Two years ago I came across a very beautiful geometric construction (figure 1), on a tee-shirt I received from the Upstate New York math team. The shirt also had a property of the construction written on it, just waiting to be proven. The coaches told me that the proof they had was six pages long, and very complex. Such a beautiful problem, should have a beautiful answer. So, I came up with a very nice proof which took less than a minute to present to the team. Since then I have delved further into the construction, learning some very interesting things. This paper proves some nice properties of the construction.

Conventions:

All triangles will have a Δ in front of it.

P(x,n) means a polygon with n sides and subscript x.

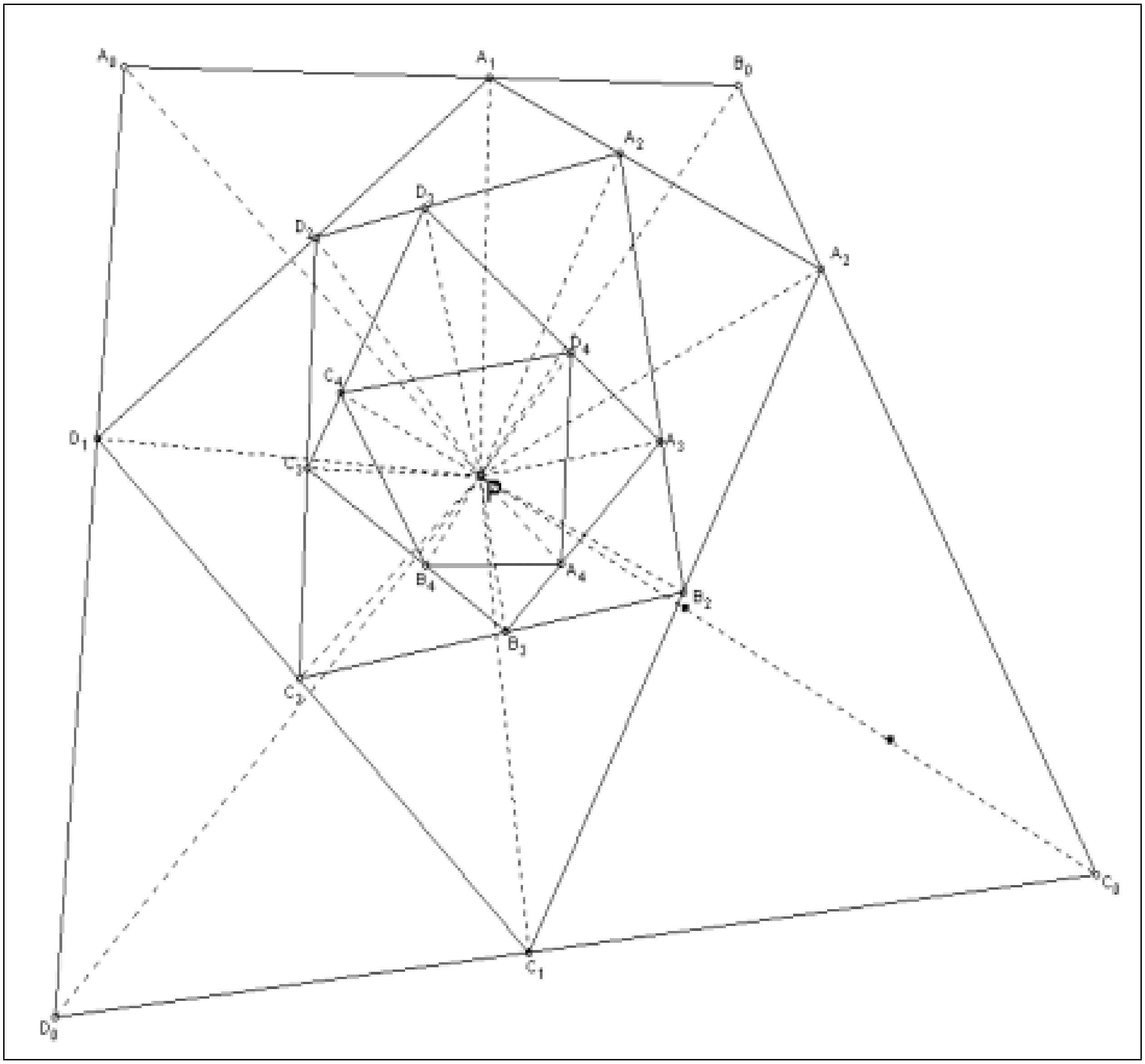

Figure 1

Quadrilateral $A_0B_0C_0D_0$ is constructed, and a point P is placed in the interior $A_0B_0C_0D_0$. From that point to the four sides of quadrilateral $A_0B_0C_0D_0$ perpendicular lines are drawn. The intersection points with the sides of $A_0B_0C_0D_0$ are labeled A_1, B_1, C_1, D_1, and are connected to form $A_1B_1C_1D_1$. The same process with interior point P is repeated on $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ is formed. After twice repeating the construction, $A_4B_4C_4D_4$ is created.

Prove: $A_0B_0C_0D_0 \sim A_4B_4C_4D_4$

