

REPORT ON: GEODESICS ON SURFACES OF CONSTANT  
GAUSS CURVATURE USING MATHEMATICA  
BY  
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EVALUATION AND RECOMMENDATION

I think that the subject of geodesics on surfaces and their depiction via computer algebra software is a good one for an undergraduate journal. The idea that solutions to differential equations are curves is an important one for undergraduates to see. Furthermore, the notion that shapes are best described by parametrizations and that parametrizations lead to differential equations in the parameters is crucial to developing an understanding of higher-dimensional geometric analysis. However, the paper in its present form is not suitable for publication in an undergraduate journal. The exposition is too unfocussed and an undergraduate might easily ask exactly what the point of the article is. I *do* believe that the article can be salvaged, but this will take some work on the part of the author. I encourage the author to undertake this effort because, in the process of creating a better paper, I think the author will learn some interesting things. To summarize, I think the paper must be rewritten before a final judgement on its suitability for publication can be made.

COMMENTS ON MATHEMATICS AND EXPOSITION

- The very first paragraph refers to great circles as shortest distance arcs on the Earth (thought of as a sphere). Actually, this is just implied, but it would be better to make the last sentence of the paragraph explicit in this regard. Indeed, the later proof that great circles *are* the sphere's geodesics must appear to be magic to an undergrad reading it. Using isometries is unfair because it must then be proved that isometries preserve geodesics. This is stated several times in the article, but no proof is given — and the proof isn't trivial. Even the *notion* of a surface isometry is non-trivial. This is an example of what I mean when I say that the article is unfocussed. Why not just focus on the geodesic equations and their solutions? Find geodesics on the sphere by solving the geodesic equations. This is a great lesson for undergrads because, here, "solving" means not finding an explicit formula, but rather displaying a relation which serves to characterize geodesics as great circles. This may be the

first time undergrads will see this. But it *is* explicit and it *does* fit with the author's exposition of the geodesic equations. If the author hasn't seen the geodesic equation solution, consult Oprea, *Differential Geometry and its Applications*, Chapter 5.

- This brings up another point. Why talk about the exponential map here? The results quoted are very standard, but I would venture to say meaningless to most undergraduates. But here is where computers can give meaning to mathematical theorems. Why not analyze a situation on the sphere of two cities at the same latitude? Compute the distance along the latitude circle (by hand even) and then find the exact great circle joining the cities. We know (now) that geodesics are great circles, so find it using vector calculus alone. Here is what I mean. Take vectors from the center of the sphere to the cities and take the cross product of these. This is then the normal of the plane intersecting the sphere in the great circle through the two cities. Take the cross product of the normal with one of the city vectors to get an orthogonal basis for the plane and then normalize and parametrize the circle in the plane (with cos and sin) which is the great circle on the sphere. Then compute the distance along the great circle and compare to the latitude distance. Of course, all of this should be done with Mathematica. This would be infinitely more illustrative and interesting than simply spouting results from books. (By the way, in my copy of the article, there is a missing proof of the exp-neighborhood theorem anyways.) So I would delete all exponential map stuff and focus on Mathematica and the geodesic equations — the things in the title of the paper.
- I would also get rid of the material on the derivative map and frames. The only thing that matters about the surface is that a unit normal can be consistently, smoothly defined and *that* is guaranteed by  $\mathbf{x}_u \times \mathbf{x}_v \neq 0$ . Then there is no need for  $E_1, E_2, E_3$ . Just take  $\mathbf{x}_u$  (or the corresponding unit vector),  $\mathbf{x}_v$  and unit normal

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}.$$

Also, I'm not really sure why constant curvature is relevant here. Is it used in any way? I don't think so. If not, then there is really no need for the shape operator or a mention of Gauss curvature. The important points are parametrization, solving the geodesic equations (by hand or numerically) and displaying the solutions. So why in the paper do we see plots of a sphere and a plane (which are certainly familiar to students), but only a few plots of actual geodesics? We should see geodesic plots on cones, surfaces of revolution (where the Clairaut relation gives interesting bounces) etc. By the way, are the curves on the sphere really great circles? They look funny to me when they curve over the top or under the bottom. Please check

these. Also, geodesics on the flat torus can't be illustrated with a picture of curves on the torus of revolution.

