Two Quasi 2-Groups*

By
Ben Harwood
Department of Mathematics and Computer Science
Northern Kentucky University
Highland Heights, KY 41099
e-mail: kymagician@yahoo.com

Section One: Introduction

As part of an undergraduate research project, I set out to classify all the quasi p-groups of order less than 24. There are 59 groups of order less than 24: the group consisting of the identity, 33 abelian groups, and 25 nonabelian groups. This work is summarized in [Hwd]. Many of the groups are semidirect products, and that structure was exploited in the classification. A brief introduction to the semidirect product may be found in [AbC]. Two of the groups provide nice examples of the techniques that were used to classify the groups of order less than 24 – a group of order 20 $Z_5 \rtimes Z_4$ and a group of order 18 $(Z_3 \times Z_3) \rtimes Z_2$. We will examine these two groups in the sections below. We will show that each of these groups is a quasi 2-group and that each of these groups is not a quasi p-group for $p \neq 2$.

Section Two: Quasi p-Groups

Abhyankar defined quasi p-groups in [Ab]. His definition was:

Definition (2.1) If G is a finite group, then G is a quasi p-group if G is generated by all of its p-Sylow subgroups.

By $p(G)$ Abhyankar denoted the subgroup of G generated by the p-Sylow subgroups. So, a finite group is a quasi p-group if $G = p(G)$. It is easy to see that $p(G)$ is a normal subgroup of G. We denote this by $p(G) \triangleleft G$.

The following lemma is proved in [Hwd].

Lemma (2.2) G is a finite group. The following are equivalent:
1. G is a quasi p-group.
2. G is generated by all of its elements whose orders are powers of p.
3. G has no nontrivial quotient group whose order is prime to p.

2 was most useful to prove that a finite group is a quasi p-group, and 3 was most useful to prove that a finite group was not a quasi p-group.

Section Three: $Z_5 \rtimes Z_4$

In terms of generators and relations, $Z_5 \rtimes Z_4 = \langle x, y | x^4 = y^5 = 1, x^{-1}yx = y^{-1} \rangle$. So, $x \in 2(Z_5 \rtimes Z_4)$. If we can get $y \in 2(Z_5 \rtimes Z_4)$, we will be done because then $2(Z_5 \rtimes Z_4) = Z_5 \rtimes Z_4$. Notice that because $x^{-1}yx = y^{-1}$, $yx = xy^{-1} = xy^4$. Now consider the order of xy. $(xy)^2 = xyxy = xxy^4y = x^2$. So, the order of xy is 4, and, therefore, $xy \in 2(Z_5 \rtimes Z_4)$. Because $x, xy \in 2(Z_5 \rtimes Z_4)$, $y = x^3xy \in 2(Z_5 \rtimes Z_4)$, and we can conclude that $Z_5 \rtimes Z_4$ is a quasi 2-group.

Because all the elements of order 5 in $Z_5 \rtimes Z_4$ are in the factor Z_5, $5(Z_5 \rtimes Z_4)$ is a proper subgroup of $Z_5 \rtimes Z_4$. Therefore, $Z_5 \rtimes Z_4$ is only a quasi 2-group.

We have proved

Proposition (3.1) $Z_5 \rtimes Z_4$ is a quasi 2-group, and it is not a quasi p-group for any prime $p \neq 2$.

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Section Four: $(Z_3 \times Z_3) \rtimes Z_2$

To prove that $(Z_3 \times Z_3) \rtimes Z_2$ is a quasi 2-group, we will find enough elements of order two to generate the group. This method requires understanding of the semidirect product. The semidirect product requires a homomorphism $\phi : Z_2 \to \mathrm{Aut}(Z_3 \times Z_3)$. For $(Z_3 \times Z_3) \rtimes Z_2$, $\phi(0)$ is the identity and $\phi(1)$ maps elements to their inverses. We will use addition for the group operation and write elements of $(Z_3 \times Z_3) \rtimes Z_2$ as $[(x, y), z]$. For $g_1 = [(x_1, y_1), z_1], g_2 = [(x_2, y_2), z_2] \in (Z_3 \times Z_3) \rtimes Z_2$, $g_1 + g_2 = [(x_1, y_1), z_1] + [(x_2, y_2), z_2] = [(x_1, y_1) + \phi(z_1)(x_2, y_2), z_1 + z_2]$. Notice that $\phi(0)(x, y) = (x, y)$ and $\phi(1)(x, y) = (-x, -y)$.