Choose angle A_0 of angles A_0, B_0, C_0, D_0. Observe that in $D_1A_0A_1P$ angles A_0D_1P and A_0A_1P are both right angles, and add up to 180°. Therefore $D_1A_0A_1P$ is a cyclic quadrilateral. Since $D_1A_0A_1P$ is a cyclic quadrilateral it can be inscribed in a circle (as shown in figure 2).

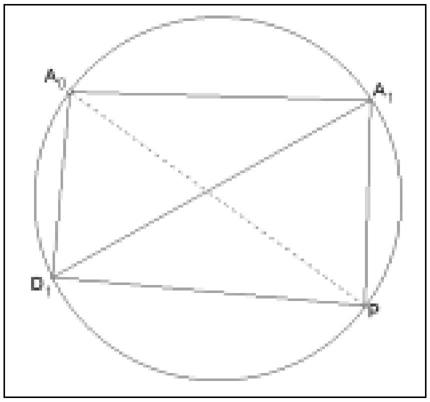

Figure 2

In figure 2, observe that angle D_1A_0P (or D_0A_0P) subtends the same arc as angle D_1A_1P, therefore $D_0A_0P \approx D_1A_1P$. The same holds for A_1A_0P (B_0A_0P) $\approx A_1D_1P$, and the same for each B_0, C_0 and D_0.

So, extending this to $A_1B_1C_1D_1$ and on:

$D_0A_0P \approx D_1A_1P \approx D_2A_2P \approx D_3A_3P \approx D_4A_4P$

And for $B_0A_0P \approx A_1D_1P \approx D_2C_2P \approx C_3B_3P \approx B_4A_4P$.

The basic idea is that the angles are rotating in opposite directions each construction.

Angle $D_0A_0B_0 \approx D_0A_0P + B_0A_0P$. Since $D_0A_0P \approx D_4A_4P$, and $B_0A_0P \approx B_4A_4P$ syllogism produces that $D_0A_0B_0 \approx D_4A_4P + B_4A_4P \approx D_4A_4B_4$. This means angle $A_0 \approx A_4$. Had we chosen angle B_0, C_0 or D_0, the same result would hold. Therefore $A_0 \approx A_4$, $B_0 \approx B_4$, $C_0 \approx C_4$, and $D_0 \approx D_4$. Therefore, $A_0B_0C_0D_0 \sim A_4B_4C_4D_4$. □

This same repeat construction of extending perpendiculars, and then connecting them can be used on an polygon of n sides.

