THREE TERM IDENTITIES FOR THE COEFFICIENTS OF CERTAIN INFINITE PRODUCTS

WEI REN

1. Introduction

In a recent paper [1], Farkas and Kra presented five three-term identities for the coefficients of certain infinite products. Let N be an integer. For non-negative integers n, define integers $P_N(n)$ by

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^N} = \sum_{n=0}^{\infty} P_N(n) q^n.$$

The following was proved in the paper. Note that we define $P_N(n) := 0$ when n is not a non-negative integer.

Theorem 1. *For all non-negative integers n, we have*

$$P_{-24}(2n+1) = -2^{11} P_{-24}\left(\frac{n-1}{2}\right) - 2^3 \cdot 3 P_{-24}(n), \tag{1.1}$$

$$P_{-12}(3n+1) = -3^5 P_{-12}\left(\frac{n-1}{3}\right) - 2^2 \cdot 3 P_{-12}(n), \tag{1.2}$$

$$P_{-6}(5n+1) = -5^2 P_{-6}\left(\frac{n-1}{5}\right) - 2 \cdot 3 P_{-6}(n), \tag{1.3}$$

$$P_{-4}(7n+1) = -7 P_{-4}\left(\frac{n-1}{7}\right) - 2^2 P_{-4}(n), \tag{1.4}$$

$$P_{-2}(13n+1) = -P_{-2}\left(\frac{n-1}{13}\right) - 2P_{-2}(n). \tag{1.5}$$

Also in the paper, Farkas and Kra mentioned that Mordell [2] proved that for all primes l and all positive integers n, we have

$$P_{-24}(ln-1) = P_{-24}(l-1) P_{-24}(n-1) - l^{11} P_{-24}\left(\frac{n}{l} - 1\right). \tag{1.6}$$

It is clear that (1.1) is a special case of (1.6) with $l = 2$ and n replaced by $n+1$. The authors state that "it is not at all clear whether (1.2), (1.3), (1.4) and (1.5) are also special

The author thanks The Colgate Division of Natural Sciences and Mathematics for its generous support.

cases of more general identities." The purpose of this paper is to show that each of the above-mentioned three-term identities is just one of an infinite family of identities. Recall that $\left(\frac{a}{p}\right)$ is the Legendre symbol defined for primes p and integers a as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p \text{ and } p \nmid a, \\ 0 & \text{if } p \mid a, \\ -1 & \text{otherwise.} \end{cases}$$

Our main result is

Theorem 2. *For all primes l and all positive integers n, we have*

(1.7)
$$P_{-12}\left(\frac{ln-1}{2}\right) = P_{-12}\left(\frac{l-1}{2}\right) P_{-12}\left(\frac{n-1}{2}\right) - \left(\frac{4}{l}\right) l^5 P_{-12}\left(\frac{\frac{n}{l}-1}{2}\right),$$

(1.8)
$$P_{-6}\left(\frac{ln-1}{4}\right) = P_{-6}\left(\frac{l-1}{4}\right) P_{-6}\left(\frac{n-1}{4}\right) - \left(\frac{-4}{l}\right) l^2 P_{-6}\left(\frac{\frac{n}{l}-1}{4}\right),$$

(1.9)
$$P_{-4}\left(\frac{ln-1}{6}\right) = P_{-4}\left(\frac{l-1}{6}\right) P_{-4}\left(\frac{n-1}{6}\right) - \left(\frac{36}{l}\right) l P_{-4}\left(\frac{\frac{n}{l}-1}{6}\right),$$

(1.10)
$$P_{-2}\left(\frac{ln-1}{12}\right) = P_{-2}\left(\frac{l-1}{12}\right) P_{-2}\left(\frac{n-1}{12}\right) - \left(\frac{-36}{l}\right) P_{-2}\left(\frac{\frac{n}{l}-1}{12}\right).$$

Remarks.

(i) Our technique also gives a proof of (1.6).

(ii) The statements in (1.2)-(1.5) are all special cases of Theorem 2. For example, if we let $l = 5$ and replace n with $4n+1$ in (1.8), we get (1.3).

