R_i(2i-1)\left[\, i + 2 + \frac{2(i-2)}{R_i(1)} \,\right].$$

Simplification shows that both sides are equal.

Let k = 6

Row					Value	Adjusting Factor
1 →	1^6	=	(1*1)(4 +-3/1)	=	1	(4 +-3/1)
2 →	2^6	=	(2*4)(9 + -2/2)	=	64	(9 + -2/2)
3 →	3^6	=	(5*9)(16 + 1/5)	=	729	(16 + 1/5)
4 →	4^6	=	(10*16)(25 + 6/10)	=	4096	(25 + 6/10)
5 →	5^6	=	(17*25)(36 + 13/17)	=	15 625	(36 + 13/17)

The whole number begins to follow the pattern of squares beginning at 2. I formulate another equation to test whether this idea is valid:

$$i^6 = (i^2 - (2i-2))(i^2)\left[\, (i+1)^2 + \frac{(i-1)^2 - 3}{i^2 - (2i-2)} \,\right] \text{ or}$$

$$i^6 = R_i(1) * R_i(2i-1)\left[\, (i+1)^2 + \frac{(i-1)^2 - 3}{R_i(1)} \,\right].$$

The right side simplifies to the left and it is shown that the sixth power can be written in terms of a pattern using squares different than the one illustrated above.

Conclusion

I have shown that there are certain geometric and algebraic patterns that exist in exponents, from the third to the sixth power, for integers. I have represented any power, k = 3, 4, 5, and 6, of positive integers as a product of two endpoints multiplied by an adjusting factor. The whole number or the first term in the adjusting factor and the exponent, k, of the positive integer increase together. The pattern follows the chart:

k	Pattern of first term in adjusting factor	Pattern of numerator in adjusting factor
3	i^{-1}	$2(i-1)\ i^{-1}$
4	i^{0}	$2(i-1)\ i^{0}$
5	i^{1}	$2(i-1)\ i^{1}$
6	i^{2}	$2(i-1)\ i^{2}$

The generalized formula is as follows;
Let a be a positive integer, define R_a to be the a-th row and let $R_a(b)$ represent the b-th entry in the a-th row.

$$a^n = (a^2 - (2a-2))(a^2)\left[\, a^{n-4} + \frac{2(a-1)a^{n-4}}{a^2 - (2a-2)} \right] \text{ or}$$

$$a^n = R_a(1) * R_a(2a-1)\left[\, a^{n-4} + \frac{2(a-1)a^{n-4}}{R_a(1)} \right].$$

The above formula is valid for any positive integer and any integral exponent. I have shown that with some curiosity, arithmetic and basic algebra that exponentials have patterns and a special relationship with natural numbers. In the future I plan to look at the plane of all integers, negative and positive, and slice certain two-dimensional geometries out of it. The next step is to create a three dimensional lattice of all integers and Gaussian integers and take three-dimensional cuts of that space and determine whether there exist relationships between geometries and mathematical functions. Ultimately, my goal is to work to understand the underlying relationship between form and function.