Prove for a polygon with n sides called $P(_0,n)$ with $_0$ as the subscript, $P(_0,n) \sim P(n,n)$ or $A_0B_0C_0....Z_0... \sim AnBnCn....Zn....$

This proof follows almost the same as the first proof. Every angle, $D_0A_0B_0$, $A_0B_0C_0$.... can be split up into two angles. These two angles move in opposite directions in each subsequent polygon constructed, until they eventually join up again. Since there are 2n positions, and together the angles move two apart every construction, after n constructions, the two angles will join up again. This holds for each set of two angles, and therefore the polygon's angles will be the same as the original after n constructions, and will also be in the same position and same order. Therefore for a polygon with n sides, P($_0$,n) ~ P(n,n) or $A_0B_0C_0$…..Z_0…. ~ $AnBnCn$…..Zn….□

Now that we have examined the angles under the constructions, we should look at the sides. Because P($_0$,n) ~ P (n,n) we know that the sides' dimensions are diminished by some factor.

Prove that the dimensions of the sides of P($_0$,n) are diminished by the factor $\sin(B_0A_0P)*\sin(C_0B_0P)*\sin(D_0C_0P)*\ldots*\sin(Z_0Y_0P)*\ldots$ to form P(n,n)

Take any edge of P($_0$,n), say A_0B_0. That is a side on the triangle ΔA_0B_0P. ΔA_0B_0P is similar to $\Delta AnBnP$, because it has the same three angles. So, it is sufficient to show than any side of ΔA_0B_0P is diminished by the given factor to form $\Delta AnBnP$.

Take edge A_0P. $A_0P*\sin(B_0A_0P) = A_1P$. $A_1P*\sin(B_1A_1P) = A_2P = A_0P*\sin(B_0A_0P)*\sin(B_1A_1P)$, yet since $B_1A_1P \approx C_0B_0P$, this is really $A_0P*\sin(B_0A_0P)*\sin(C_0B_0P)$. This continues n times. The leg of $\Delta AnBnP$ is $\sin(B_0A_0P)*\sin(C_0B_0P)*\sin(D_0C_0P)*\ldots*\sin(Z_0Y_0P)*\ldots$times the length of ΔA_0B_0P side. Therefore the dimensions of the sides of P($_0$,n) are diminished by a factor $\sin(B_0A_0P)*\sin(C_0B_0P)*\sin(D_0C_0P)*\ldots*\sin(Z_0Y_0P)*\ldots$ to form P(n,n). □

The next question to ask is about the orientation of the polygon P(n,n), form from the n construction of P($_0$,n). Basically, how much did the polygon rotate?

Prove that the polygon P(n,n) has rotated $90° (n) – B_0A_0P – C_0B_0P – ...Z_0Y_0P – ...$ degrees around P from P($_0$,n).

I will first illustrate this with a quadrilateral, or n=4. It is sufficient to show that a single segment A_0P rotates the given amount of degrees, because all of the other segments move the same angle due to the similarity in the polygons. So, A_0P initially rotates through the angle A_0PA_1 to A_1P.

$A_0PA_1 \approx 180° – A_1A_0P – A_0A_1P \approx 180° – A_1A_0P – 90° \approx 90° – A_1A_0P$. Now, A_1P also rotates through an angle of A_1PA_2, which by similar syllogism, $A_1PA_2 \approx 90° – A_2A_1P$. This process occurs four times until the total angle rotated by A_0P to A_1P to A_2P to A_3P to A_4P is $90° – A_1A_0P + 90° – A_2A_1P + 90° – A_3A_2P + 90° – A_4A_3P \approx 360° – B_0A_0P – C_0B_0P – D_0C_0P – A_0D_0P$ (the replacement comes from the congruency discussed earlier between each angle).

For P(n,n), there are n rotations which A_0P goes through until it reaches AnP. The total rotation is $90° – A_1A_0P + 90° – A_2A_1P\ldots\ldots 90° – A_0AnP \approx 90° (n) – B_0A_0P – C_0B_0P – D_0C_0P \ldots.- Z_0Y_0P\ldots..$ Therefore the polygon P(n,n) has rotated $90° (n) – B_0A_0P – C_0B_0P -\ldots..Z_0Y_0P -\ldots..$ degrees around P from P($_0$,n). □

At first I believed that this number would always be 180°, but using geometer's sketchpad (the source of the graphics) I saw that in fact that is not always, or usually true.