It immediately follows that

Corollary 1. *For all primes l and all positive integers n,*

$$\text{if } l \not\equiv 1 \pmod 4, \text{ then } \quad P_{-6}\left(\frac{ln-1}{4}\right) = -\left(\frac{-4}{l}\right) l^2 P_{-6}\left(\frac{\frac{n}{l}-1}{4}\right),$$
$$\text{if } l \not\equiv 1 \pmod 6, \text{ then } \quad P_{-4}\left(\frac{ln-1}{6}\right) = -\left(\frac{36}{l}\right) l P_{-4}\left(\frac{\frac{n}{l}-1}{6}\right),$$
$$\text{if } l \not\equiv 1 \pmod{12}, \text{ then } \quad P_{-2}\left(\frac{ln-1}{12}\right) = -\left(\frac{-36}{l}\right) P_{-2}\left(\frac{\frac{n}{l}-1}{12}\right).$$

2. Preliminaries

We now briefly introduce modular forms. For complete details, see [3]. Let $\mathbb{H}$ denote the upper half of the complex plane, $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The set $\mathrm{SL}_2(\mathbb{Z})$ is defined as $\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1; a, b, c, d \in \mathbb{Z} \right\}$, and is sometimes denoted Γ. Let N be a positive integer. One important subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{N} \right\}.$$

Given an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define the following transformation for $z \in \mathbb{H}$: $\gamma z := \frac{az+b}{cz+d}$.

Let f be a holomorphic function on the upper half-plane $\mathbb{H}$, and let k be an integer. If γ is a 2×2 matrix with rational entries and positive determinant, then we define $f(z) \mid [\gamma]_k$ as

$$f(z) \mid [\gamma]_k := (\det \gamma)^{k/2} (cz + d)^{-k} f(\gamma z).$$

It is easy to verify that

$$f(z) \mid [\gamma_1 \gamma_2]_k = (f(z) \mid [\gamma_1]_k) \mid [\gamma_2]_k. \tag{2.1}$$

Let $f(z)$ be a holomorphic function on $\mathbb{H}$. Let k be an integer and N be a positive integer. Recall that a *Dirichlet character mod* N is a function $\chi(n) : \mathbb{Z} \to \mathbb{C}$, not identically zero, which satisfies, for all integers n and m,

(i) $\chi(n) = \chi(m)$ if $n \equiv m \pmod{N}$,
(ii) $\chi(n) = 0$ if $\gcd(n, N) > 1$,
(iii) $\chi(nm) = \chi(n)\chi(m)$.

Let χ be a Dirichlet character mod N. Then $f(z)$ is called a *modular form of weight* k *for* $\Gamma_0(N)$ *with character* χ if

$$f(z) \mid [\gamma]_k = \chi(d) f \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \tag{2.2}$$

and if for any $\gamma_0 \in \Gamma$,

$$f(z) \mid [\gamma_0]_k \text{ has a Fourier expansion of the form } \sum_{n=0}^{\infty} a(n) q_N^n, \quad \text{where } q_N := e^{2\pi i z/N}. \tag{2.3}$$

A modular form is called a *cusp form* if in addition we have

$$a(0) = 0 \ \text{ in (2.3) for all } \gamma_0 \in \Gamma. \tag{2.4}$$

The set of such modular forms is denoted $M_k(\Gamma_0(N),\chi)$, and the set of cusp forms is denoted $S_k(\Gamma_0(N),\chi)$. Every modular form has a Fourier expansion at infinity,

$$f(z)=\sum_{n=0}^{\infty}a(n)q^n,\qquad \text{where } q:=e^{2\pi iz}.$$

We will identify f with its expansion.

Note that the conditions (2.2), (2.3) and (2.4) are preserved under addition and scalar multiplication, so the sets of modular forms and the sets of cusp-forms of fixed weight, character and N are complex vector spaces.

Another important ingredient in our study are the Hecke operators. For each space $M_k(\Gamma_0(N),\chi)$, there exists a family of Hecke operators $T(p)$, one operator for each prime p. These are linear operators which preserve the space of cusp forms:

$$T(p): S_k(\Gamma_0(N),\chi)\to S_k(\Gamma_0(N),\chi). \tag{2.5}$$

They can be defined explicitly as follows: If $f(z)=\sum_{n=0}^{\infty}a(n)q^n\in M_k(\Gamma_0(N),\chi)$, then

$$f(z)\mid T(p):=\sum_{n=0}^{\infty}(a(pn)+\chi(p)p^{k-1}a(n/p))q^n,\quad \text{where}\quad a(n/p)=0 \text{ if } p\nmid n.$$