The order of $(Z_3 \times Z_3) \rtimes Z_2$ is 18. We will determine the orders of each of the 18 elements.

Obviously, the order of $[(0, 0), 0]$ is 1.

Now we consider the case of elements of the form $[(x, y), 0]$ with $(x, y) \neq (0, 0)$. We have that

$$2[(x, y), 0] = [(x, y) + \phi(0)(x, y), 0 + 0] = [(x, y) + (x, y), 0] = [(2x, 2y), 0] \neq [(0, 0), 0]$$
$$3[(x, y), 0] = [(2x, 2y) + \phi(0)(x, y), 0 + 0] = [(2x, 2y) + (x, y), 0] = [(3x, 3y), 0] = [(0, 0), 0]$$

So, the order of $[(x, y), 0]$ with $(x, y) \neq (0, 0)$ is 3.

Next consider the element $[(0, 0), 1]$.

$$2[(0, 0), 1] = [(0, 0), 1] + [(0, 0), 1] = [(0, 0) + \phi(1)(0, 0), 1 + 1] = [(0, 0) + (0, 0), 0] = [(0, 0), 0]$$

So, the order of $[(0, 0), 1]$ is 2.

Finally, consider elements of the form $[(x, y), 1]$ with $(x, y) \neq (0, 0)$.

$$2[(x, y), 1] = [(x, y), 1] + [(x, y), 1] = [(x, y) + \phi(1)(x, y), 1 + 1] = [(x, y) + (-x, -y), 0] = [(0, 0), 0]$$

So, the order of $[(x, y), 1]$ with $(x, y) \neq (0, 0)$ is 2.

Therefore, each of the elements of $(Z_3 \times Z_3) \rtimes Z_2$ has order 1, 2, or 3. There is one element of order 1: $[(0, 0), 0]$. There are 8 elements of order 3: $[(1, 0), 0]$, $[(2, 0), 0]$, $[(1, 1), 0]$, $[(2, 1), 0]$, $[(1, 2), 0]$, $[(2, 2), 0]$, $[(0, 1), 0]$, $[(0, 2), 0]$. The remaining 9 elements each have order 2: $[(0, 0,), 1]$, $[(1, 0), 1]$, $[(2, 0), 1]$, $[(1, 1), 1]$, $[(2, 1), 1]$, $[(1, 2), 1]$, $[(2, 2), 1]$, $[(0, 1), 1]$, $[(0, 2), 1]$.

We note two ways to see that the elements of order 2 generate $(Z_3 \times Z_3) \rtimes Z_2$. First, because the 9 elements of order 2 and the identity must be in $2((Z_3 \times Z_3) \rtimes Z_2)$, by Lagrange's theorem, $2((Z_3 \times Z_3) \rtimes Z_2)$ must be all of $(Z_3 \times Z_3) \rtimes Z_2$. Alternatively, we notice that $[(2, 0), 1]$, $[(1, 0), 1]$, $[(0, 2), 1]$, and $[(0, 1), 1]$ are each elements of order 2, and that

$$[(2, 0), 1] + [(1, 0), 1] = [(2, 0) + (-1, 0), 1 + 1] = [(1, 0), 0]]$$

and

$$[(0, 2), 1] + [(0, 1), 1] = [(0, 2) + (0, -1), 1 + 1] = [(0, 1), 0]]$$

.

So, the generators of $(Z_3 \times Z_3) \rtimes Z_2$ – $[(1, 0), 0]$, $[(0, 1), 0]$, and $[(0, 0), 1]$ – are all in $2((Z_3 \times Z_3) \rtimes Z_2)$

Because all the elements of order 3 in $(Z_3 \times Z_3) \rtimes Z_2$ are in the factor $Z_3 \times Z_3$, $3((Z_3 \times Z_3) \rtimes Z_2)$ is a proper subgroup of $(Z_3 \times Z_3) \rtimes Z_2$. Therefore, $(Z_3 \times Z_3) \rtimes Z_2$ is only a quasi 2-group.

We have proved

Proposition (4.1) $(Z_3 \times Z_3) \rtimes Z_2$ is a quasi 2-group, and it is not a quasi p-group for any prime $p \neq 2$.

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