- This brings up another point. The description of geodesics on the hyperbolic plane (or Poincaré half-plane) is again by magic. But why? The geodesic equations are easy to solve for the Poincaré plane (see Oprea, Ch. 5), so this should be the method. A paper should have a point, a moral — and this paper's moral (I guess from the title) is that Mathematics can show us geodesics by solving the geodesic equations. So, there are some cases where solutions are explicit and Mathematica verifies those. In a way, those are the test cases. Then, a surface of revolution is experimented with, but the geodesic equations can't be solved by hand, so Mathematica provides new information in terms of the pictures it gives. This is the scientific method as the moral of the story. Of course, how do you verify things on the Poincaré plane without fixing the results by using the hand solution? Just fix the Mathematica procedures to handle conformal metrics. This is done in Oprea, Ch. 5 at the end with Maple, but it should be easily convertible to Mathematica. Then the circles in the Poincaré plane are verifications of the by-hand solutions of the geodesic equations. (By the way, there are a few misprints in Oprea, so check his website for these.)
- Of course, this then begs the question: how is a geodesic defined on a surface not in  $\mathbb{R}^3$  (like the Poincaré plane)? This is basically ignored in the article. There is no unit normal, so what can be done? Simply define a geodesic as *a solution to the geodesic equations!* Voila! Again, this emphasizes the importance of the geodesic equations by providing an extension to surfaces not in  $\mathbb{R}^3$  of the very notion of geodesic. By the way, it should be noted when geodesics are defined that the very definition implies that they are constant speed by the usual calculation

$$(\alpha' \cdot \alpha')' = 2\alpha'' \cdot \alpha' = 0.$$

- I don't want to forget one thing. Geodesics connecting two points are not unique. Think of the North and South poles on the sphere. There are an infinite number of longitudes connecting them. Or on the cylinder, a helix winds up above a point and a vertical line also goes up above the point. Geodesics are unique only in the sense of an initial value problem — that is, they start out uniquely. The problem at  $\pi/2$  on the sphere is not a parametrization problem, but a conjugate point problem and I don't think the paper wants to get into that. Remember, the definition of geodesic says nothing about far away uniqueness. A point on the equator (say) has two geodesics connecting it to the North pole, but one is shorter than the other. At any rate, Mathematica will draw either depending on which tangent vector (i.e. direction) is given as input.

- I've already mentioned that using the torus of revolution to illustrate the flat torus is problematical, but let me encourage the author to look at the cylinder (another flat surface) in more depth. What I mean is, if you look in Oprea's book, he unrolls the (flat) cone so that we see the cone's geodesics going to straight lines in the plane. He gives the cylinder as an exercise. Why not use Mathematica to unroll the cylinder to see the helices unroll to lines? This is infinitely more illustrative than a discussion of the flat torus.
- Why are the geodesic equations converted to a first order system? Won't Mathematica solve second order systems numerically? I don't care, but the reader may want to know. For instance, in Oprea, Maple solves the 2nd order systems. Also, I would explain at least one line of code in the geodesic equations in Mathematica, for example. I don't think someone non-versed in Mathematica could follow the code otherwise.
- To summarize: stick with the geodesic equations, solve some by hand, verify by Mathematica and use Mathematica to numerically solve others and graph. Include pictures of interesting things like geodesics rather than a picture of a plane! If the Poincaré plane is really necessary, then modify the Mathematica procedures to find geodesic equations with a conformal metric and graph the geodesics in the plane. Remember, an article like this should be like De Vinci's Last Supper; the point of perspective placed exactly over the most important element.

#### MISPRINTS ETC.

- (1) Pages should be numbered.
- (2) p. 1, paragraph 2; looks like
- (3) p. 1, paragraph 2; thought of as a sphere
- (4) p.1; the parametrization of the sphere is good for  $0 \leq u < 2 * \pi$  and  $-\pi/2 < v < \pi/2$ .
- (5) p. 1; Actually do the calculation (with Mathematica if you like) showing that the sphere parametrization is orthogonal.
- (6) p. 2, paragraph 1; inner product is defined which varies smoothly (or differentiably) on the tangent planes of  $M$  ...
- (7) p. 3, paragraph 1;  $\alpha(t_0) = p$ , not  $\alpha(t) = p$
- (8) p. 3; make sure you say this definition of geodesic gives constant speed
- (9) p. 4; the dot product with  $\mathbf{x}_u$  and  $\mathbf{x}_v$  was forgotten in the first terms of the geodesic equations.
- (10) p.4; An immediate consequence is the following theorem.
- (11) p. 12, two lines above beginning of last paragraph; instead of "Now", say "The geodesic equations reduce to ..."

- (12) p. 12, last paragraph; there are too many “note’s” in this paragraph. Reword if you include.
- (13) p. 14; code at top is too small and blurry
- (14) p. 16; cylinder “below” is actually above. Just say cylinders in Figures 8-10.
- (15) p. 17; We have *not* seen that geodesics are isometric invariants. You’ve only said that.