A modular form is called a *primitive eigenform* if for every prime p, there exists a scalar λ_p such that

$$f(z)\mid T(p)=\lambda_p f(z). \tag{2.6}$$

3. Proof of Theorem 2

We first establish the relationship between the coefficients of modular forms and the coefficients of the functions $P_N(n)$. We will use Dedekind's η-function,

$$\eta(z)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)^{24},\qquad q=e^{2\pi iz}.$$

For non-negative integers n, define integers $a(n),b(n),c(n)$ and $d(n)$ by

$$\sum_{n=1}^{\infty}a(n)q^n:=\eta^{12}(2z)=q\prod_{n=1}^{\infty}(1-q^{2n})^{12}=q-12q^3+54q^5-88q^7-99q^9+\cdots,$$
$$\sum_{n=1}^{\infty}b(n)q^n:=\eta^{6}(4z)=q\prod_{n=1}^{\infty}(1-q^{4n})^{6}=q-6q^5+9q^9+10q^{13}-30q^{17}+\cdots,$$
$$\sum_{n=1}^{\infty}c(n)q^n:=\eta^{4}(6z)=q\prod_{n=1}^{\infty}(1-q^{6n})^{4}=q-4q^7+2q^{13}+8q^{19}-5q^{25}+\cdots,$$
$$\sum_{n=1}^{\infty}d(n)q^n:=\eta^{2}(12z)=q\prod_{n=1}^{\infty}(1-q^{12n})^{2}=q-2q^{13}-q^{25}+2q^{37}+q^{49}+\cdots.$$

We now state a Theorem that implies Theorem 2.

Theorem 3. *For all primes l and non-negative integers n, we have*

$$a(ln) = a(l)a(n) - l^5 \left(\frac{4}{l}\right) a\left(\frac{n}{l}\right), \tag{3.1}$$

$$b(ln) = b(l)b(n) - l^2 \left(\frac{-4}{l}\right) b\left(\frac{n}{l}\right), \tag{3.2}$$

$$c(ln) = c(l)c(n) - l \left(\frac{36}{l}\right) c\left(\frac{n}{l}\right), \tag{3.3}$$

$$d(ln) = d(l)d(n) - \left(\frac{-36}{l}\right) d\left(\frac{n}{l}\right). \tag{3.4}$$

The following Lemma shows that Theorem 3 implies Theorem 2.

Lemma 1. *For all non-negative integers n, we have*

$$P_{-12}\left(\frac{n-1}{2}\right) = a(n),$$
$$P_{-6}\left(\frac{n-1}{4}\right) = b(n),$$
$$P_{-4}\left(\frac{n-1}{6}\right) = c(n),$$
$$P_{-2}\left(\frac{n-1}{12}\right) = d(n).$$

Proof of Lemma 1. By the definitions of $P_N(n)$ and $a(n)$, we have

$$\eta^{12}(2z) = q\prod_{n=1}^{\infty}(1-q^{2n})^{12} = q\sum_{n=0}^{\infty}P_{-12}(n)q^{2n} = \sum_{n=0}^{\infty}P_{-12}(n-1)q^{2n} = \sum_{n=1}^{\infty}a(n)q^n.$$

Therefore, we get $a(n) = P_{-12}\left(\frac{n-1}{2}\right)$. The same technique can be applied to $\eta^6(4z), \eta^4(4z)$ and $\eta^2(12z)$ and the other three statements in Lemma 1 can be proved similarly. □

We now define a family of characters which will be used in our proofs. If $4 \mid N$, then define χ_N by

$$\chi_N(d) := \begin{cases} (-1)^{\frac{d-1}{2}} & \text{if } \gcd(d,N)=1, \\ 0 & \text{if } \gcd(d,N)\neq 1. \end{cases}$$

We also define the trivial character mod N as the following:

$$\chi_N^{\text{triv}}(d) := \begin{cases} 1 & \text{if } \gcd(d,N)=1, \\ 0 & \text{if } \gcd(d,N)\neq 1. \end{cases}$$

Clearly, χ_N (as well as the trivial character) is a Dirichlet character mod N.

To prove Theorem 3, we need the following two Lemmas:

Lemma 2.

$$\begin{aligned}
&\eta^2(12z) \in S_1\left(\Gamma_0(144), \chi_{144}\right), \\
&\eta^4(6z) \in S_2\left(\Gamma_0(36), \chi_{36}^{\text{triv}}\right), \\
&\eta^6(4z) \in S_3\left(\Gamma_0(16), \chi_{16}\right), \\
&\eta^{12}(2z) \in S_6\left(\Gamma_0(4), \chi_{4}^{\text{triv}}\right).
\end{aligned}$$

Lemma 3. *All four functions in Lemma 2 are primitive eigenforms.*

Proof of Lemma 2. Let $f(z) = \eta^6(4z)$. By the definition of cusp-forms, we have to show that with $N = 16$ and $\chi = \chi_{16}$, the statements in (2.2), (2.3) and (2.4) hold. We will use the famous transformation formula for the η-function [4]:

$$\text{If } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \text{ and } c \text{ is positive, then } \eta(\gamma z) = \varepsilon(a,b,c,d)(cz+d)^{1/2}\eta(z),$$

$$\text{where} \quad \varepsilon(a,b,c,d) := \begin{cases} \pm\exp\left(\frac{2\pi i}{24}(-3c - bd(c^2-1) + c(a+d))\right) & \text{if } c \text{ is odd,} \\ \pm\exp\left(\frac{2\pi i}{24}(3d - 3 - ac(d^2-1) + d(b-c))\right) & \text{if } d \text{ is odd.} \end{cases}$$

We remark that in the definition of ε, $\pm$ is explicitly determined, but the value is not important for our purpose.

Suppose $\gamma \in \Gamma_0(16)$, and $c > 0$. Then d is odd, and we have

$$\begin{aligned}
f(\gamma z) &= \eta^6(4\gamma z) \\
&= \eta^6\left(\frac{a \cdot 4z + 4b}{(c/4)(4z) + d}\right) \\
&= \varepsilon^6(a, 4b, c/4, d)(cz+d)^3\eta^6(4z),
\end{aligned} \tag{3.5}$$

where

$$\varepsilon^6(a, 4b, c/4, d) = \exp\left(\frac{\pi i}{2}(3d - 3 - c/4 \cdot a(d^2-1) + 4b \cdot d - d \cdot c/4\right) = (-1)^{\frac{d-1}{2}} \tag{3.6}$$

That is, we get $\varepsilon^6(a, 4b, c/4, d) = \chi_{16}(d)$. Using (3.5) and (3.6), we have verified (2.2) when $c > 0$:

$$\text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(16), \quad \text{then } f(z) \mid [\gamma]_3 = \chi_{16}(d) f(z). \tag{3.7}$$

Note. We still have to deal the cases when $c = 0$ and $c < 0$. If $c = 0$, then $a = d = \pm 1$. Therefore, (3.7) still holds.

If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $c < 0$, then we have

$$\begin{aligned}
\left(f(z) \mid \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_3\right) \mid [\gamma]_3 &= f(z) \mid \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}_3 && \text{by (2.1)} \\
&= \chi_{16}(-d) f(z) && \text{by (3.7)} \\
&= -\chi_{16}(d) f(z).
\end{aligned}$$

While on the other hand, we also have

$$f(z) \mid \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -f(z).$$

Therefore, we proved that $f(z) \mid [\gamma]_3 = \chi_{16}(d) f(z)$ holds for all matrices $\gamma \in \Gamma_0(16)$.

Next we have to check conditions in (2.3) and (2.4). Let $g(z) = \eta^6(z)$. So we have

$$f(z) = g(4z) = \frac{1}{4^{3/2}} \cdot g(z) \mid \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}_3 .$$

Then for $\gamma_0 = \left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, where as above we may assume $c_0 > 0$, (2.1) gives

$$\begin{aligned} f(z) \mid [\gamma_0]_3 &= \frac{1}{8} \left(g(z) \mid \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}_3 \right) \mid [\gamma_0]_3 \\ &= \frac{1}{8} \cdot g(z) \mid \begin{pmatrix} 4a_0 & 4b_0 \\ c_0 & d_0 \end{pmatrix}_3 . \end{aligned}$$

To find the Fourier expansion of $f(z) \mid [\gamma_0]_3$, we first prove that there exist $A, D \in \mathbb{Z}^+$, such that

$$\gamma := \begin{pmatrix} 4a_0 & 4b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} \in \mathrm{SL}_2(\mathbb{Z}). \tag{3.8}$$

By multiplying out, we get

$$\gamma = \begin{pmatrix} 4a_0/A & (4b_0 A - 4a_0 B)/AD \\ c_0/A & (d_0 A - c_0 B)/AD \end{pmatrix}.$$

Let $A = \gcd(c_0, 4)$ and $D = 4/A$, then $A, D \in \mathbb{Z}^+$ and $AD = 4$, so

$$\gamma = \begin{pmatrix} a_0 D & b_0 A - a_0 B \\ c_0/A & (d_0 A - c_0 B)/4 \end{pmatrix},$$

and $\det \gamma = 1$. Also it is clear that c_0/A, the lower-left entry of γ, is an integer. Now we show that the there exists B such that the lower-right entry is also an integer. Since $A = \gcd(c_0, 4)$, we have $\gcd(c_0/A, 4/A) = 1$. Thus there is a solution to the congruence $(c_0/A)\, x \equiv d_0 \pmod{4/A}$. So there exists $B \in \mathbb{Z}$ such that $(d_0 A - c_0 B)/4 \in \mathbb{Z}$. This shows that matrices as in (3.8) exist. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Note that since $c_0 > 0$, we have $c > 0$. Then we have

$$\begin{aligned}
f(z) \mid [\gamma_0]_3 &= \frac{1}{8} \cdot g(z) \mid \begin{pmatrix} 4a_0 & 4b_0 \\ c_0 & d_0 \end{pmatrix}_3 \\
&= \frac{1}{8} \left(g \mid [\gamma]_3\right) \mid \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}_3 \\
&= \frac{1}{8} \left(\eta^6(\gamma z)(cz+d)^{-3}\right) \mid \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}_3 \\
&= \frac{1}{8}\varepsilon(a,b,c,d)^6 \left(\eta^6(z) \mid \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}_3\right) \\
&= \frac{1}{8}\varepsilon(a,b,c,d)^6 \left(4^{3/2} D^{-3} \eta^6\left(\frac{Az+B}{D}\right)\right) \\
&= M \cdot \eta^6\left(\frac{Az+B}{D}\right) \quad \text{where } M = \frac{1}{8}\varepsilon(a,b,c,d)^6 4^{3/2} D^{-3} \\
&= M \cdot \left(e^{2\pi i \frac{Az+B}{D}}\right)^{1/4} \prod_{n=1}^{\infty} (1 - \left(e^{2\pi i \frac{Az+B}{D}}\right)^n)^6 \\
&= e^{\pi i B/2D} M \cdot q_{4D}^{A} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i Bn/D} q_{4D}^{4An}\right)^6, \quad \text{where } q_D = e^{2\pi i z/D}.
\end{aligned}$$

It is now clear that the Fourier expansion of $f \mid [\gamma_0]_3$ has $a(n) = 0$ for $n \leq 0$. Therefore, conditions in (2.2), (2.3) and (2.4) are satisfied, and $f(z) = \eta^6(4z)$ is a cusp-form of weight 3 on $\Gamma_0(16)$ with the character χ_{16}. In other words,

$$f(z) \in S_3(\Gamma_0(16), \chi_{16}).$$

The same techniques can be applied to $\eta^{12}(2z), \eta^4(6z)$ and $\eta^2(12z)$, and the other three statements in Lemma 2 can be proved similarly. □

Proof of Lemma 3. Since the four functions are all cusp-forms, we thus can use the following formula [5] to compute the dimension of their spaces. Let $\dim M_k(\Gamma_0(N), \chi)$ and $\dim S_k(\Gamma_0(N), \chi)$ denote the dimension of the space of corresponding modular forms and cusp-forms, respectively. Then we have

$$\dim S_k\left(\Gamma_0(N), \chi\right) - \dim M_{2-k}\left(\Gamma_0(N), \chi\right) =$$
$$\frac{k-1}{12} N \prod_{p\mid N}\left(1+\frac{1}{p}\right) - \frac{1}{2}\prod_{p\mid N}\lambda(r_p, s_p, p) + \varepsilon_k \sum_{\substack{x \mod N \\ x^2+1\equiv 0 \mod N}} \chi(x) + \mu_k \sum_{\substack{x \mod N \\ x^2+1\equiv 0 \mod N}} \chi(x),$$

where r_p (resp. s_p) denotes the exponent of p in the factorization of N (resp. of the conductor of the character χ), and $\lambda(r_p, s_p, p)$, ε and μ are defined as

$$\lambda(r_p, s_p, p) := \begin{cases} p^{r'} + p^{r'-1} & \text{if } 2s_p \leq r_p = 2r', \\ 2p^{r'} & \text{if } 2s_p \leq r_p = 2r' + 1, \\ 2p^{r_p - s_p} & \text{if } \ 2s_p \geq r_p, \end{cases}$$

$$\varepsilon_k := \begin{cases} 0 & \text{if } k \text{ is odd}, \\ -\frac{1}{4} & \text{if } k \equiv 2 \pmod 4, \\ \frac{1}{4} & \text{if } k \equiv 0 \pmod 4, \end{cases} \qquad \mu_k := \begin{cases} 0 & \text{if } k \equiv 0 \pmod 3, \\ -\frac{1}{4} & \text{if } k \equiv 1 \pmod 3, \\ \frac{1}{4} & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Note that

$$\dim M_k(\Gamma_0(N), \chi) = 0 \quad \text{if } k < 0 \text{ or } k = 0 \text{ and } \chi \text{ is not the trivial character } \chi^{\text{triv}};$$
$$\dim M_0(\Gamma_0(N), \chi^{\text{triv}}) = 1.$$

And we also know that the conductor of the trivial character is 1, and the conductor of χ_N, where $4 \mid N$, is 4.

Since $\eta^6(4z) \in S_3(\Gamma_0(16), \chi_{16})$, we have $\dim M_{-1}(\Gamma_0(16), \chi_{16}) = 0$, $r_2 = 4$, $s_2 = 2$ and $\varepsilon_3 = \mu_3 = 0$. We then get

$$\dim S_3(\Gamma_0(16), \chi_{16}) = (1/6) \cdot 16 \cdot (3/2) - (1/2)\left(2^2 + 2\right) = 1.$$

Similarly, we can find that $\dim S_2(\Gamma_0(36), \chi_{36}) = 1$ and $\dim S_6(\Gamma_0(4), \chi_4) = 1$. In addition, by (2.5), there must exist a scalar λ_p for every prime p such that $f(z) \mid T(p) = \lambda_p f(z)$, i.e. they are all primitive eigenforms. By [6], we know that $\eta^2(12z)$ is also a primitive eigenform. □

Since Theorem 2 immediately follows Theorem 3, we will conclude by proving Theorem 3.

Proof of Theorem 3. Let $f(z) = \eta^6(4z)$. Since it is a cusp-form of weight 3 on $\Gamma_0(16)$ with the character χ_{16} and it is proved to be a primitive eigenform, by (2.6), we know that for each prime l, there exists a scalar λ_l such that

$$f(z) \mid T(l) = \sum_{n=1}^{\infty} \left(b(ln) + \chi_{16}(l) l^2 b(n/l)\right) q^n = \lambda_l f(z) = \lambda_l \sum_{n=1}^{\infty} \lambda_l b(n) q^n.$$

It is clear that for all non-negative integers n and primes l, we have

$$\lambda_l b(n) = b(ln) + \chi_{16}(l) l^2 b(n/l).$$

Since $b(1) = 1$, we get $\lambda_l = b(l)$. Notice that $\chi_{16}(l) = \left(\frac{-4}{l}\right)$, so for all primes l, we have

$$b(l)b(n) = b(ln) + \left(\frac{-4}{l}\right) l^2 b(n/l),$$

which is (3.2). By applying the same technique, (3.1), (3.3) and (3.4) can be proved in a similar way. □

References

[1] H.M. Farkas and I. Kra, *Three term theta identities*, Contemp. Math. **256** (2000), 95-96.
[2] L.J. Mordell, *Note on certain modular relations considered by Messers, Ramanujan, Darling, and Rogers*, Proc. Lond. Math. Soc. **20** (1922), 408-416.
[3] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer-Verlag, 1993.
[4] D. Dummit, H. Kisilevsky and J. Mckay, *Multiplicative products of η-functions*, Contemp. Math. **45** (1985), 89-94.
[5] H. Cohen and J. Oesterlé, *Dimensions des espaces de formes modulaires*, Modular functions of one variable, VI. Springer Lect. Notes **627** (1977), 70-73.
[6] Jean-Pierre Serre, *Sur la lacunarité des puissances de η*, Glasgow Math. J. **27** (1985), 203-221.

Department of Mathematics, Colgate University, Hamilton, NY 13346
E-mail address: wren@mail.colgate.edu